operational methods

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$$e^{B}e^{A} = \sum \frac{\left(\overset{1}{A}+\overset{2}{B}\right)^{n}}{n!}$$

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CONTENTS

PREFACE

Operational methods are such methods, which make it possible to reduce differential problems to algebraic problems. This is why these methods are of particular use to specialists dealing with applied mathematics.

This book is devoted to one, but sufficiently general operational method, which absorbs many operational methods known to date and allows for the uniform solution of both classical problems, involving differential equations with partial derivatives, and the absolutely new problems of mathematical physics, including those connected with non-linear equations in partial derivatives.

We shall proceed to describe this general method after studying the methods well-known in mathematics for calculating operators (mainly self-adjoint operators), for which Heaviside's method served as a source.

The main theorem stated in this book belongs to the theory of operators and is proved in the last chapter, but its formulation is given in Introduction as well. This theorem may lead, in particular, to the existence and uniqueness theorems for hyperbolic, elliptic and parabolic equations with variable coefficients and allows to reduce them to integral equations of the second kind with smooth kernels, i.e., to provide an effective solution for these equations. We have in mind that by detaching from the solution the non-smooth (or rapidly oscillating) components, we can thus reduce the problem to the one that is easily solved on an electronic computer. When the rapidly oscillating part detached from the solutions was compared with the exact solution by means of a numerical experiment in the one-dimensional example of Sec. 8 of Introduction, their near coincidence was revealed. It turns out, however, that the computer is unable to perform such an experiment even in a two-dimensional case because of the enormous number of operations. Under these circumstances the only thing to do is using suitable asymptotics and reducing the initial problem to an integral equation. Thus, the approximate solution constructed in the main theorem,

detaching the non-smooth or rapidly oscillating part, serves as a kind of natural supplement to the electronic computer: together they provide a numerical answer.

This book on operational methods should be accessible to senior-course students of mathematics and physics faculties at universities and departments of applied mathematics. This means that only a knowledge of classical analysis is required of the reader. The book provides explanations in sufficient volume of such concepts as the theory of Banach algebras of distributions (Chapter I), the theory of linear differential and difference equations (Secs. 1, 2, and 3 of Introduction), the theory of non-linear equations of the first order with partial derivatives (Chapter IV).

This material may be also of use to the reader who is already familiar with these questions, because rather often it is not presented in traditional style, and adapted for further reference. The reader who studies the book thoroughly will be equipped to carry on independent research in the modern theory of linear, non-linear differential and differential-difference equations with partial derivatives. Besides, a study of the concrete problems presented here may serve as an excellent springboard for further investigation, although from a certain viewpoint, of the theory of representations, topology and the theory of sheafs. An example of the latter is the theory of V-objects (Chapter III). As to applying the described methods in physics, quite obviously, it is not limited to the examples given in the book. Additional mention should be made of the asymptotics of solutions in the band solid-state theory, in problems of molecular collisions, in the theory of laser resonators, in chain-reaction equations in chemistry, in problems of refraction and diffraction, in derivation of integral equations of the Lippmann-Schwinger and Faddeev type, in calculating quasi-classical amendments to the Thomas-Fermi equation, amendments to electronic plasma equations, asymptotic solutions of Hartree's equations, in electronic optics, in problems of above-barrier reflection and in many other problems of modern mathematical physics.

This book has been written in such a way as to serve the widest possible circle of readers. It is suitable for two methods of study. The reader, who seeks to avoid fine assessments and passing to the limit and only wishes to master the practical techniques for obtaining asymptotic solutions, may omit that part of the book which is devoted to functional analysis. The "Introduction to Operational Calculus" has been written with this purpose in view; having studied it, the reader can tackle operator techniques (omitting Chapters I-II) learning how to reduce concrete problems to an integral equation, detaching the non-smooth part of the solution.

In following the second method of learning, which consists in the gradual and deeper study of operational methods, it is best

to begin with the first chapter and to read Introduction only after Chapter II.

The most effective way of mastering the subject, however, consists rather in first reading Introduction and then reading all the book in succession.

The reader should nevertheless be warned that all these methods are not at all easy, because the book provides a new operational calculus—the calculus of ordered operators.

This book is actually a synopsis of the course of lectures delivered by the author in the duration of three years at the Department of Applied Mathematics at the Moscow Institute of Electronic Machine-Building, consecutively to third-, fourth- and fifth-year students. The material of the last chapters was also given in lectures (in the duration of three years) to fifth-year (graduating) students specializing in mathematics at the Physics Department of the Moscow State University. Besides, this course of lectures (an abridged version) was also delivered by the author at several mathematical schools (at the International Mathematics School at Sopot in 1971, in Pushchino in 1971, in Voronezh in 1972, and other places).

The experience gained in delivering this course of lectures showed that, despite expectations, operational calculus, with the parallel examination of comparatively few examples at seminar sessions, is grasped more easily by students than questions connected with traditional functional analysis, fine assessments, functional spaces and passing to the limit.

The students of senior courses and post-graduates can quickly learn to solve complex problems connected with operational methods. In this book, for instance, the reader will find important formulas obtained by M. V. Karasev (Theorems 4.4 and 6.6 of Introduction) and V. G. Danilov (Theorem 1.1 in Chapter III).

If we should draw an analogy between the exposition of operational calculus given in the "Introduction to Operational Calculus" and the hypothetical exposition of differential calculus (see the left column) the approximate result would consist in the following:

Secs. 1-3 deal with the ring of polynomials.	Secs. 1-3 deal with the Heaviside method and operator calculus with constant coefficients.
Secs. 4-5. The rules of the formal differentiation of polynomials are given; various formulas of differentiation for polynomials are derived: formulas for differentiating the product; the composite function, expansion in a Taylor series.	Secs. 4-5. The rules of the new calculus are derived on the example of formal power series of ordered operators. Formulas are derived for commutators of series, composite function and expansion in a Newton series.

In Sec. 6 the system of axioms is introduced and the same formulas are derived for arbitrary functions in the form of theorems, thus determining all formulas of the techniques of differentiation.	In Sec. 6 the system of axioms is introduced and the formulas for arbitrary functions in the form of theorems are the same, thus determining all formulas of the techniques of the calculation of ordered operators.
Sec. 7 gives a simple example.	Sec. 7 gives a simple example.
Sec. 8 shows how a new solution for a classical physics problem is obtained with the use of introduced techniques.	Sec. 8 shows how new physical effects are obtained with the use of introduced techniques in studying the classical problem of deriving the wave equation from equations of the oscillations of a crystal lattice.
Sec. 9 sets a problem on the simplest differential equation. The fundamental concept of a characteristic polynomial is introduced and the physical meaning of initial conditions is discussed. The main theorem of existence is formulated and the formula for the solution of an equation in a particular case is given.	The main problem is formulated in Sec. 9, the fundamental concept of characteristics for functions of an ordered set of operators is introduced, and the physical meaning of absorption conditions is discussed. The main theorem is formulated and the explicit formula is given for the solution of the main problem in a particular case (the general formula is given in the last chapter).

Further (if we are to continue the analogy) the theory of the derivative as a limit (Chapter I) and partial derivatives as multiple limits (Chapter II) is conducted in succession.

The theory of functions in Chapter I is built by systematic use of the conception of "completing with limits", just as real numbers in Cantor's theory are elements of the completion of a set of sequences of rational numbers. This point of view is in accord with the initial conception of physicists with regard to Dirac's delta function as the limit of bell-shaped functions.

Secs. 3-9 of Chapter I are devoted to calculating the functions of one operator. These sections introduce the concepts of generating and regular operators generalizing the concepts of self-adjoint and normal operators, respectively. The theorem is proved for regular operators which shows their generality in the case of a discrete spectrum: the regularity of an operator is a necessary and sufficient

condition for the completeness of eigen and associated elements. The known calculation of self-adjoint operators is derived as a consequence.

At the beginning of Chapter II the calculation of functions of two noncommutative ordered operators, their joint spectrum and spectral expansion are studied in detail. Then the functions of several regular operators are examined and formulas for them are derived, which had been obtained from axioms in Introduction. All the techniques for calculating noncommutative operators are thus built, but on a functional basis now. Operational calculus as such ends with this.

The rest of the book is devoted to a special transformation making it possible to prove the main theorem. This transformation is called the canonical operator.

A great deal of preparatory work is conducted in Chapter III. A canonical operator is introduced in the simplest real one-dimensional case (depending on ordered operators).

Chapter IV calls for an examination in greater detail. This part of the book is required for the ultimate construction of a complex canonical operator, but, generally speaking, the chapter is quite detached and in no way connected with the techniques of ordered operators. It may even be read all at once. The chapter is devoted to constructing the solution, in the large, of equations generalizing Hamilton-Jacobi equations.

We introduce the notion of Lagrangean manifold with a complex germ which results in a geometric interpretation for solutions of equations of the Hamilton-Jacobi type.

The following physical analogy may be cited to illustrate the point. When a stone is dropped into water it causes the waves to spread over the surface in even circles at first; when the wave is reflected the picture that is produced can hardly be given a geometric interpretation. The same may be said of the Hamilton-Jacobi equation (a particular solution of a certain Hamilton-Jacobi equation will be precisely the one that causes these circles) which on intervals of time not exceeding a certain t_1 has a simple and smooth solution. Further, when $t > t_1$ there occurs a similarity of imposition of waves reflected many times. In order to gain a clear understanding of multi-valued functions (branching of the solution), it is necessary to make them uniform in much the same way as this is done by means of Riemann's sheets for analytical functions with branching. It is precisely the construction of the object in the phase space of the Lagrangean manifold with a complex germ that makes it possible to "unravel" multi-valued solutions of Hamilton-Jacobi type equations (with absorption). The concept "index"—as a whole number—introduced here makes it possible to designate these solutions (sheets) on the Lagrangean manifold with a complex germ.

Besides, the indices of closed paths on the Lagrangean manifold with a complex germ constitute an important characteristic of this object (the characteristic class of an object).

After this the complex canonical operator is built in Chapter V and the main theorem is proved.

The results of Chapters I-V and Secs. 4-9 of Introduction mainly belong to the author (with the exception of the theorems in Chapter I concerning self-adjoint operators, and the theorems of Karasev and Danilov mentioned above).

The starting point of this research was Feinman's remark* to the effect that if the order of action of the operators is determined by indices, then the operators become as if commuting.

The theory of the Lagrangean manifold with a complex germ originated as a result of the study of Leray's works on the Cauchy problem.

A. A. Kirillov, who edited the book, made a number of valuable suggestions in principle concerning the structure of the book as a whole.

I benefited largely from my consultations with A. A. Samarsky, discussions with D. V. Anosov and V. V. Kucherenko, for which I am deeply grateful to them. I am also very grateful to P. P. Mosolov, G. A. Voropaeva, V. L. Dubnov who made several valuable remarks pertaining to the author's manuscript.

This book, however, would never have been written as a school aid if not for my pupils, who made notes and worked at this course of lectures. Moreover, they did not even have detailed synopses of Chapters III and V at their disposal. The first two chapters were jotted down by V. L. Dubnov, the third by M. V. Karasev and V. L. Dubnov, the fourth by V. G. Danilov, the fifth by V. G. Danilov and M. V. Karasev. The chapters IV and V were edited by G. A. Voropayeva; A. G. Prudkovsky made all the calculations on a computer. Besides this, our numerous discussions with V. L. Dubnov, M. V. Karasev, V. G. Danilov, S. Yu. Dobrokhotov, A. G. Prudkovsky and G. A. Voropayeva rendered me invaluable assistance. In preparing the manuscript for the publisher I was also helped by A. G. Davtyan and G. Yu. Malysheva. My gratitude to them all is boundless.

V. P. Maslov

* In the article "Operational Calculus Relating to Quantum Electrodynamics" (*Phys. Rev.*, 84, 1951) in the section "Description of Method of Designation". By the recommendation of the editor, Feinman's calculus has not been included in the book, just as the applications to physics enumerated on page 8.

INTRODUCTION TO OPERATIONAL CALCULUS

Sec. 1. Solution of Ordinary Differential Equations by the Heaviside Operational Method

Let $C^\infty(\mathbf{R})$ (or C^∞) denote the set of infinitely differentiable functions $\varphi(x)$, $-\infty < x < +\infty$. *The operation (operator), d/dx, of differentiation* with respect to x is defined in C^∞; this operation will henceforth be denoted by D:

$$D\varphi(x) = \frac{d}{dx}\varphi(x) = \varphi'(x).$$

Clearly, the operator

$$P_n(D) : C^\infty \to C^\infty \text{ given by } P_n(D) = \sum_{i=0}^{n} a_i D^i, \quad a_i D^0 = a_i$$

is defined for every polynomial $P_n(x)$, where a_i are complex numbers, $i = 0, 1, 2, \ldots, n$. That is, by definition,

$$P_n(D)\varphi(x) \overset{\text{def}}{=} \sum_{i=0}^{n} a_i \varphi^{(i)}(x), \quad \varphi^{(0)} = \varphi.$$

Let $K_1[D]$ denote the set of operators of the form $P_n(D)$; for any two operators in $K_1[D]$ their sum and product are defined by the formulas

$$\sum_{i=0}^{n} a_i D^i + \sum_{i=0}^{n} b_i D^i = \sum_{i=0}^{n} (a_i + b_i) D^i,$$

$$\left(\sum_{i=0}^{n} a_i D^i\right) \cdot \left(\sum_{j=0}^{n} b_j D^j\right) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_i b_j D^{i+j}.$$

Let $K_1[x]$ denote the set of polynomials over the complex number field

$$P_n \in K_1[x], \quad P_n(x) = \sum_{i=0}^{n} a_i x^i.$$

There is a one-to-one correspondence between the operator $P_n(D) \in K_1[D]$, $P_n(D) = a_0 + a_1 D + \ldots + a_n D^n$ and the polynomial

$P_n \in K_1[x]$. (To prove that this is a one-to-one correspondence, it suffices to show that if $\sum_{n=0}^{N} a_n f^{(n)}(x) \equiv 0$ for every $f \in C^\infty$, then $a_n \equiv 0$; but this is obvious.) We shall call the polynomial $P(x)$ corresponding to the operator $P(D)$ *the symbol of the operator* $P(D)$. Under this correspondence the sum of the polynomials goes to the sum of the two corresponding operators and the product of two polynomials goes to the product of the two corresponding operators. We shall say that $K_1[x]$ and $K_1[D]$ are *isomorphic* by this construction.

By the main theorem of algebra, any polynomial can be written in the form

$$P_n(x) = a_n \prod_{i=1}^{m} (x-\alpha_i)^{k_i}, \quad \sum_{i=1}^{m} k_i = n, \tag{1.1}$$

where α_i are the complex roots of the polynomial and k_i is the multiplicity of the corresponding root.

It follows from the isomorphism of $K_1[x]$ and $K_1[D]$ that for every operator $P_n(D)$ we have the formula

$$P_n(D) = a_n \prod_{i=1}^{m} (D-\alpha_i)^{k_i}, \tag{1.2}$$

where α_i is the operation (operator) of multiplication by the constant α_i.

We shall now solve an ordinary differential equation with constant coefficients

$$P_n(D)\, y(x) = \sum_{i=0}^{n} a_i y^{(1)}(x) = f(x), \quad f \in C^\infty. \tag{1.3}$$

Suppose for the sake of simplicity that all the roots α_i of the polynomial $P_n(x)$ are simple. Then we can put equation (1.3) into the form

$$\prod_{i=1}^{n} (D-\alpha_i)\, y(x) = \frac{f(x)}{a_n}.$$

To solve equation (1.3) we need the following lemma.

Lemma 1.1. *Let* $P_n(D) \in K_1[D]$, $f \in C^\infty$. *The following formula is valid*:

$$P_n(D)\, e^{\alpha x} f(x) = e^{\alpha x} P_n(D+\alpha)\, f(x). \tag{1.4}$$

We prove this by induction. We compute $De^{\alpha x}f(x)$:

$$De^{\alpha x}f(x) = \alpha e^{\alpha x}f(x) + e^{\alpha x}f'(x) = e^{\alpha x}(\alpha f(x) + f'(x)) = \\ = e^{\alpha x}(D+\alpha) f(x).$$

Suppose that formula (1.4) is valid for the operator D^{n-1}. We prove that it is valid for the operator D^n as well. We have

$$D^n e^{\alpha x}f(x) = D[D^{n-1}e^{\alpha x}f(x)] = DF(x),$$

where $F(x) = D^{n-1}e^{\alpha x}f(x)$. By the induction assumption

$$D^{n-1}e^{\alpha x}f(x) = e^{\alpha x}(D+\alpha)^{n-1}f(x);$$

hence

$$DF(x) = De^{\alpha x}[(D+\alpha)^{n-1}f(x)] = \alpha e^{\alpha x}[(D+\alpha)^{n-1}f(x)] + \\ + e^{\alpha x}D[(D+\alpha)^{n-1}f(x)] = e^{\alpha x}(D+\alpha)^n f(x).$$

Thus for any integer m we have obtained the formula

$$D^m e^{\alpha x}f(x) = e^{\alpha x}(D+\alpha)^m f(x). \tag{1.5}$$

Equation (1.4) follows immediately from (1.5), and the lemma is proved.

Now consider the equation

$$(D-\alpha)y(x) = f(x), \quad f \in C^\infty. \tag{1.6}$$

Using the lemma we obtain

$$(D-\alpha)y(x) = (D-\alpha)e^{\alpha x}e^{-\alpha x}y(x) = e^{\alpha x}De^{-\alpha x}y(x).$$

Thus equation (1.6) is equivalent to

$$De^{-\alpha x}y(x) = f(x)e^{-\alpha x}. \tag{1.6'}$$

It is well known that the general solution of the equation $Dy(x) = f(x)$ has the form $y(x) = \int f(x)\,dx + C$, where C is a constant. Introduce the notation

$$\int f(x)\,dx + C = \frac{1}{D}f(x). \tag{1.7}$$

We note that $1/D$ maps an element $f \in C^\infty$ into an entire class of functions $1/D\ f \subset C^\infty$; if $y_0 \in \frac{1}{D}f$, then $z \in \frac{1}{D}f$ if and only if $z - y_0 = \text{const}$.

We can now write down a solution of equations (1.6) and (1.6′) in the form

$$y = e^{\alpha x}\frac{1}{D}e^{-\alpha x}f(x). \tag{1.8}$$

We return to equation (1.3). We have

$$P_n(D)\,y(x)=a_n\prod_{i=1}^{n}(D-\alpha_i)\,y(x)=a_n(D-\alpha_1)\prod_{i=2}^{n}(D-\alpha_i)\,y(x)=$$

$$=a_ne^{\alpha_1x}De^{-\alpha_1x}\prod_{i=2}^{n}(D-\alpha_i)\,y(x)=\ldots=$$

$$=a_ne^{\alpha_1x}De^{-\alpha_1x}\ldots e^{\alpha_nx}De^{-\alpha_nx}\,y(x)=f(x).$$

Successively applying (1.8), we obtain the formula

$$y=e^{\alpha_nx}\frac{1}{D}e^{-\alpha_nx}\ldots e^{\alpha_1x}\frac{1}{D}e^{-\alpha_1x}\cdot\frac{f(x)}{a_n}\stackrel{\text{def}}{=}\frac{1}{P_n(D)}f(x). \tag{1.9}$$

We have obtained a formula which gives a solution of equation (1.3) in terms of integrals of the right-hand side. Thus *the existence of a solution* is proved. It is easy to verify that the result is independent of the order of the roots $\alpha_1, \ldots, \alpha_n$ of the polynomial $P_n(x)$. It is clear from the form of the solution that the general solution of equation (1.3) given by (1.9) depends on not more than n constants (in fact on exactly n).

Formula (1.9) defines a mapping from an element $f \in C^\infty$ to a class of functions. If the equation $P_n(x) = 0$ has simple roots $\alpha_1, \ldots, \alpha_n$, then the difference of two elements of the class has the form $c_1e^{\alpha_1x} + c_2e^{\alpha_2x} + \ldots + c_ne^{\alpha_nx}$, where $c_1, c_2, \ldots, c_n$ are constants. The method of proof shows that formula (1.9) is valid for multiple roots as well; in this case the difference of two elements has the form

$$P_1(x)\,e^{\alpha_1x}+P_2(x)\,e^{\alpha_2x}+\ldots+P_m(x)\,e^{\alpha_mx},$$

where $P_1, P_2, \ldots, P_m$ are suitable polynomials.

We shall denote this mapping (or operator) by $\frac{1}{P_n(D)}$.

We now show that, to obtain *uniqueness of the solution*, it is sufficient to specify the initial Cauchy data

$$y(0)=c_0,\ \ldots,\ y^{(n-1)}(0)=c_{n-1}. \tag{1.10}$$

To prove this, it suffices to show that the problem

$$P_n(D)\,y(x)=0, \tag{1.11}$$

$$y(0)=0,\ \ldots,\ y^{(n-1)}(0)=0 \tag{1.12}$$

has the unique solution $y(x)\equiv 0$. Note that the problem $(D-\alpha)\,y(x)=0$, $y(0)=0$ has the unique solution $y(x)\equiv 0$.

We rewrite equation (1.11):

$$P_n(D)\,y(x)=\prod_i(D-\alpha_i)^{k_i}\,y(x)=(D-\alpha_1)\,W(x)=0.$$

Here $W(x)$ is obviously a linear combination of the derivatives of $y(x)$ of order $\leqslant n-1$. Hence $W(0)=0$ by (1.12), and therefore, $W(x)=0$. It is clear that by continuing this process we obtain $y(x)=0$.

We now give another method of calculating the class of functions $\frac{1}{P_n(D)}f$, which uses an extension of one variable x.

Let $R(x) = P(x)/Q(x)$, where $P(x)$, $Q(x)$ are polynomials. We shall correspond to the function $R(x)$ the mapping $R(D)$, which maps function $f \in C^\infty$ to the class of functions $P(D)\frac{1}{Q(D)}f \subset C^\infty$. The function $R(x)$ will be called *the symbol of the mapping* $R(D)$. We verify that the mapping $R(D)$ is independent of the way we write its symbol $R(x)$ in the form $P(x)/Q(x)$. In fact, let $\overline{P}(x) = P(x)S(x)$ and $\overline{Q}(x) = Q(x)S(x)$, where $S(x)$ is a polynomial. We now prove that for any $f \in C^\infty$ we have the equation

$$P(D)\frac{1}{Q(D)}f = \overline{P}(D)\frac{1}{\overline{Q}(D)}f. \tag{1.13}$$

Let $y \in P(D)\frac{1}{Q(D)}f$. Then $y(x)$ may be written in the form $y(x) = P(D)z(x)$, where $z(x)$ is a solution of the equation $Q(D)z(x) = f(x)$. Let $u(x)$ be a solution of the equation $S(D)u(x) = z(x)$; then $\overline{Q}(D)u(x) = Q(D)S(D)u(x) = Q(D)z(x) = f(x)$ so that $u \in \frac{1}{\overline{Q}(D)}f$. Hence $y(x) = P(D)z(x) = P(D)S(D)u(x) = \overline{P}(D)u(x)$. Thus

$$y \in \overline{P}(D)\frac{1}{\overline{Q}(D)}f.$$

Conversely suppose $y \in \overline{P}(D)\frac{1}{\overline{Q}(D)}f$. Then $y(x) = \overline{P}(D)u(x)$, where $u(x)$ satisfies the equation $\overline{Q}(D)u(x) = f(x)$. Let $z(x) = S(D)u(x)$. Then $z(x)$ satisfies the equation $Q(D)z(x) = f(x)$ and $y(x) = P(D)z(x) \in P(D)\frac{1}{Q(D)}f(x)$; hence $y \in P(D)\frac{1}{Q(D)}f$, Q.E.D.

Let $F[D]$ be a set of operators $R(D)$. We define an operation of a sum of two elements in $F[D]$, induced by the corresponding operation in the field $F[x]$ of rational functions. By definition, $R(D) = R_1(D) + R_2(D)$ if, and only if, $R(x) = R_1(x) + R_2(x)$. The following lemma holds.

Lemma 1.2. *Let $R_1(x)$, $R_2(x)$ be elements of the field $F[x]$, $R(x) = R_1(x) + R_2(x)$. A function y belongs to a class $R(D)f$ if, and only if, it may be expressed in the form $y(x) = y_1(x) + y_2(x)$, where $y_1 \in R_1(D)f$, $y_2 \in R_2(D)f$.*

Proof. Let $P_i(x)$, $Q_i(x)$ $(i = 1, 2)$ be polynomials and $R_i(x) = P_i(x)/Q_i(x)$ $(i = 1, 2)$. Then $R(x) = R_1(x) + R_2(x) = P(x)/Q(x)$, where $P(x) = P_1(x)Q_2(x) + Q_1(x)P_2(x)$, $Q(x) = Q_1(x)Q_2(x)$. Denote the class of functions of the form $y(x)$ by

$$R_1(D)f(x) + R_2(D)f(x).$$

We shall prove that $R(D)f \subset R_1(D)f + R_2(D)f$. Let $y(x) \in R(D)f(x)$. Then $y(x) = [P_1(D)\,Q_2(D) + Q_1(D)P_2(D)]\,z(x)$, where $z(x)$ satisfies the equation

$$Q_1(D)Q_2(D)z(x) = f(x). \tag{1.14}$$

Let $P_1(D)Q_2(D)z(x) = y_1(x)$, $P_2(D)Q_1(D)z(x) = y_2(x)$. The function $u_1(x) = Q_2(D)z(x)$ satisfies the equation $Q_1(D)u_1(x) = f(x)$ and the function $u_2(x) = Q_1(D)z(x)$ satisfies the equation $Q_2(D)u_2(x) = f(x)$. Hence

$$y_1(x) \in R_1(D)f(x), \quad y_2(x) \in R_2(D)f(x)$$

and this proves the lemma.

Consider an example of the calculation of $1/P_n(D)$ by the homomorphism obtained $R(x) \to R(D)$. We express the symbol $1/P_n(x)$ as the sum of simple fractions

$$\frac{1}{P_n(x)} = \sum_{i=1}^{m} \frac{a_i}{(x-\alpha_i)^{k_i}}, \quad \sum_{i=1}^{m} k_i = n,$$

where α_i are complex roots of multiplicity k_i of the polynomial $P_n(x)$, a_i are complex polynomials. It follows from Lemma 1.2 that

$$\frac{1}{P_n(D)} f \subset \sum_{i=1}^{m} \frac{a_i}{(D-\alpha_i)^{k_i}} f.$$

Hence, any solution $y(x)$ of equation (1.3) can be expressed in the form

$$y(x = \sum_{i=1}^{m} a_i(D)\, e^{\alpha_i x} \underbrace{\frac{1}{D} \cdots \frac{1}{D}}_{k_i} e^{-\alpha_i x} f(x), \tag{1.15}$$

which is immediately seen from (1.6), (1.6′) and (1.8). In the particular case of the simple roots, (1.15) has the form

$$y(x) = \sum_{i=1}^{n} \left[a_i\, e^{\alpha_i x} \int_0^x e^{-\alpha_i p} f(p)\,dp + C_i e^{\alpha_i x} \right]. \tag{1.15′}$$

It is not difficult to verify*, that the general solution of (1.3) has exactly n constants. Hence, (1.15) provides *the general solution* $\left(\text{i.e., any element of the class } \frac{1}{P(D)} f\right)$ of equation (1.3).

We can also use formula (1.9) to find the solution of the homogeneous equation.

Example. Find the general solution of the equation

$$(D - \alpha_1)(D - \alpha_2)\, y(x) = 0, \qquad \alpha_1 \neq \alpha_2.$$

Formula (1.9) implies

$$y(x) = e^{\alpha_2 x} \frac{1}{D} e^{-\alpha_2 x} e^{\alpha_1 x} \frac{1}{D} e^{-\alpha_1 x} 0 =$$

$$= e^{\alpha_2 x} \frac{1}{D} e^{-\alpha_2 x} e^{\alpha_1 x} C_1 = e^{\alpha_2 x} \frac{1}{D} e^{(\alpha_1 - \alpha_2)x} C_1 =$$

$$= e^{\alpha_2 x} C_1 \int e^{(\alpha_1 - \alpha_2)x}\, dx + C_2 e^{\alpha_2 x} = C_1' e^{\alpha_1 x} + C_2 e^{\alpha_2 x}.$$

Now we shall consider the case of multiple roots.

Example. Find the general solution of the equation

$$(D - \alpha)^3\, y(x) = 0.$$

Equation (1.9) implies:

$$y(x) = e^{\alpha x} \frac{1}{D} e^{-\alpha x} e^{\alpha x} \frac{1}{D} e^{-\alpha x} e^{\alpha x} \frac{1}{D} e^{-\alpha x} 0 =$$

$$= e^{\alpha x} \frac{1}{D} \frac{1}{D} \frac{1}{D} 0 = e^{\alpha x} (C_0 + C_1 x + C_2 x^2).$$

Note. Now consider the equation

$$P_n(D)\, y(x) = P_k(x), \qquad P_n(0) \neq 0, \tag{1.16}$$

where $P_k(x)$ is a polynomial of degree k. Let $P_{n,s}(x)$ be a sum of $s + 1$ terms of the Taylor expansion of the function $1/P_n(x)$ at $x = 0$. The equation

$$P_n(x) \left[P_{n,s}(x) + \sum_{k=s+1}^{\infty} \left(\frac{1}{P_n(x)} \right)^{(k)}_{x=0} \frac{x^k}{k!} \right] = 1$$

implies

$$P_n(x)\, P_{n,s}(x) = 1 + x^{s+1} \cdot P_{n-1}(x), \tag{1.17}$$

* The statement is immediate since a solution of equation (1.3) may be expressed in the form of a sum of the solution of this equation and a solution of the homogeneous equation. On the other hand, any term of the expansion of $1/P_n(D)$ into simple fractions applied to zero satisfies a homogeneous equation $P_n(D)\, y = 0$, since $P_n(D)$ may be expressed in the form $P_k(D)(D - \alpha_k)^{\beta_k}$, where $(D - \alpha_k)^{\beta_k}$ is a denominator of the term.

where $P_{n-1}(x)$ is a polynomial of degree equal to, or less than, $n-1$.

We shall show that

$$y(x) = P_{n,s}(D)\, P_k(x) \tag{1.18}$$

is a solution of equation (1.16) for $s \geqslant k$. Indeed, equation (1.17) implies

$$\begin{aligned} P_n(D)\,\bar{y}(x) = P_n(D)\,P_{n,s}(D)\,P_k(x) = \\ = (1 + P_{n-1}(D)\,D^{s+1}) \cdot P_k(x) = P_k(x) + \\ + P_{n-1}(D) \cdot D^{s+1} \cdot P_k(x). \end{aligned}$$

But the last term equals zero for $s \geqslant k$. Thus formula (1.18) gives a solution of equation (1.16) for any $s \geqslant k$ and any solution of equation (1.16) is expressed by the formula

$$y(x) = P_{n,s}(D)\, P_k(x) + \varphi(x),$$

where $\varphi \in \frac{1}{P_n(D)}\, 0$.

Sec. 2. Difference Equations

Let e^{hD} be *a translation operator* defined by the formula

$$e^{hD} f(x) \overset{\text{def}}{=} f(x+h),$$

where h is a real number.

The definition is justified by the following heuristic argument: let $f(x)$ be an analytic function defined in the $2h$-neighborhood of a point x, $-\infty < x < \infty$. Then

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^k}{k!} f^{(k)}(x).$$

On the other hand, a formal expansion of e^{hD} in a power series provides

$$e^{hD} f(x) = \sum_{k=0}^{\infty} \frac{h^k D^k}{k!} f(x) = \sum_{k=0}^{\infty} \frac{h^k}{k!} f^{(k)}(x) = f(x+h).$$

Thus, the expansion in the Taylor series of $f(x+h)$ is compared with the formal expansion of the exponent e^{hD}.

Consider an equation

$$\sum_{k=0}^{n} a_k y(x+kh) = f(x), \quad f \in C_0^\infty, \tag{2.1}$$

where C_0^∞ is a space of infinitely differentiable finite functions on $\mathbf{R}$ (i.e., with a compact support). Write it in the form

$$P_n(e^{hD})\,y(x) = f(x),$$

where

$$P_n(e^{hD}) = \sum_{k=0}^{n} a_k e^{khD}.$$

We assume for simplicity that all the roots of the equation $P_n(x) = \sum_{k=0}^{n} a_k x^k$ are different. Then an equation is true

$$\frac{1}{P_n(x)} = \sum_{i=1}^{n} \frac{A_i}{x-\alpha_i}, \tag{2.2}$$

where A_i are constants.

If $y(x)$ is a solution of an equation

$$(e^{hD} - \alpha)\,y(x) = f(x) \tag{2.3}$$

then $y(x)$ is a solution of an equation

$$(e^{hD}-\alpha)\,e^{\frac{x}{h}\ln\alpha}\,e^{-\frac{x}{h}\ln\alpha}\,y(x) = f(x).$$

It is easy to verify that

$$(e^{hD}-\alpha)\,f(x) = \alpha e^{\frac{x}{h}\ln\alpha}(e^{hD}-1)\,e^{-\frac{x}{h}\ln\alpha} f(x).$$

Hence, (2.3) may be expressed in the form

$$(e^{hD}-1)\,e^{-\frac{x}{h}\ln\alpha}\,y(x) = \frac{1}{\alpha}\,e^{-\frac{x}{h}\ln\alpha} f(x).$$

Consider that the series $-\sum_{k=0}^{\infty} e^{khD}$ is a formal power series of the function $(e^{hD}-1)^{-1}$. It is easily seen that

$$e^{-\frac{x}{h}\ln\alpha}\,y(x) = -\frac{1}{\alpha}\sum_{k=0}^{\infty} e^{khD} e^{-\frac{x}{h}\ln\alpha} f(x) + y_1(x),$$

$$y(x) = -\frac{1}{\alpha}\,e^{\frac{x}{h}\ln\alpha}\sum_{k=0}^{\infty} f(x+kh)\,e^{-\frac{x+kh}{h}\ln\alpha} + y_1(x)\,e^{\frac{x}{h}\ln\alpha},$$

where $y_1(x)$ is a solution of a homogeneous equation $(e^{Dh}-1)\,y_1 = 0$; $y_1(x+h) = y_1(x)$. Hence $y_1(x)$ is a periodic function of a period h. It can be proved that a solution of equation (2.1) can

be stated in the form

$$y(x)=\sum_{i=1}^{n}\frac{A_i}{e^{hD}-\alpha_i}f(x) \tag{2.4}$$

along the same lines as in the preceding section.

If $y(x)$ is defined by (2.4) we state by definition

$$y(x)\overset{\text{def}}{=}\frac{1}{P_n(e^{hD})}f(x).$$

Thus the problem is reduced to the equation (2.3).

Problem. Find a partial solution of a differential-difference equation

$$y''(x)+y(x+1)=x^k e^{\alpha x},$$

where α satisfies the condition $\alpha = ie^{\alpha/2}$.

Sec. 3. Solution of Systems of Differential Equations by the Heaviside Operational Method

A system of differential equations with constant coefficients may be written in the form

$$B(D)\cdot Y(x)=F(x), \tag{3.1}$$

where $Y(x)=(y_1(x), \ldots, y_n(x))$, $F(x)=(f_1(x), \ldots, f_n(x))$ are vector functions, and $B(D)$ are $n\times n$ matrices having as elements the operators considered in Sec. 1.

It is easy to verify that the operators $B(D)$ constitute a noncommutative algebra, which we shall denote by $M[D]$. Let $M[x]$ be a matrix algebra having as elements polynomials $P(x)=\sum_{i=0}^{m} a_i x^i$. The isomorphism $P(x)\to P(D)$ of the algebras $K_1[x]$ and $K_1[D]$ defined in Sec. 1 can be extended onto an isomorphism of algebras $M[x]$ and $M[D]$.

Let

$$B\in M[x],\quad B(x)=\|P^{ij}(x)\|.$$

Let $A_{ij}(x)$ be the minor of the matrix B, corresponding to the ith line and the jth column and consider the matrix

$$A\in M[x],\quad A(x)=\|A_{ij}\|^t=\|A_{ji}\|.$$

The following formula is a corollary of a familiar theorem of linear algebra

$$A(x)B(x)=E\cdot\Delta(x), \tag{3.2}$$

where E is a unit matrix, $\Delta(x) = \det B(x)$. Equation (3.2) for the corresponding operators implies

$$A(D)\, B(D) = E\Delta(D). \tag{3.3}$$

Equation (3.1) can be expressed in the form

$$A(D)\, B(D)\, Y(x) = A(D)\, F(x)$$

using equation (3.3). Let $A(D)\, F(x) = G(x)$, then

$$E\Delta(D)\, Y(x) = G(x) \tag{3.4}$$

or

$$\Delta(D)\, y_i(x) = g_i(x), \quad i = 1, 2, \ldots, n. \tag{3.4'}$$

Equations of this type have already been considered. Their solutions have the form

$$y_i(x) = \frac{1}{\Delta(D)} g_i(x), \quad i = 1, 2, \ldots, n. \tag{3.5}$$

We find by the argument stated above, that if $Y(x)$ satisfies the system of equations (3.1), then its components, $y_i(x)$, satisfy equation (3.4′). Note, nonetheless, that if $y_i(x)$ is a general solution of equation (3.4′), the function $Y(x) = (y_1(x), \ldots, y_n(x))^t$ does not, in general, satisfy the system of equations (3.1). It is necessary to impose conditions on the integration constants contained in the solution of equation (3.5) for the function $Y(x) = (y_1(x), \ldots$ $\ldots, y_n(x))^t$ to be a solution of system (3.1). If

$$z_i(x) = \frac{1}{\Delta(D)} f_i(x),$$

then

$$Y(x) = A(D)\, Z(x),$$

where $Z(x) = (z_1(x), \ldots, z_n(x))$ obviously verifies system (3.1):

$$B(D)\, Y(x) = B(D)\, A(D)\, Z(x) = E\Delta(D)\, Z(x) = F(x).$$

It is easy to see that we have obtained the general solution of system (3.1).

Example. Solve the system of equations

$$Dy_1(x) + y_1(x) + Dy_2(x) = 0,$$
$$D^2 y_1(x) - y_1(x) + D^2 y_2(x) + y_2(x) = 0.$$

Put the system in the form

$$(D + 1)\, y_1(x) + Dy_2(x) = 0,$$
$$(D^2 - 1)\, y_1(x) + (D^2 + 1)\, y_2(x) = 0.$$

We see that in this case $\Delta(D) = (D+1)^2$. Hence, the equation has the form

$$y_1(x) = ae^{-x} + bxe^{-x},$$
$$y_2(x) = ce^{-x} + dxe^{-x}.$$

We obtain, on substituting these functions into the first equation and dividing by e^{-x}

$$b - c - dx + d = 0,$$

i.e., $d = 0$, $b = c$. The substitution of $y_1(x)$, $y_2(x)$ into the second equation provides the same conditions for the coefficients. Thus,

$$y_1(x) = ae^{-x} + bxe^{-x},$$
$$y_2(x) = be^{-x}.$$

It has thus been shown that the operational Heaviside method reduces the problem of ordinary differential and differential-difference equations with constant coefficients to the purely algebraic problem of linear algebra, the familiar methods of which have been thoroughly investigated. Even if they were not known, it would be only natural to reduce the analytical problem to an algebraic one, familiar or not, and then to try and solve it.

Equations with variable coefficients are far more difficult, since the corresponding algebraic problem has not been studied. In this case we understand the operational method not only as a kind of reduction of a differential problem to an algebraic one, but also as a method for the solution of the latter.

Sec. 4. Algebra of Convergent Power Series of Noncommutative Operators

Heaviside's operational calculus enables us to solve linear differential equations with constant coefficients. We shall now turn to equations with variable coefficients

$$\sum_{j=0}^{n} a_j(x) \frac{d^j y}{dx^j} = f(x).$$

If D is an operator of differentiation, then the equation can be written in the form

$$P(x, D)\, y(x) = f(x),$$

where P is a polynomial with respect to the second argument

$$P(x, p) = \sum_{j=0}^{n} a_j(x) p^j.$$

The main difficulty is that the operators of multiplication and differentiation with respect to x *do not commute*. Consider the following example. Let $P(x, p)$ be a polynomial in x, p

$$P(x, p) = xp.$$

It is not clear how to define $P(x, D)$ since different substitutions of x and D in $P(x, p)$ produce different results

$$xDy(x) = xy'(x), \qquad Dxy(x) = (xy)' = y(x) + xy'(x).$$

It is obvious that the definition

$$P(x, D) \overset{\text{def}}{=} \sum a_i(x) D^i,$$

where $P(x, \xi) = \sum a_i(x) \xi^i$, makes no sense if we do not assume that the operator D in $P(x, D)$ acts first. Therefore we define the operator correctly if we assume an order of action of the operators x and D.

$$P\left(\overset{2}{x}, \overset{1}{D}\right) = \sum a_i(x) D^i.$$

Thus a natural generalization of the definition of a polynomial in D in Heaviside's method affords a definition of polynomials in ordered operators.

We have introduced rules of addition and multiplication for polynomials in the operator D in the case of constant coefficients. They turned out to be the same rules as for ordinary polynomials. In this section we shall introduce similar rules for polynomials in ordered operators. The rules will lead us to new algebraic concepts which we shall axiomatize later.

We shall consider convergent power series of operators as kind of "infinite polynomials". From the example of Heaviside's method, we saw that in some cases a solution can be represented as a convergent power series (see Sec. 2). The student unfamiliar with the concept of bounded operators may consider operators as ordinary matrices.

Besides, in Secs. 1-3 we saw that the concept of polynomials in D is generally speaking not sufficient to solve differential equations. Hence in this case as well we shall generalize the obtained rules for more general functions of ordered operators (see Sec. 6).

Let Op be a noncommutative algebra with a unit. Elements A, B, C, . . . of Op will be called *operators*. Let $\mathscr{A}$ be an algebra of the formal power series of variables x, y, z, ... The set of variables being infinite, every function is a power series of finite number of variables.

Let $\mathscr{S}$ be a set of formal power series of elements of the algebra Op. Such series are defined along the same lines as formal power series

of commutative variables but now two monomials, for instance, A^2B and ABA, are equal by definition if they differ only by the order of multipliers.

An operation of multiplication is defined in the natural way in the set $\mathscr{S}$. Consider an example of the product of two monomials:

$$2A^2BC\cdot 3CABA = 6\cdot A^2BC^2ABA.$$

The multiplication in $\mathscr{S}$ is noncommutative.

Now let the letters a, b, c denote some numbers. For the sake of simplicity we shall introduce the concepts of operational calculus, for example, only for three noncommuting operators, the generalization for any number of operators being obvious.

We shall introduce some definitions and rules of computation of formal power series of operators of the algebra Op. The rules will depend on the order of elements of Op.

Put numbers over the operators, for example,

$$\overset{1}{A},\ \overset{2}{B},\ \overset{3}{C}.$$

We shall say that the operator A acts first, B—second, C—third. Consider the following mapping of the set $\mathscr{A}$ in $\mathscr{S}$, which, for every series

$$f(x, y, z) = \sum_{i,\,j,\,k=0}^{\infty} a_{ijk}x^iy^jz^k,$$

provides

$$\mu: f(x, y, z) \to f\left(\overset{1}{A}, \overset{2}{B}, \overset{3}{C}\right) \overset{\text{def}}{=} \sum_{i,\,j,\,k=0}^{\infty} a_{ijk}C^kB^jA^i.$$

Let $\mathfrak{U}\left(\overset{1}{A}, \overset{2}{B}, \overset{3}{C}\right)$ be an image of $\mathscr{A}$ by the mapping μ. Algebraic operations in $\mathscr{A}$ induce by the mapping μ algebraic operations in $\mathfrak{U}\left(\overset{1}{A}, \overset{2}{B}, \overset{3}{C}\right)$ which provide $\mathfrak{U}\left(\overset{1}{A}, \overset{2}{B}, \overset{3}{C}\right)$ with a structure of commutative algebra.

Note. The injection $\mathfrak{U}\left(\overset{1}{A}, \overset{2}{B}, \overset{3}{C}\right) \subset \mathscr{S}$ induces in $\mathfrak{U}\left(\overset{1}{A}, \overset{2}{B}, \overset{3}{C}\right)$ an operation of multiplication, which differs from multiplication in $\mathfrak{U}\left(\overset{1}{A}, \overset{2}{B}, \overset{3}{C}\right)$ defined above. Multiplication in $\mathscr{S}$ is not commutative, as distinct from multiplication in $\mathfrak{U}\left(\overset{1}{A}, \overset{2}{B}, \overset{3}{C}\right)$.

Example.

$$\left(\overset{1}{A} + \overset{2}{B}\right)^2 = \overset{1}{A^2} + 2\overset{1}{A}\overset{2}{B} + \overset{2}{B^2} = A^2 + 2BA + B^2.$$

Note that there is $\overset{1}{A} + \overset{2}{B} = A + B$ and, nonetheless, the element $\left(\overset{1}{A}+\overset{2}{B}\right)^2$ of the algebra $\mathfrak{U}\left(\overset{1}{A}, \overset{2}{B}\right)$ does not coincide generally with $(A + B)^2$:

$$(A + B)^2 = A^2 + AB + BA + B^2 \neq \left(\overset{1}{A}+\overset{2}{B}\right)^2.$$

The elements A, B, C have appeared as formal symbols until now. Let us now take into consideration that there exist *relations* in Op: two different polynomials containing elements of Op may be equal to each other as operators. Then we shall call them equivalent, which is denoted by the symbol $\sim$ between them. Now, let Op be Banach, and let $\mathcal{A}'$ be the subalgebra of $\mathcal{A}$, consisting of series convergent everywhere. The set of all μ-operations restricted on $\mathcal{A}'$ generates a subalgebra $\mathcal{S}'$ of $\mathcal{S}$. For each series f belonging to $\mathcal{S}'$ there is a unique element denoted by $[f]$ which is its sum.

We shall call two elements of $\mathcal{S}'$ equivalent if and only if their sums are equal.

If

$$f\left(\overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{D}\right) \sim f\left(\overset{1}{A}, \overset{3}{B}, \overset{2}{C}, \overset{4}{D}\right),$$

then we shall denote each of the equivalent series by $f\left(\overset{1}{A}, \overset{2}{B}, \overset{2}{C}, \overset{3}{D}\right)$.

Example. $\left(\overset{1}{A}+\overset{1}{B}\right)\overset{2}{C}$ denotes $\overset{1}{A}\overset{3}{C}+\overset{2}{B}\overset{3}{C}$ or $\overset{2}{A}\overset{3}{C}+\overset{1}{B}\overset{3}{C}$.

The mapping μ has the following properties:

(1) $f\left(\overset{1}{A}, \overset{2}{B}, \overset{3}{C}\right) \sim f\left(\overset{n_1}{A}, \overset{n_2}{B}, \overset{n_3}{C}\right)$, if $n_1 < n_2 < n_3$. Here $f\left(\overset{n_1}{A}, \overset{n_2}{B}, \overset{n_3}{C}\right)$ means (by definition) that if $f(x, y, z) = \sum_{i,j,k} a_{ijk}x^iy^jz^k$,

$$f\left(\overset{n_1}{A}, \overset{n_2}{B}, \overset{n_3}{C}\right) \overset{\text{def}}{=} \sum_{i,j,k} C^k 1 \ldots 1 B^j 1 \ldots 1 A^i 1 \ldots 1,$$

where the units take all the places except the n_1th, n_2th and n_3th one in the right-hand side of each monomial.

(2) If $B = 0$, then

$$f\left(\overset{1}{A}, \overset{3}{C}\right)\overset{2}{B} \sim 0, \qquad \sum a_{kj}C^kBA^j \sim 0.$$

(3) If A and B commute, then

$$f\left(\overset{1}{A}, \overset{2}{B}, \overset{3}{C}\right) \sim f\left(\overset{2}{A}, \overset{1}{B}, \overset{3}{C}\right)$$

and thus both sides can be denoted by $f\left(\overset{1}{A}, \overset{1}{B}, \overset{2}{C}\right)$. This definition makes sense also if $B = A$

$$f\left(\overset{1}{A}, \overset{1}{A}, \overset{2}{C}\right) = g\left(\overset{1}{A}, \overset{2}{C}\right),$$

where $g(x, z) = f(x, x, z)$.
Let

$$f(x, y, z) = \sum_{i, j, k=0}^{\infty} a_{ijk} x^i y^j z^k.$$

We may substitute operators $\overset{1}{A}$ and $\overset{3}{B}$ for x, z and leave y as a formal variable

$$f\left(\overset{1}{A}, y, \overset{3}{B}\right) \overset{\text{def}}{=} \sum_{i, j, k=0}^{\infty} a_{ijk} B^k y^j A^i.$$

Consider a substitution of *an operational expression* acting second for y into $f\left(\overset{1}{A}, y, \overset{3}{B}\right)$. Let $\varphi\left(\overset{1}{C}, \overset{2}{D}, \ldots, \overset{k}{E}\right)$ be an element of $\mathfrak{U}\left(\overset{1}{C}, \overset{2}{D}, \ldots, \overset{k}{E}\right)$, where some of the operators $C, D, \ldots, E$ may coincide with each other and with A and B as well. Take by definition

$$f\left(\overset{1}{A}, \overset{2}{[\![}\varphi\left(\overset{1}{C}, \overset{2}{D}, \ldots, \overset{k}{E}\right)]\!], \overset{3}{B}\right) = \sum_{i, j, k=0}^{\infty} a_{ijk} B^k H^j A^i,$$

where H is an element of $\mathscr{S}'$ equal to $\varphi\left(\overset{1}{C}, \overset{2}{D}, \ldots, \overset{k}{E}\right)$, H^j is the jth power of H in the sense of ordinary noncommutative multiplication in $\mathscr{S}'$, i.e., indices over the operators in the expression in the double brackets $[\![\ ]\!]$ do not define the order of action of operators outside the brackets (and vice versa). That is why we shall call the brackets $[\![\ ]\!]$ *autonomous*. In the same way we define an operation of the substitution of operational expression of higher order.

Example. (1) Let e^x be a convergent power series

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}.$$

Then

$$\overset{2}{[\![e^B]\!]} \cdot \overset{1}{[\![e^A]\!]} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{B^j A^k}{j! k!} =$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} B^k A^{n-k} \sim \sum_{n=0}^{\infty} \frac{\left(\overset{1}{A} + \overset{2}{B}\right)^n}{n!} = e^{\overset{1}{A} + \overset{2}{B}}.$$

(2) Take an expression of the product $T = e^{B^2} \sum_{n=0}^{\infty} \frac{2^n B^n A^n}{n!} e^{A^2}$ in powers of homogeneous binomials in A and B using the operational method. We have

$$T = e^{B^2} \llbracket e^{2\overset{2}{B}\overset{1}{A}} \rrbracket e^{A^2} = \overset{3}{\llbracket} e^{B^2} \rrbracket \, \overset{2}{\llbracket} e^{\overset{1}{A}\left(2\overset{2}{B}+\overset{1}{A}\right)} \rrbracket =$$

$$= \overset{4}{\llbracket} e^{B^2} \rrbracket \, e^{2\overset{3}{B}\overset{2}{A}} \, \overset{1}{\llbracket} e^{A^2} \rrbracket = e^{\overset{4}{B^2}+2\overset{3}{B}\overset{2}{A}+\overset{1}{A^2}} = e^{\overset{2}{B^2}+2\overset{2}{B}\overset{1}{A}+\overset{1}{A^2}} =$$

$$= e^{\left(\overset{1}{A}+\overset{2}{B}\right)^2} = \sum \frac{\left(\overset{1}{A}+\overset{2}{B}\right)^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=0}^{n} C_{2n}^{i} B^{i} A^{2n-i},$$

where C_{2n}^{i} is the corresponding binomial coefficient.

(3) $\llbracket \overset{1}{A} + \overset{2}{B} \rrbracket^2 = \left(\overset{1}{A} + \overset{2}{B}\right)\left(\overset{3}{A} + \overset{4}{B}\right) = A^2 + AB + BA + B^2.$

(4) $\overset{1}{A}\overset{2}{\llbracket}(A+B)\rrbracket^2 \overset{3}{C} = \overset{1}{A}\left(\overset{1}{A} + \overset{2}{B}\right)\left(\overset{3}{A}+\overset{4}{B}\right)\overset{5}{C}.$

Now we shall introduce an operation of *extraction of the autonomous brackets* (operation "prime"). Let

$$f(x,\ y,\ z) = \sum_{i,\ j,\ k} a_{ijk} x^i y^j z^k.$$

Then we shall consider

$$f\left(\overset{1}{A},\ \overset{3}{\llbracket} \varphi \left(\overset{2'}{B},\ \overset{1}{C},\ \overset{2}{D}\right) \rrbracket,\ \overset{4}{E}\right) \tag{4.1}$$

as a convergent power series of $\mathscr{S}'$ obtained by the following procedure. First consider $f\left(\overset{1}{A},\ \overset{2}{\llbracket} \varphi \left(\overset{3}{B_1},\ \overset{1}{C},\ \overset{2}{D}\right)\rrbracket,\ \overset{3}{E}\right)$, where the operator B_1 is supposed to be non-commuting with any of the operators $A,\ C,\ D,\ E$. Denote by S the element $\varphi\left(\overset{3}{B_1},\ \overset{1}{C},\ \overset{2}{D}\right) \in \mathscr{S}'$. Then

$$f\left(\overset{1}{A},\ \overset{2}{\llbracket} \varphi \left(\overset{3}{B_1},\ \overset{1}{C},\ \overset{2}{D}\right)\rrbracket,\ \overset{3}{E}\right) = \sum_{i,\ j,\ k} a_{ijk} E^k S^j A^i.$$

Now in every term of the last sum take the operator B_1 (contained in S^j) from its place to the right so that it might act immediately after A^i. The obtained convergent power series will be denoted by

$$f\left(\overset{1}{A},\ \overset{3}{\llbracket} \varphi \left(\overset{2'}{B_1},\ \overset{1}{C},\ \overset{2}{D}\right)\rrbracket,\ \overset{4}{E}\right).$$

We need only substitute B for B_1 everywhere in order to define the series (4.1).

Example.

$$\overset{2}{\llbracket}\left(A+\overset{1'}{BC}\right)^{2}\rrbracket=\left(\overset{2}{A}+\overset{1}{B}\overset{3}{C}\right)\left(\overset{4}{A}+\overset{1}{B}\overset{5}{C}\right).$$

We shall consider an example where the operation μ immediately solves the problem on being introduced. Let us take a set of all homomorphisms of the vector space C_0^∞ as Op and put

$$X\varphi(\xi)=\xi\varphi(\xi),\qquad P\varphi(\xi)=-i\,d\varphi(\xi)/d\xi,\qquad \varphi\in C_0^\infty.$$

Let A be an operator of Op; $f(x, y)$, $S(x, y)$, $g(x, y)$ be polynomials

$$f(x, y)=\sum_{j,\,l=0}^{n} a_{jl}x^jy^l,\qquad g(x, y)=\sum_{j,\,l=0}^{m} c_{jl}x^jy^l,$$

$$e^{iS(x,\,y)}\overset{\text{def}}{=\!=}e^{iS(\xi,\,y)},\qquad S(x, y)\in C^\infty.$$

Then

$$\llbracket f\left(\overset{2}{X},\ \overset{1}{P}\right)\rrbracket\,\llbracket e^{iS\left(\overset{2}{X},\,y\right)}g\left(\overset{2}{X},\ \overset{1}{A}\right)\rrbracket=$$
$$=\sum_{i,\,l=0}^{m} a_{jl}\overset{4}{X}{}^j\,\overset{3}{\llbracket}\overset{2}{P}{}^l e^{iS\left(\overset{1}{X},\,y\right)}\rrbracket\,g\left(\overset{2}{X},\ \overset{1}{A}\right).$$

Hence, using the identity

$$\overset{2}{P}{}^l e^{iS\left(\overset{1}{X},\,y\right)}=e^{iS\left(\overset{2}{X},\,y\right)}\overset{1}{\llbracket}P+\frac{\partial S}{\partial X}(X, y)\rrbracket^l, \tag{4.2}$$

we obtain

$$\llbracket f\left(\overset{2}{X},\ \overset{1}{P}\right)\rrbracket\,\llbracket e^{iS\left(\overset{2}{X},\,y\right)}\,g\left(\overset{2}{X},\ \overset{1}{A}\right)\rrbracket=$$
$$=\sum_{j,\,l=0}^{m} a_{jl}\overset{4}{X}{}^j e^{iS\left(\overset{4}{X},\,y\right)}\overset{3}{\llbracket}P+\frac{\partial S}{\partial X}(X, y)\rrbracket^l g\left(\overset{2}{X},\ \overset{1}{A}\right)=$$
$$=e^{iS\left(\overset{4}{X},\,y\right)}f\left(\overset{4}{X},\overset{3}{\llbracket}P+\frac{\partial S}{\partial X}(X, y)\rrbracket\right)g\left(\overset{2}{X},\ \overset{1}{A}\right).$$

Formula (4.2) in the case of the operators P and X can be obtained similarly to 1.4. Thus we have proved the following theorem.

Theorem 4.1. *Let X, P be the operators introduced above; then*

$$\llbracket f\left(\overset{2}{X},\ \overset{1}{P}\right)\rrbracket\,\llbracket e^{iS\left(\overset{2}{X},\,y\right)}\,g\left(\overset{2}{X},\ \overset{1}{P}\right)\rrbracket\sim$$
$$\sim e^{iS\left(\overset{4}{X},\,y\right)}f\left(\overset{4}{X},\overset{3}{\llbracket}P+\frac{\partial S}{\partial X}(X, y)\rrbracket\right)g\left(\overset{2}{X},\ \overset{1}{P}\right). \tag{4.3}$$

Problem. Substitute $\overset{1'}{P}$ for y into (4.2)

$$\overset{3}{P}{}^s\, e^{iS\left(\overset{2}{X},\ \overset{1}{P}\right)} \sim e^{iS\left(\overset{4}{X},\ \overset{1}{P}\right)}\, \overset{2}{[\![} P+\frac{\partial S}{\partial X}\left(X,\ \overset{1'}{P}\right)]\!]^s. \qquad (4.4)$$

Obtain an analogue of formula (4.3) from (4.4) using the rules and the definitions for the μ-operation.

The operator $f\left(\overset{2}{X},\ \overset{1}{P}\right)$ in formula (4.3) is called *the Hamiltonian.* Formula (4.3) will be called *the formula of commutation of Hamiltonian and exponential.*

Theorem 4.2. *If* $f\left(\overset{1}{A},\ \overset{2}{C}_1,\ \ldots,\ \overset{n+1}{C}_n,\ \overset{n+2}{B}\right) \sim 0$, *then*

$$\varphi\left(\overset{1}{A},\ \overset{n+2}{B}\right) f\left(\overset{1}{A},\ \overset{2}{C}_1,\ \ldots,\ \overset{n+1}{C}_n,\ \overset{n+2}{B}\right) \sim 0$$

for any $\varphi(x,\ y)$.

Proof.

$$\varphi\left(\overset{1}{A},\ \overset{n+2}{B}\right) f\left(\overset{1}{A},\ \overset{2}{C}_1,\ \ldots,\ \overset{n+1}{C}_n,\ \overset{n+2}{B}\right) \sim$$
$$\sim \varphi\left(\overset{1}{A},\ \overset{n+4}{B}\right) f\left(\overset{2}{A},\ \overset{3}{C}_1,\ \ldots,\ \overset{n+2}{C}_n,\ \overset{n+3}{B}\right) \sim$$
$$\sim \varphi\left(\overset{1}{A},\ \overset{3}{B}\right) \overset{2}{[\![} f\left(\overset{1}{A},\ \overset{2}{C}_1,\ \ldots,\ \overset{n+1}{C}_n,\ \overset{n+2}{B}\right)]\!] \sim 0.$$

Corollary. *The relation*

$$[\![f\left(\overset{2}{X},\ \overset{1}{P}\right)]\!]\, [\![e^{iS\left(\overset{2}{X}\right)} g\left(\overset{2}{X},\ \overset{1}{A}\right)]\!] \sim 0$$

is equivalent to the relation

$$f\left(\overset{4}{X},\ \overset{3}{[\![} P+S'(X)]\!]\right) g\left(\overset{2}{X},\ \overset{1}{A}\right) \sim 0.$$

The proof follows from Theorem 4.2, the formula of commutation of Hamiltonian and exponential and an obvious formula

$$e^{-iS\left(\overset{4}{X}\right)} e^{iS\left(\overset{4}{X}\right)} f\left(\overset{4}{X},\ \overset{3}{[\![} P+S'(X)]\!]\right) g\left(\overset{2}{X},\ \overset{1}{A}\right) \sim$$
$$\sim f\left(\overset{4}{X},\ \overset{3}{[\![} P+S'(X)]\!]\right) g\left(\overset{2}{X},\ \overset{1}{A}\right).$$

We shall introduce two formulas of importance in the theory of differential equations.

Difference derivatives, being similar to derivatives of the conventional calculus, are very important in the calculus of non-commutating operators. We are therefore introducing corresponding formulas which will be very useful in the sequel.

Let $f(x_1, x_2, \ldots, x_n)$ be a function of n variables. *An operator of difference differentiation* δ_1 transforms this function to the following function of $(n+1)$ variables

$$\delta_1 f(x_1', x_1'', x_2, \ldots, x_n) \overset{\text{def}}{=\!=} \frac{f(x_1', x_2, \ldots, x_n) - f(x_1'', x_2, \ldots, x_n)}{x_1' - x_1''},$$

the first argument doubling itself. An operator δ_j of difference differentiation with respect to the jth argument is defined similarly. Powers of the iterated operator δ_j act with respect to any argument reproduced by the previous differentiations (the reader may himself see that the result does not depend on the choice of the jth argument).

Example. If f is a function of one variable x, then

$$\delta f(x'; x'') = \frac{f(x') - f(x'')}{x' - x''},$$

$$\delta^2 f(x'; x''; x''') = \frac{f(x')}{(x'-x'')(x'-x''')} + \frac{f(x'')}{(x''-x')(x''-x''')} + \frac{f(x''')}{(x'''-x')(x'''-x'')};$$

if $f(x, y)$ is a function of two variables, then

$$\delta_1\delta_2 f(x'; x'', y'; y'') = \frac{f(x', y') - f(x', y'') + f(x'', y'') - f(x'', y')}{(x''-x')(y''-y')}.$$

We shall use the notation $\delta/\delta x$, $\delta/\delta y$ instead of δ_1, δ_2.

Theorem 4.3. *The following formula of a change of the arguments' order is true*:

$$\llbracket f(\overset{2}{A}, \overset{1}{B})\rrbracket - \llbracket f(\overset{1}{A}, \overset{2}{B})\rrbracket \sim \left[A \overset{3}{,} B\right] \delta_1\delta_2 f(\overset{1}{A}; \overset{5}{A}, \overset{2}{B}; \overset{4}{B}),$$

where $\left[A \overset{3}{,} B\right] = \overset{3}{\llbracket} AB - BA\rrbracket$ *is a commutator of the operators* A *and* B.

Proof. We obtain by the properties of the μ-operation

$$\overset{3}{\llbracket} AB - BA\rrbracket \cdot \frac{f(\overset{1}{A}, \overset{2}{B}) - f(\overset{1}{A}, \overset{4}{B}) + f(\overset{5}{A}, \overset{4}{B}) - f(\overset{5}{A}, \overset{2}{B})}{(\overset{1}{A} - \overset{5}{A})(\overset{2}{B} - \overset{4}{B})} \sim$$

$$\sim \left(\overset{3}{A}\overset{2}{B} - \overset{4}{B}\overset{3}{A}\right) \cdot \frac{f(\overset{1}{A}, \overset{2}{B}) - f(\overset{1}{A}, \overset{4}{B}) + f(\overset{5}{A}, \overset{4}{B}) - f(\overset{5}{A}, \overset{2}{B})}{(\overset{1}{A} - \overset{5}{A})(\overset{2}{B} - \overset{4}{B})} \sim$$

$$\sim \overset{3}{A}\, \frac{f(\overset{1}{A}, \overset{2}{B}) - f(\overset{5}{A}, \overset{2}{B})}{\overset{1}{A} - \overset{5}{A}} - \overset{3}{A}\, \frac{f(\overset{1}{A}, \overset{4}{B}) - f(\overset{5}{A}, \overset{4}{B})}{\overset{1}{A} - \overset{5}{A}} \sim$$

$$\sim \overset{3}{A}\,\frac{f\left(\overset{1}{A},\overset{2}{B}\right)-f\left(\overset{3}{A},\overset{2}{B}\right)}{\overset{1}{A}-\overset{3}{A}}-\overset{1}{A}\,\frac{f\left(\overset{1}{A},\overset{2}{B}\right)-f\left(\overset{3}{A},\overset{2}{B}\right)}{\overset{1}{A}-\overset{3}{A}}\sim$$

$$\sim \frac{\overset{3}{A}-\overset{1}{A}}{\overset{1}{A}-\overset{3}{A}}\left[f\left(\overset{1}{A},\overset{2}{B}\right)-f\left(\overset{3}{A},\overset{2}{B}\right)\right]\sim [\![f\left(\overset{2}{A},\overset{1}{B}\right)]\!]-[\![f\left(\overset{1}{A},\overset{2}{B}\right)]\!].$$

Note. Similar formulas are true for the permutation of an order of action of two operators in the case of two operational arguments. For example

$$f\left(\overset{1}{A},\overset{2}{B},\overset{3}{C}\right)-f\left(\overset{1}{B},\overset{2}{A},\overset{3}{C}\right)\sim\left[\overset{3}{B},A\right]\delta_1\delta_2 f\left(\overset{1}{A};\overset{5}{A},\overset{2}{B};\overset{4}{B},\overset{6}{C}\right).$$

Theorem 4.4. *The following formula is true*

$$\overset{2}{[\![}f\left(A+B\overset{1'}{C}\right)]\!]\sim\overset{2}{[\![}f\left(\overset{1}{A}+\overset{2}{B}\overset{1'}{C}\right)]\!]+$$
$$+[\![\overset{1}{C}\left[\overset{4}{A},B\right]\delta^2 f\left(\overset{2}{A}+\overset{6}{B}\overset{1}{C},\ [\![A+B\overset{1'}{C}]\!],\ \overset{5}{[\![}A+B\overset{1'}{C}]\!]\right)]\!].$$

Proof. It is sufficient to prove that

$$f(A+B)\sim[\![f\left(\overset{1}{A}+\overset{2}{B}\right)]\!]+[\![\left[\overset{3}{A},B\right]\delta^2 f\left(\overset{1}{A}+\overset{5}{B},\ A\overset{2}{+}B,\ A\overset{4}{+}B\right)]\!].$$

We have a relation $f(z)\sim f(x+y)+(z-x-y)\,\delta f(x+y,z)$. By substituting the operators $\overset{1}{A},\overset{3}{B},A\overset{2}{+}B$ for x, y, z, respectively, we obtain

$$f(A+B)\sim[\![f\left(\overset{1}{A}+\overset{2}{B}\right)]\!]+[\![\left(A\overset{2}{+}B-\overset{1}{A}-\overset{3}{B}\right)\delta f\left(\overset{1}{A}+\overset{3}{B},\ A\overset{2}{+}B\right)]\!].$$

Using Theorem 4.3 and the note, we obtain

$$f(A+B)\sim[\![f\left(\overset{1}{A}+\overset{2}{B}\right)]\!]+[\![\left(A\overset{2}{+}B-\overset{3}{B}\right)\delta f\left(\overset{1}{A}+\overset{3}{B},\ A\overset{2}{+}B\right)]\!]-$$
$$-[\![\overset{1}{A}\delta f\left(\overset{1}{A}+\overset{3}{B},\ A\overset{2}{+}B\right)]\!]\sim[\![f\left(\overset{1}{A}+\overset{2}{B}\right)]\!]+$$
$$+[\![\left(A\overset{2}{+}B-\overset{3}{B}\right)\delta f\left(\overset{1}{A}+\overset{3}{B},\ A\overset{2}{+}B\right)]\!]-$$
$$-[\![\overset{3}{A}\delta f\left(\overset{1}{A}+\overset{4}{B},\ A\overset{2}{+}B\right)]\!]+$$
$$+[\![[\overset{3}{A},A+B]\,\delta^2 f\left(A\overset{2}{+}B,\ A\overset{4}{+}B,\ \overset{1}{A}+\overset{5}{B}\right)]\!].$$

We have made use of the fact that for the function

$$g(x,y,z,v)=x\delta f(z+v;\,y)=x\,\frac{f(z+v)-f(y)}{z+v-y}$$

the difference derivative with respect to the first two arguments is equal to

$$\delta_1\delta_2 g(x_1;\, x_2,\, y_1;\, y_2,\, z,\, v) = \\ = \frac{\left[\dfrac{f(z+v)-f(y_1)}{z+v-y_1} - \dfrac{f(z+v)-f(y_2)}{z+v-y_1}\right](x_1-x_2)}{(x_1-x_2)(y_1-y_2)} = \\ = \delta^2 f(y_1,\, y_2,\, z+v).$$

Now we have only to note that $[A,\, A+B] \sim [A,\, B]$ and that

$$[\![\left(A \overset{2}{+} B - \overset{3}{B}\right) \delta f\left(\overset{1}{A} + \overset{3}{B},\, A \overset{2}{+} B\right)]\!] - [\![\overset{3}{A} \delta f\left(\overset{1}{A} + \overset{4}{B},\, A \overset{2}{+} B\right)]\!] =$$

$$= \left(A \overset{4}{+} B - \overset{5}{B}\right) \delta f\left(\overset{1}{A} + \overset{5}{B},\, A \overset{2}{+} B\right) - \overset{3}{A} \delta f\left(\overset{1}{A} + \overset{5}{B},\, A \overset{2}{+} B\right) =$$

$$= \left(A \overset{4}{+} B - \overset{3}{A} - \overset{5}{B}\right) \delta f\left(\overset{1}{A} + \overset{5}{B},\, A \overset{2}{+} B\right) \sim 0$$

by virtue of Theorem 4.2 (or, more exactly, its generalization), Q.E.D. It will be of use to prove the following corollary.

Corollary. *An expansion is true*

$$f(A+B) = f\left(\overset{1}{A} + \overset{2}{B}\right) + \frac{1}{2}\left[A,\, \overset{2}{B}\right] f''\left(\overset{1}{A} + \overset{3}{B}\right) + R_2, \tag{4.5}$$

where R_2 is given by a formula, containing difference derivatives up to the order 4.

$$\begin{aligned} R_2 = {} & [A,\, \overset{2}{[A,\, B]}]\, \delta^3 f\left(\overset{1}{A} + \overset{4}{B},\, \overset{1}{A} + \overset{4}{B},\, \overset{1}{A} + \overset{4}{B},\, \overset{3}{A} + \overset{4}{B}\right) + \\ & + [[A,\, B],\, \overset{3}{B}]\, \delta^3 f\left(\overset{1}{A} + \overset{2}{B},\, \overset{1}{A} + \overset{4}{B},\, \overset{1}{A} + \overset{6}{B},\, \overset{5}{A} + \overset{6}{B}\right) + \\ & + [A,\, \overset{3}{B}]\, [A,\, \overset{6}{B}]\, \delta^4 f\left(\overset{1}{A} + \overset{5}{B},\, \overset{1}{A} + \overset{8}{B},\, A \overset{2}{+} B,\, A \overset{1}{+} B,\, A \overset{7}{+} B\right) + \\ & + [A,\, \overset{3}{B}]\, [A,\, \overset{6}{B}]\, \delta^4 f\left(\overset{1}{A} + \overset{2}{B},\, \overset{1}{A} + \overset{8}{B},\, \overset{4}{A} + \overset{8}{B},\, A \overset{5}{+} B,\, A \overset{7}{+} B\right) + \\ & + [A,\, \overset{2}{B}]\, [A,\, \overset{5}{B}]\, \delta^4 f\left(\overset{1}{A} + \overset{4}{B},\, \overset{1}{A} + \overset{6}{B},\, \overset{1}{A} + \overset{8}{B},\, \overset{3}{A} + \overset{8}{B},\, \overset{7}{A} + \overset{8}{B}\right). \end{aligned} \tag{4.6}$$

It is easy to write out the next term of expansion (4.5):

$$\begin{aligned} f(A+B) = {} & f\left(\overset{1}{A} + \overset{2}{B}\right) + \frac{1}{2}[A,\, \overset{2}{B}]\, f''\left(\overset{1}{A} + \overset{3}{B}\right) + \frac{1}{6}\left([A,\, \overset{2}{[A,\, B]}] + \right. \\ & \left. + [[A,\, B],\, \overset{2}{B}]\right) f'''\left(\overset{1}{A} + \overset{2}{B}\right) + \frac{1}{8}[A,\, \overset{2}{B}]^2 f^{(4)}\left(\overset{1}{A} + \overset{3}{B}\right) + R_3, \end{aligned} \tag{4.7}$$

where R_3 is expressed by the commutators of the third order just as in (4.6).

Theorem 4.5. *The following formula is valid*

$$f\left(\overset{2}{C}, A \overset{1}{+} B\right) = f\left(\overset{2}{C}, \overset{1}{A}\right) + \overset{2}{B}\frac{\delta f}{\delta x_2}\left(\overset{4}{C}, \overset{1}{A}, \overset{3}{A}\right) +$$
$$+ \overset{2}{B}\overset{4}{B}\frac{\delta^2 f}{\delta x_2^2}\left(\overset{6}{C}, \overset{1}{A}, \overset{3}{A}, A \overset{5}{+} B\right).$$

Proof. Consider the equality

$$\varphi(x_1, x_2, x_3) = \varphi(x_1, x_2, y_3) + (x_3 - y_3)\frac{\delta\varphi}{\delta x_3}(x_1, x_2; x_3, y_3).$$

We use a substitution $x_1 \to \overset{6}{C_1}$, $x_2 \to \overset{1}{C_2}$, $x_3 \to A \overset{5}{+} B$, $y_3 \to \overset{3}{A}$. Then we have, for any $T \in \mathrm{Op}$,

$$\overset{2}{T}\varphi\left(\overset{6}{C_1}, \overset{1}{C_2}, A \overset{5}{+} B\right) =$$
$$= \overset{2}{T}\varphi\left(\overset{6}{C_1}, \overset{1}{C_2}, \overset{3}{A}\right) + \overset{2}{T}\frac{\delta\varphi}{\delta x_3}\left(\overset{6}{C_1}, \overset{1}{C_2}; \overset{3}{A}, A \overset{5}{+} B\right)\left(A \overset{5}{+} B - \overset{3}{A}\right) =$$
$$= \overset{2}{T}\varphi\left(\overset{6}{C_1}, \overset{1}{C_2}, \overset{3}{A}\right) + \overset{2}{T}\overset{4}{B}\frac{\delta\varphi}{\delta x_3}\left(\overset{6}{C_1}, \overset{1}{C_2}; \overset{3}{A}, A \overset{5}{+} B\right). \qquad (4.8)$$

Hence, in particular, when φ is independent of the second argument and the function $\varphi(x_1, y_2, x_2)$ is equivalent to $f(x_1, x_2)$ and $C_1 \sim C$, $T \sim 1$ we have

$$f\left(\overset{2}{C}, A \overset{1}{+} B\right) = f\left(\overset{2}{C}, \overset{1}{A}\right) + \overset{2}{B}\frac{\delta f}{\delta x_2}\left(\overset{6}{C}; \overset{1}{A}, A \overset{5}{+} B\right).$$

We apply formula (4.8) to the last term in the case

$$\varphi(x_1, x_2, x_3) = \frac{\delta f}{\delta x_2}(x_1, x_2, x_3), \quad T \sim B, \quad C_1 \sim C, \ C_2 \sim A.$$

Then the sought-for expansion follows from (4.8), Q.E.D.

Sec. 5. Spectrum of a Pair of Ordered Operators

We shall first consider the spectrum of matrices. Let A be a matrix. Let K be a set of polynomials P with coefficients in $\mathbf{C}$ such that $P(A) = 0$, where the zero denotes a zero matrix.

Definition. *The spectrum $\sigma(A) \subset \mathbf{C}$ of a matrix A is an intersection of the sets of zeros of the polynomials belonging to K: $(z \in \sigma) \Leftrightarrow (P(z) = 0$ for any $P \in K)$.*

Let $k_P(\lambda)$ be a multiplicity of a zero of polynomial $P(z) \in K$ at a point $\lambda \in \sigma(A)$. Define *the multiplicity of the point* λ *of the spectrum* as $\inf\limits_{P \in K} k_P(\lambda)$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be the eigenvalues of a matrix A, and $Q(z) = \prod\limits_1^s (z-\lambda_j)^{k_j}$ is its characteristic polynomial. Linear algebra provides the equation

$$Q(A) = 0.$$

Hence we see that the spectrum of a matrix coincides with the familiar eigenvalues set, the multiplicity of the eigenvectors k_j being λ_j. Take for the sake of simplicity $s = 3$. For any polynomial $P(z)$ there exists a formula

$$P(z) = \sum_{i=0}^{k_1-1} \frac{1}{i!} P^{(i)}(\lambda_1)(z-\lambda_1)^i + (z-\lambda_1)^{k_1} P_1(z),$$

where $P_1(z)$ is a polynomial with coefficients dependent on λ_1. Similarly

$$P_1(z) = \sum_{i=0}^{k_2-1} \frac{1}{i!} P_1^{(i)}(\lambda_2)(z-\lambda_2)^i + (z-\lambda_2)^{k_2} P_2(z),$$

$$P_2(z) = \sum_{i=0}^{k_3-1} \frac{1}{i!} P_2^{(i)}(\lambda_3)(z-\lambda_3)^i + (z-\lambda_3)^{k_3} P_3(z),$$

where $P_2(z)$, $P_3(z)$ are polynomials, the coefficients of P_2 being dependent on λ_1, λ_2. Then, taking into account that

$$(A-\lambda_1)^{k_1}(A-\lambda_2)^{k_2}(A-\lambda_3)^{k_3} = 0,$$

we obtain

$$P(A) = \sum_{i=0}^{k_1-1} \frac{1}{i!} P^{(i)}(\lambda_1)(A-\lambda_1)^i +$$
$$+ (A-\lambda_1)^{k_1} \sum_{i=0}^{k_2-1} \frac{1}{i!} P_1^{(i)}(\lambda_2)(A-\lambda_2)^i +$$
$$+ (A-\lambda_1)^{k_1}(A-\lambda_2)^{k_2} \sum_{i=0}^{k_3-1} \frac{1}{i!} P_2^{(i)}(\lambda_2)(A-\lambda_3)^i. \tag{5.1}$$

Hence $P(A)$ is a polynomial of the order $k_1 + k_2 + k_3 - 1$ and the matrix $P(A)$ depends only on the values of the polynomial and a finite set of its derivatives at points of the spectrum. For this

reason formula (5.1) is called *the spectral expansion of the operator* $P(A)$.

Let A, B be elements of the algebra of operators. Consider a set K of polynomials $P(z_1, z_2)$ with such coefficients in $\mathbf{C}$ that $P\left(\overset{1}{A}, \overset{2}{B}\right) = 0$.

We wish to extend the given definition of the spectrum and of the spectral expansion into the case of ordered sets of operators. For the sake of simplicity we confine ourselves to the spectrum of a pair of operators $\overset{1}{A}$, $\overset{2}{B}$.

Definition. *The spectrum* $\sigma\left(\overset{1}{A}, \overset{2}{B}\right) \subset \mathbf{C}^2$ *of an ordered pair of operators* $\overset{1}{A}$, $\overset{2}{B}$ *is an intersection of sets of all zeroes of polynomials belonging to* K, *i.e.*, $\left(z_1, z_2 \in \sigma\left(\overset{1}{A}, \overset{2}{B}\right)\right) \Longleftrightarrow (P(z_1, z_2) = 0)$ *for all* $P(z_1, z_2) \in K$.

Note. Theorem 4.2 states that if $Q(z_1, z_2) \in K$ then $Q(z_1, z_2) \times P(z_1, z_2) \in K$, where $P(z_1, z_2)$ is a polynomial.

Let A, B, C be elements of the algebra of operators. Consider a set $\mathcal{S}(z_1, z_2)$ of polynomials $P(z_1, z_2)$ with such coefficients in $\mathbf{C}$ that

$$\overset{2}{C}P\left(\overset{1}{A}, \overset{3}{B}\right) = 0.$$

Definition. *The spectrum* $\sigma_C\left(\overset{1}{A}, \overset{3}{B}\right) \subset \mathbf{C}^2$ *of a pair* $\overset{1}{A}$, $\overset{3}{B}$ *relative to* $\overset{2}{C}$ *is an intersection of sets of zeroes of polynomials belonging to* $\mathcal{S}(z_1, z_2)$; i.e.,

$$\left(z_1, z_2 \in \sigma_C\left(\overset{1}{A}, \overset{3}{B}\right)\right) \Longleftrightarrow (P(z_1, z_2) = 0)$$

for any $P(z_1, z_2) \in \mathcal{S}(z_1, z_2)$.

Note, that if $Q(z_1, z_2) \in \mathcal{S}(z_1, z_2)$, then $Q(z_1, z_2)P(z_1, z_2) \in \mathcal{S}(z_1, z_2)$, where $P(z_1, z_2)$ is a polynomial.

Consider an important example of the spectral expansion of the function $\overset{2}{C}f\left(\overset{1}{A}, \overset{2}{B}\right)$. We shall use formula (4.3) of the commutation of Hamiltonian and exponential. Take in this formula

$$X = x, \quad P = -i\frac{\partial}{\partial x}, \qquad S(X, P) = S(x), \qquad g(X, P) = g(x)$$

(cf. (4.31)). Thus, in the right-hand side we obtain

$$e^{iS\left(\overset{4}{x}\right)} f\left(\overset{4}{x}, \overset{3}{[\![}P + \frac{\partial S}{\partial x}]\!]\right) g\left(\overset{2}{x}\right) = e^{iS(x)} [\![f\left(\overset{2}{x}, \overset{1}{[\![}P + \frac{\partial S}{\partial x}]\!]\right)]\!] g(x).$$

By virtue of Theorem 4.5 we get

$$f\left(\overset{2}{x}, \overset{1}{[\![}P+\frac{\partial S}{\partial x}]\!]\right) g(x)=$$

$$=f\left(x, \frac{\partial S}{\partial x}\right) g(x)+\overset{2}{P} \frac{f\left(\overset{4}{x}, \overset{1}{\frac{\partial S}{\partial x}}\right)-f\left(\overset{4}{x}, \overset{3}{\frac{\partial S}{\partial x}}\right)}{\overset{1}{\frac{\partial S}{\partial x}}-\overset{3}{\frac{\partial S}{\partial x}}} g(x)+$$

$$+\overset{2}{P}\overset{4}{P} \frac{\delta^2 f}{\delta p^2}\left(\overset{6}{x}; \overset{1}{\frac{\partial S}{\partial x}}, \overset{3}{\frac{\partial S}{\partial x}}, \overset{5}{[\![}P+\frac{\partial S}{\partial x}\right)]\!] g(x).$$

Calculate the middle term of the equation. Consider to the effect

$$i \overset{2}{\frac{\partial}{\partial x}}\left(\overset{1}{x}-\overset{3}{x}\right)^2 \varphi(x)=$$

$$=i\left\{\frac{\partial}{\partial x} x^2\varphi(x)-2x \frac{\partial}{\partial x} x\varphi(x)+x^2 \frac{\partial}{\partial x} \varphi(x)\right\}=0.$$

Hence the spectrum $\left(\overset{1}{x}, \overset{3}{x}\right)$ with respect to $ih \overset{2}{\frac{\partial}{\partial x}}$ is located in $\mathbf{R}^2$ by the formula $x_1 = x_2$. Hence

$$i \overset{2}{\frac{\partial}{\partial x}}\left(\overset{1}{x}-\overset{3}{x}\right)^n=0 \text{ for } n \geqslant 2$$

by Theorem 4.2.

We have the following "spectral expansion" as a corollary:

$$i \overset{2}{\frac{\partial}{\partial x}} P\left(\overset{1}{x}, \overset{3}{x}\right) \sim iP\left(\overset{3}{x}, \overset{3}{x}\right) \overset{2}{\frac{\partial}{\partial x}}+i\overset{2}{\frac{\partial}{\partial x}} P'_x\left(\overset{3}{x}, \overset{3}{x}\right)\left(\overset{1}{x}-\overset{3}{x}\right) \sim$$

$$\sim iP(x, x) \frac{\partial}{\partial x}-i\left[P'_x(x, x)\, x \frac{\partial}{\partial x}-P'_x(x, x) \frac{\partial}{\partial x}\, x\right] \sim$$

$$\sim iP(x, x) \frac{\partial}{\partial x}+iP'_x(x, x).$$

Hence

$$\overset{2}{P} \frac{f\left(\overset{4}{x}, \overset{1}{\partial S/\partial x}\right)-f\left(\overset{4}{x}, \overset{3}{\partial S/\partial x}\right)}{\overset{1}{\partial S/\partial x}-\overset{3}{\partial S/\partial x}} \varphi(x) \sim$$

$$\sim -i\left[\left.\frac{\partial f}{\partial p}\right|_{p=\partial S/\partial x} \frac{\partial}{\partial x}+\frac{1}{2} \left.\frac{\partial^2 f}{\partial p^2}\right|_{p=\frac{\partial S}{\partial p}} \frac{\partial^2 S}{\partial x^2}\right] \varphi(x).$$

An operator P such that

$$P_\varphi=\left.\frac{\partial f}{\partial p}\right|_{p=\frac{\partial S}{\partial x}} \frac{\partial \varphi}{\partial x}+\frac{1}{2} \frac{\partial^2 S}{\partial x^2} \left.\frac{\partial^2 f}{\partial p^2}\right|_{p=\frac{\partial S}{\partial x}} \varphi$$

is called a transfer operator (for the Hamiltonian $f(x, p)$ and the given $S(x)$).

Note that we do not use only the concept of spectrum in spectral expansions, the number of terms in the Taylor expansion being important as well. This number is connected with a multiplicity of spectrum in the case of matrices. In the example considered above it is natural to define the multiplicity of spectrum located on the bisector of the first coordinate angle as equal to two. The problem of the multiplicity of spectrum (i.e., the problem of the number of terms in the Taylor expansion) is very hard in general. This problem is reduced to the study of a "subsidiary" Banach space B_{mid} in Chapters I and II.

We shall point out some analogies between the given example and the spectral expansion of matrices to clarify the direction taken in investigating the properties of operators in Chapters I and II; these are the following:

1. The operator $\overset{1}{x}$ acts in the space of differentiable functions because the second operator is $i\frac{\partial}{\partial x}$. Since the following inequality is valid

$$\left|\frac{d}{dx}(e^{ixt}g(x))\right| + |e^{ixt}g(x)| \leqslant (1+|t|)\max(|g'|+|g|)$$

the operator of multiplication by e^{ixt} increases as the first power of t in the space of differentiable functions.

2. A matrix A, with a maximal length of adjoint elements equal to 1, satisfies the condition

$$|e^{iAt}g| \leqslant (1+|t|)|g|,$$

where g is a vector, $|g|$ is its modulus.

3. The Taylor formula in both cases is reduced to two terms of the spectral expansion. In Chapter II we shall see that the number of terms in spectral expansion is closely connected with the estimates of growth of the operator e^{iAt}.

Note. In the sequel it is important to bear in mind the following properties of matrices:

(1) any matrix A may be put in the form

$$A = A_1 + iA_2, \tag{5.2}$$

where A_1, A_2 commute, A_2 and A_1 have a real spectrum.

(2) We can define the matrix spectrum in the following way; let K be a set of polynomials $P(x, y)$, such that $P(A_1, A_2) = 0$, *the spectrum of the matrix* $\sigma(A)$ is an intersection of sets of zeroes

of polynomials belonging to K, i.e., $x, y \in \sigma(A) \Leftrightarrow P(x, y) = 0$ for any $P(x, y) \in K$, the eigenvalues of A in this case are equal to $x + iy$, where $x, y \in \sigma(A)$.

Thus we may consider only real roots of polynomials of two variables. Therefore, though formula (5.2) is itself a kind of spectral expansion, it is worthwhile remembering when studying a number of non-commuting operators, that each of the operators may be put in the form of a function of two operators with a real spectrum.

We shall return to the problem in Chapter I.

Sec. 6. Algebras with μ-Structures

The introduced calculus is still insufficient to solve differential equations with variable coefficients. We shall introduce the necessary algebraic constructions to the effect.

We shall first consider the most simple case and then turn to more complicated cases (cf. axioms (μ_4), (μ_6) below). We shall prove a number of theorems and, on their basis, demonstrate in Sec. 7 how they can be used.

Next in Sec. 8 we shall demonstrate the power of the operational calculus by the classical example of the deduction of the wave equation. We shall see that the operational calculus provides an adequate means for studying mathematical and physical effects proper to the transition from the system of equations of oscillations of a lattice to the wave equation.

The operational calculus of convergent series of operators is introduced in the axiomatic way, i.e., the main points are formulated as axioms, then formulas are derived and the axioms are verified to be valid for the basic operators necessary for the solution of differential equations. The method is convenient as well from the didactic point of view, i.e., it makes easy to become familiar with the ordered operators calculus techniques. We shall work within the framework of the basic algebraic structures.

Let $\mathscr{A}$ be an algebra with a unit over $\mathbf{R}$, $\mathscr{A}$ is generally noncommutative. The elements of $\mathscr{A}$ will be called *operators*.

Let $\mathscr{S}^\infty$ be a set of infinite differentiable functions $f(x)$, $x \in \mathbf{R}^k$ (k is not fixed) growing together with all their derivatives as $|x|^l$ or less at infinity (l is defined for every function f separately). Functions belonging to $\mathscr{S}^\infty$ are called *symbols.* A symbol is called of *rank* k if the corresponding function depends on k variables. *An algebra* $\mathscr{A}$ *is provided with a* μ*-structure* if for any finite set A_1, $A_2, \ldots, A_k$ of operators belonging to a set $M \subset \mathscr{A}$ and any set of numbers $n_1, n_2, \ldots, n_k$ (so that $n_i \neq n_j$ if A_i, A_j do not commute), the following operation is defined

$$\mu: (x_1 \to A_1, \ldots, x_k \to A_k),$$

which substitutes an operator $A \in \mathcal{A}$ written in the form

$$A = [\![f \left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}\right)]\!]$$

for the symbol $f(x_1, \ldots, x_k)$. We shall cancel out the brackets $[\![\ ]\!]$, where it will not lead to ambiguity. The operation μ satisfies the following axioms:

(μ_1) *The homogeneity axiom*: if $\alpha \in \mathbf{R}$, then

$$[\![\alpha f \left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}\right)]\!] = \alpha [\![f \left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}\right)]\!];$$

in particular, if $f(x_1, \ldots, x_k) = 0$, then

$$[\![f \left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}\right)]\!] = 0.$$

(μ_2) *The shifting of indices axiom*. Let $n_1, n_2, \ldots, n_k$ and $m_1, m_2, \ldots, m_k$ be such two sets of indices that, if $i \neq j$ and $(n_i < n_j) \Rightarrow (m_i < m_j)$, then

$$[\![f \left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}\right)]\!] = [\![f \left(\overset{m_1}{A_1}, \ldots, \overset{m_k}{A_k}\right)]\!],$$

and if $n_i = n_j$, $A_i = A_j = A$, then

$$[\![f \left(\overset{n_1}{A_1}, \ldots, \overset{n_i}{A_i}, \ldots, \overset{n_j}{A_j}, \ldots, \overset{n_k}{A_k}\right)]\!] = [\![g \left(\overset{n_1}{A_1}, \ldots, \overset{n_i}{A_i}, \ldots, \overset{n_k}{A_k}\right)]\!],$$

where

$$g(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k) = [f(x_1, x_2, \ldots, x_k)]_{x_j = x_i}.$$

Example. If $f(x, y) = x^2 y$, $g(x, y) = xy^2$, then

$$f\left(\overset{1}{A}, \overset{2}{B}\right) = g\left(\overset{2}{B}, \overset{1}{A}\right).$$

Example. Let $A, B \in M$, then

$$4 \sin \overset{1}{A} \cos \overset{2}{A} \sin \overset{3}{B} \cos \overset{4}{B} = \sin 2\overset{1}{A} \sin 2\overset{2}{B}.$$

(μ_3) *The correspondence axiom*: the μ-operation transfers the unit symbol into the unit operator and $[\![\overset{1}{A}]\!] = A$.

(μ_4) *The sum axiom*: if $n_i \neq m_j$ for any i, j, then

$$[\![f \left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}\right) + g \left(\overset{m_1}{B_1}, \ldots, \overset{m_l}{B_l}\right)]\!] =$$
$$= [\![f \left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}\right)]\!] + [\![g \left(\overset{m_1}{B_1}, \ldots, \overset{m_l}{B_l}\right)]\!], \quad B_i \in M.$$

Problem 6.1. If $f(x_1, \ldots, x_n) \equiv g(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ when σ is the permutation of the set $\{1, \ldots, n\}$ then $f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}\right) = g\left(\overset{\sigma(1)}{A_{\sigma(1)}}, \ldots\right.$

$\ldots, \overset{\sigma(n)}{A}_{\sigma(n)}\Big)$ $\Big[$hint: apply the operation $\mu\colon (x_1 \to \overset{1}{A}_1, \ldots, x_n \to \overset{n}{A}_n, y_1 \to \overset{\sigma(1)}{A}_{\sigma(1)}, \ldots, y_n \to \overset{\sigma(n)}{A}_{\sigma(n)}\Big)$ to the symbol

$$F(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_1, \ldots, x_n) - g(y_1, \ldots, y_n)].$$

Problem 6.2. If $f(x_1, \ldots, x_{n+1}) \equiv g(x_1, \ldots, x_n)$ then $f\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{A}_{n+1}\right) = g\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n\right)$ $\Big[$hint: apply the operation $\mu\colon \left(x_1 \to \overset{1}{A}_1, \ldots, x_n \to \overset{n}{A}_n, x_{n+1} \to \overset{n+1}{A}_{n+1}, y_1 \to \overset{1}{A}_1, \ldots, y_n \to \overset{n}{A}_n\right)$ to the symbol $F(x_1, \ldots, x_{n+1}, y_1, \ldots, y_n) = f(x_1, \ldots, x_{n+1}) - g(y_1, \ldots, y_n)\Big]$.

(μ_5) *The product axiom*: if $m_i < n_j$ for any i, j, then

$$[\![f\left(\overset{n_1}{A}_1, \ldots, \overset{n_k}{A}_k\right) g\left(\overset{m_1}{B}_1, \ldots, \overset{m_l}{B}_l\right)]\!] = [\![f\left(\overset{n_1}{A}_1, \ldots, \overset{n_k}{A}_k\right)]\!] \, [\![g\left(\overset{m_1}{B}_1, \ldots, \overset{m_l}{B}_l\right)]\!].$$

Example. Under the conditions of the preceding example

$$\sin 2\overset{1}{A} \sin 2\overset{2}{B} = [\![\sin 2B]\!] \, [\![\sin 2A]\!].$$

(μ_6) *The zero axiom*: if

$$[\![f\left(\overset{n_1}{A}_1, \ldots, \overset{n_k}{A}_k\right)]\!] = 0$$

and $p_1, \ldots, p_l, r_1, \ldots, r_m$ are such numbers that $p_i < n_j$, $r_i > n_j$ for all i, j, then for any symbol $g(x_1, \ldots, x_{l+m})$ and operators $B_1, \ldots, B_l, C_1, \ldots, C_m, C_i \in M$,

$$[\![f\left(\overset{n_1}{A}_1, \ldots, \overset{n_k}{A}_k\right) g\left(\overset{p_1}{B}_1, \ldots, \overset{p_l}{B}_l, \overset{r_1}{C}_1, \ldots, \overset{r_m}{C}_m\right)]\!] = 0,$$

It follows from the stated axioms that the sum axiom is in fact valid for any n_i, m_j. Indeed, the following theorem is true.

Theorem 6.1. (The first sum theorem.) *For any n_i, m_j there exists an equation*

$$[\![f\left(\overset{n_1}{A}_1, \ldots, \overset{n_k}{A}_k\right) + g\left(\overset{m_1}{B}_1, \ldots, \overset{m_l}{B}_l\right)]\!] = [\![f\left(\overset{n_1}{A}_1, \ldots, \overset{n_k}{A}_k\right)]\!] + [\![g\left(\overset{m_1}{B}_1, \ldots, \overset{m_l}{B}_l\right)]\!].$$

Proof. Let $n_s \leqslant m_s \leqslant n_{s+1} \leqslant m_{s+1}$ for any s. Note, that if $n_s = m_s$, then B_s, A_s commute by the definition of the μ-structure.

Consider a sum

$$[\![f\begin{pmatrix} n_1 & & n_k \\ A_1, & \ldots, & A_k \end{pmatrix} + g\begin{pmatrix} m_1 & & m_l \\ B_1, & \ldots, & B_l \end{pmatrix}]\!]. \tag{6.1}$$

This sum is equal to the sum (by the shifting of indices axiom)

$$[\![f\begin{pmatrix} n_1' & & n_k' \\ A_1, & \ldots, & A_k \end{pmatrix} + g\begin{pmatrix} m_1' & & m_l' \\ B_1, & \ldots, & B_l \end{pmatrix}]\!], \tag{6.2}$$

where n_i', m_j' are such indices that

$$m_{i+1}' > n_{i+1}' > m_i' > n_i.$$

Besides, there is an equality

$$\begin{aligned} f\begin{pmatrix} n_1 & & n_k \\ A_1, & \ldots, & A_k \end{pmatrix} &= f\begin{pmatrix} n_1' & & n_k' \\ A_1, & \ldots, & A_k \end{pmatrix}, \\ g\begin{pmatrix} m_1 & & m_l \\ B_1, & \ldots, & B_l \end{pmatrix} &= g\begin{pmatrix} m_1' & & m_l' \\ B_1, & \ldots, & B_l \end{pmatrix} \end{aligned} \tag{6.3}$$

for any term of the sum as well. Then the sum (6.2) is equal to

$$[\![f\begin{pmatrix} n_1' & & n_k' \\ A_1, & \ldots, & A_k \end{pmatrix}]\!] + [\![g\begin{pmatrix} m_1' & & m_l' \\ B_1, & \ldots, & B_l \end{pmatrix}]\!] \tag{6.4}$$

by the sum axiom since $n_2' \neq m_j'$ for any i, j. The theorem follows from (6.2) and (6.3).

Theorem 6.2. (The second sum theorem.)

$$\begin{aligned} &[\![\left(A_i \overset{n_i}{+} B_i\right) f\begin{pmatrix} n_1 & & n_{i-1} & n_{i+1} & & n_k \\ A_1, & \ldots, & A_{i-1}, & A_{i+1}, & \ldots, & A_k \end{pmatrix}]\!] = \\ &= [\![\overset{n_i}{A_i} f\begin{pmatrix} n_1 & & n_{i-1} & n_{i+1} & & n_k \\ A_1, & \ldots, & A_{i-1}, & A_{i+1}, & \ldots, & A_k \end{pmatrix}]\!] + \\ &+ [\![\overset{n_i}{B_i} f\begin{pmatrix} n_1 & & n_{i-1} & n_{i+1} & & n_k \\ A_1, & \ldots, & A_{i-1}, & A_{i+1}, & \ldots, & A_k \end{pmatrix}]\!], \end{aligned}$$

where $A_i \overset{n_i}{+} B_i \overset{\text{def}}{=} [\![\overset{n_i}{A_i + B_i}]\!]$.

Proof. We may assume $|n_j - n_i| > 2$ for $j \neq i$ without loss of generality by virtue of the axiom (μ_2). Hence

$$\begin{aligned} &[\![\left(A_i \overset{n_i}{+} B_i\right) f\begin{pmatrix} n_1 & & n_{i-1} & n_{i+1} & & n_k \\ A_1, & \ldots, & A_{i-1}, & A_{i+1}, & \ldots, & A_k \end{pmatrix}]\!] - \\ &- [\![\overset{n_i}{A_i} f\begin{pmatrix} n_1 & & n_{i-1} & n_{i+1} & & n_k \\ A_1, & \ldots, & A_{i-1}, & A_{i+1}, & \ldots, & A_k \end{pmatrix}]\!] - \\ &- [\![\overset{n_i}{B_i} f\begin{pmatrix} n_1 & & n_{i-1} & n_{i+1} & n_k \\ A_1, & \ldots, & A_{i-1}, & A_{i+1}, & A_k \end{pmatrix}]\!] = \\ &= [\![\left(\overset{n_i}{A_i + B_i} - \overset{n_i+1}{A_i} - \overset{n_i+2}{B_i}\right) f\begin{pmatrix} n_1 & & n_{i-1} & n_{i+1} & & n_k \\ A_1, & \ldots, & A_{i-1}, & A_{i+1}, & \ldots, & A_k \end{pmatrix}]\!]. \end{aligned}$$

We obtain the following formula with the help of the axiom of sum, the axiom of shifting of indices and the axiom of correspondence:

$$\llbracket \overset{n_i}{\left(A_i + B_i\right)} - \overset{n_i+1}{A_i} - \overset{n_i+2}{B_i} \rrbracket = \llbracket A_i \overset{n_i}{+} B_i \rrbracket - \llbracket \overset{n_i+1}{A_i} \rrbracket - \llbracket \overset{n_i+2}{B_i} \rrbracket =$$

$$= \llbracket A_i \overset{1}{+} B_i \rrbracket - \llbracket \overset{1}{A_i} \rrbracket - \llbracket \overset{1}{B_i} \rrbracket = A_i + B_i - A_i - B_i = 0.$$

Then we apply the zero axiom and the theorem is proved.

Theorem 6.3. (The theorem of product.) *Let* $n_1, n_2, \ldots, n_k, p_1, \ldots, p_l, r_1, \ldots, r_m$ *be integers such that* $p_i < n_j$, $r_i > n_j$ *for all* i, j *and* $\llbracket f\left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}\right)\rrbracket = F$. *Then for any symbol* $g(x_1, \ldots, x_l, y_1, \ldots, y_m)$ *there is a relation*

$$\llbracket f\left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}\right) g\left(\overset{p_1}{B_1}, \ldots, \overset{p_l}{B_l}, \overset{r_1}{C_1}, \ldots, \overset{r_m}{C_m}\right)\rrbracket =$$

$$= \llbracket \overset{n_1}{F} g\left(\overset{p_1}{B_1}, \ldots, \overset{p_l}{B_l}, \overset{r_1}{C_1}, \ldots, \overset{r_m}{C_m}\right)\rrbracket.$$

Proof. We may assume without loss of generality by axiom (μ_2), that there exists such a number n that $n \neq n_1, \ldots, n_k$ and $p_i < n < r_j$ for all i, j. Then we have by axiom (μ_1) and Theorem 6.1:

$$\llbracket f\left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}\right) g\left(\overset{p_1}{B_1}, \ldots, \overset{p_l}{B_l}, \overset{r_1}{C_1}, \ldots, \overset{r_m}{C_m}\right)\rrbracket -$$

$$- \llbracket \overset{n_1}{F} g\left(\overset{p_1}{B_1}, \ldots, \overset{p_l}{B_l}, \overset{r_1}{C_1}, \ldots, \overset{r_m}{C_m}\right)\rrbracket =$$

$$= \llbracket \left(f\left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}\right) - \overset{n}{F}\right) g\left(\overset{p_1}{B_1}, \ldots, \overset{p_l}{B_l}, \overset{r_1}{C_1}, \ldots, \overset{r_m}{C_m}\right)\rrbracket.$$

It is sufficient to apply axiom (μ_6) to complete the proof.

Theorem 6.4. (The formula of the change of indices.) *If* $n_i = n_j - 1$ *and* $|n_l - n_i| > 2$ *when* $l \neq i, j$, *there exists an equality*

$$\llbracket f\left(\overset{n_1}{A_1}, \ldots, \overset{n_{i-1}}{A_{i-1}}, \overset{n_i}{A_i}, \overset{n_{i+1}}{A_{i+1}}, \ldots, \overset{n_{j-1}}{A_{j-1}}, \overset{n_j}{A_j}, \overset{n_{j+1}}{A_{j+1}}, \ldots, \overset{n_k}{A_k}\right)\rrbracket -$$

$$- \llbracket f\left(\overset{n_1}{A_1}, \ldots, \overset{n_{i-1}}{A_{i-1}}, \overset{n_j}{A_i}, \overset{n_{i+1}}{A_{i+1}}, \ldots, \overset{n_{j-1}}{A_{j-1}}, \overset{n_i}{A_j}, \overset{n_{j+1}}{A_{j+1}}, \ldots, \overset{n_k}{A_k}\right)\rrbracket =$$

$$= \llbracket \left[\overset{n_i}{A_j}, A_i\right] \frac{\delta^2 f}{\delta x_i \, \delta x_j}\left(\overset{n_1}{A_1}, \ldots, \overset{n_{i-1}}{A_{i-1}}, \overset{n_i-1}{A_i}, \overset{n_i+1}{A_i}, \ldots\right.$$

$$\left. \ldots, \overset{n_{j-1}}{A_{j-1}}, \overset{n_i-2}{A_j}, \overset{n_i+2}{A_j}, \overset{n_{j+1}}{A_{j+1}}, \ldots, \overset{n_k}{A_k}\right)\rrbracket, \tag{6.5}$$

where $[A_j, A_i] = A_jA_i - A_iA_j$ *is a commutator of* A_j *and* A_i.

Proof. Let $k = 2$, $A_1 = A$, $A_2 = B$. It is necessary to prove the equality

$$
\begin{aligned}
&\llbracket f(\overset{2}{A}, \overset{1}{B})\rrbracket - \llbracket f(\overset{1}{A}, \overset{2}{B})\rrbracket = \\
&= \llbracket \overset{3}{(A, B)}\, \frac{f(\overset{1}{A}, \overset{2}{B}) - f(\overset{1}{A}, \overset{4}{B}) + f(\overset{5}{A}, \overset{4}{B}) - f(\overset{5}{A}, \overset{2}{B})}{(\overset{1}{A} - \overset{5}{A})(\overset{2}{B} - \overset{4}{B})} \rrbracket.
\end{aligned}
$$

We shall put the right-hand side into this form with the help of axioms (μ_1), (μ_2), (μ_4) and Theorem 6.3

$$
\begin{aligned}
&\llbracket (\overset{3}{A}\overset{2}{B} - \overset{4}{B}\overset{3}{A})\, \frac{f(\overset{1}{A}, \overset{2}{B}) - f(\overset{1}{A}, \overset{4}{B}) + f(\overset{5}{A}, \overset{4}{B}) - f(\overset{5}{A}, \overset{2}{B})}{(\overset{1}{A} - \overset{5}{A})(\overset{2}{B} - \overset{4}{B})} \rrbracket = \\
&= \llbracket \overset{3}{A}\, \frac{f(\overset{1}{A}, \overset{2}{B}) - f(\overset{1}{A}, \overset{4}{B}) + f(\overset{5}{A}, \overset{4}{B}) - f(\overset{5}{A}, \overset{2}{B})}{\overset{1}{A} - \overset{5}{A}} \rrbracket = \\
&= \llbracket \overset{3}{A}\, \frac{f(\overset{1}{A}, \overset{2}{B}) - f(\overset{5}{A}, \overset{2}{B})}{\overset{1}{A} - \overset{5}{A}} \rrbracket + \llbracket \overset{3}{A}\, \frac{f(\overset{5}{A}, \overset{4}{B}) - f(\overset{1}{A}, \overset{4}{B})}{\overset{1}{A} - \overset{5}{A}} \rrbracket = \\
&= \llbracket \overset{5}{A}\, \frac{f(\overset{1}{A}, \overset{2}{B}) - f(\overset{5}{A}, \overset{2}{B})}{\overset{1}{A} - \overset{5}{A}} \rrbracket + \llbracket \overset{1}{A}\, \frac{f(\overset{5}{A}, \overset{2}{B}) - f(\overset{1}{A}, \overset{2}{B})}{\overset{1}{A} - \overset{5}{A}} \rrbracket = \\
&= \llbracket (\overset{5}{A} - \overset{1}{A})\, \frac{f(\overset{1}{A}, \overset{2}{B}) - f(\overset{5}{A}, \overset{2}{B})}{\overset{1}{A} - \overset{5}{A}} \rrbracket = \llbracket -f(\overset{1}{A}, \overset{2}{B}) + f(\overset{5}{A}, \overset{2}{B})\rrbracket = \\
&= \llbracket f(\overset{2}{A}, \overset{1}{B})\rrbracket - \llbracket f(\overset{1}{A}, \overset{2}{B})\rrbracket, \quad \text{Q.E.D.}
\end{aligned}
$$

The proof in the general case is the same.

We shall indicate a formula for a composite function.

Theorem 6.5. (K-formula.) *Let f, g be symbols of rank 1 and 2, respectively, A, B operators belonging to M such that $[A, B] \in M$, $\llbracket g(\overset{1}{A}, \overset{2}{B})\rrbracket \in M$. Then*

$$
\begin{aligned}
f(\llbracket g(\overset{1}{A}, \overset{2}{B})\rrbracket) &= \llbracket f(g(\overset{1}{A}, \overset{2}{B}))\rrbracket + \llbracket \overset{5}{[A, B]}\, \frac{\delta g}{\delta x_1}(\overset{3}{A}, \overset{7}{A}, \overset{9}{B}) \times \\
&\times \frac{\delta g}{\delta x_2}(\overset{3}{A}, \overset{4}{B}, \overset{6}{B})\, \frac{\delta^2 f}{\delta x^2} \times \\
&\times (g(\overset{1}{A}, \overset{9}{B}),\ \overset{2}{\llbracket g(\overset{1}{A}, \overset{2}{B})\rrbracket},\ \overset{8}{\llbracket g(\overset{1}{A}, \overset{2}{B})\rrbracket})\rrbracket. \qquad (6.6)
\end{aligned}
$$

The note made to Theorem 6.4 is also true of this theorem. Indeed, the following equality is true

$$\llbracket f\left(\overset{1}{C}_1, \overset{2}{C}_2, \ldots, \overset{k}{C}_k, \overset{k+1}{\llbracket} g\left(\overset{1}{A}, \overset{2}{B}\right)\rrbracket, \overset{k+2}{C}_{k+1}, \overset{k+3}{C}_{k+2}, \ldots, \overset{s+1}{C}_s\right)\rrbracket =$$

$$= \llbracket f\left(\overset{1}{C}_1, \overset{2}{C}_2, \ldots, \overset{k}{C}_k, g\left(\overset{k+1}{A}, \overset{k+2}{B}\right), \overset{k+3}{C}_{k+1}, \ldots, \overset{s+2}{C}_s\right)\rrbracket +$$

$$+ \llbracket \overset{k+5}{\left[A, B\right]} \frac{\delta g}{\delta x_1}\left(\overset{k+3}{A}, \overset{k+7}{A}, \overset{k+9}{B}\right) \frac{\delta g}{\delta x_2}\left(\overset{k+3}{A}, \overset{k+4}{B}, \overset{k+6}{B}\right) \frac{\delta^2 f}{\delta x_{k+1}^2} \times$$

$$\times \left(\overset{1}{C}_1, \overset{2}{C}_2, \ldots, \overset{k}{C}_k, g\left(\overset{k+1}{A}, \overset{k+9}{B}\right), \overset{k+2}{\llbracket} g\left(\overset{1}{A}, \overset{2}{B}\right)\rrbracket,\right.$$

$$\left.\overset{k+8}{\llbracket} g\left(\overset{1}{A}, \overset{2}{B}\right)\rrbracket, \overset{k+10}{C}_{k+1}, \ldots, \overset{s+9}{C}_s\right)\rrbracket. \tag{6.7}$$

We shall call formula (6.7), as well as a more particular formula (6.6), *K-formula*.

Proof of Theorem 6.5. There exists an equality

$$f(z) = f(g(x_1, x_2)) + (z - g(x_1, x_2)) \frac{\delta f}{\delta x}(g(x_1, x_2); z).$$

Applying the operation $\mu: \left(z \to \overset{2}{g}, x_1 \to \overset{1}{A}, x_2 \to \overset{3}{B}\right)$ to both sides of the equality, where $g = \llbracket g\left(\overset{1}{A}, \overset{2}{B}\right)\rrbracket$, and using the first theorem of sum we get

$$\llbracket f(g)\rrbracket = \llbracket f\left(g\left(\overset{1}{A}, \overset{3}{B}\right)\right)\rrbracket + \llbracket\left(\overset{2}{g} - g\left(\overset{1}{A}, \overset{3}{B}\right)\right) \frac{\delta f}{\delta x}\left(g\left(\overset{1}{A}, \overset{3}{B}\right), \overset{2}{g}\right)\rrbracket.$$

We further obtain

$$\llbracket f(g)\rrbracket = \llbracket f\left(g\left(\overset{1}{A}, \overset{2}{B}\right)\right)\rrbracket + \llbracket\left(\overset{4}{g} - g\left(\overset{2}{A}, \overset{5}{B}\right)\right) \frac{\delta f}{\delta x}\left(g\left(\overset{1}{A}, \overset{5}{B}\right), \overset{3}{g}\right)\rrbracket =$$

$$\llbracket f\left(g\left(\overset{1}{A}, \overset{2}{B}\right)\right)\rrbracket + \llbracket \overset{4}{g} \frac{\delta f}{\delta x}\left(g\left(\overset{1}{A}, \overset{5}{B}\right), \overset{2}{g}\right) -$$

$$- g\left(\overset{3}{A}, \overset{5}{B}\right) \frac{\delta f}{\delta x}\left(g\left(\overset{1}{A}, \overset{5}{B}\right), \overset{2}{g}\right)\rrbracket +$$

$$+ \llbracket\left(\overset{4}{A}, g\right) \frac{\delta g}{\delta x_1}\left(\overset{3}{A}, \overset{5}{A}, \overset{7}{B}\right) \frac{\delta^2 f}{\delta x^2}\left(g\left(\overset{1}{A}, \overset{7}{B}\right), \overset{2}{g}, \overset{6}{g}\right)\rrbracket$$

with the help of axiom (μ_2) and changing the indices of $\overset{2}{A}$ and $\overset{3}{g}$ by Theorem 6.4.

According to axiom (μ_2) and Theorem 6.3

$$\llbracket \overset{4}{g} \frac{\delta f}{\delta x}\left(g\left(\overset{1}{A}, \overset{5}{B}\right), \overset{2}{g}\right)\rrbracket = \llbracket g\left(\overset{3}{A}, \overset{4}{B}\right) \frac{\delta f}{\delta x}\left(g\left(\overset{1}{A}, \overset{5}{B}\right), \overset{2}{g}\right)\rrbracket.$$

By the latter formula and Theorems 6.3 and 6.4 we obtain

$$[\![f(g)]\!] = [\![f\left(g\left(\overset{1}{A}, \overset{2}{B}\right)\right)]\!] + [\![\left(g\left(\overset{3}{A}, \overset{4}{B}\right) - g\left(\overset{3}{A}, \overset{4}{B}\right)\right) \times$$
$$\times \frac{\delta f}{\delta x}\left(g\left(\overset{1}{A}, \overset{5}{B}\right), \overset{2}{g}\right)]\!] + [\![\left[\overset{5}{A, B}\right] \times$$
$$\times \frac{\delta g}{\delta x_1}\left(\overset{3}{A}, \overset{7}{A}, \overset{9}{B}\right) \frac{\delta g}{\delta x_2}\left(\overset{3}{A}, \overset{4}{B}, \overset{6}{B}\right) \frac{\delta^2 f}{\delta x^2}\left(g\left(\overset{1}{A}, \overset{9}{B}\right), \overset{2}{g}, \overset{8}{g}\right)]\!].$$

The proof is completed by axiom (μ_1) since $g(x_1, x_2) - g(x_1, x_2) = 0$.

Theorem 6.6. (The Newton expansion.) *Let A, $(A + B)$, $B \in M$, f be a symbol of rank 1. Then*

$$[\![f(A+B)]\!] = [\![f(A)]\!] + \sum_{k=1}^{m-1} [\![\overset{2}{B} \ldots \overset{2k}{B} \frac{\delta^k f}{\delta x^k}\left(\overset{1}{A}, \ldots, \overset{2k+1}{A}\right)]\!] +$$
$$+ [\![\overset{2}{B} \ldots \overset{2m}{B} \frac{\delta^m f}{\delta x^m}\left(\overset{1}{A}, \overset{3}{A}, \ldots, \overset{2m-1}{A}, \overset{2m+1}{A+B}\right)]\!]. \tag{6.8}$$

The note cited in the cases of two preceding theorems is valid here as well; its statement is left to the reader.

Proof. First obtain the following formula:

$$[\![\overset{2}{T_1} \ldots \overset{2r}{T_r} \varphi\left(\overset{1}{A_1}, \ldots, \overset{2r-1}{A_r}, \overset{2r+1}{A+B}\right)]\!] =$$
$$= [\![\overset{2}{T_1} \ldots \overset{2r}{T_r} \varphi\left(\overset{1}{A_1}, \ldots, \overset{2r-1}{A_r}, \overset{2r+1}{A}\right)]\!] +$$
$$+ [\![\overset{2}{T} \ldots \overset{2r}{T_r} \overset{2r+2}{B} \frac{\delta\varphi}{\delta x_{r+1}}\left(\overset{1}{A_1}, \ldots, \overset{2r-1}{A_r}, \overset{2r+1}{A}, \overset{2r+3}{A+B}\right)]\!], \tag{6.9}$$

where $B, A_1, \ldots, A_r, A, A + B, T_1, \ldots, T_r \in M$, φ is a symbol of rank $r + 1$. We cancel out $T_1, \ldots, T_r$; $A_1, \ldots, A_r$ for the sake of simplicity, i.e., we shall prove

$$[\![f(A+B)]\!] - [\![f(A)]\!] = [\![\overset{2}{B} \frac{\delta f}{\delta x}\left(\overset{1}{A}, \overset{3}{A+B}\right)]\!]. \tag{6.10}$$

The proof of general formula (6.9) is different from the proof of formula (6.10) only by more cumbersome calculations.

There exists an equality

$$f(x_2) - f(x_1) - (x_2 - x_1)\frac{\delta f}{\delta x}(x_2, x_1) = 0.$$

Apply the operation $\mu : \left(x_2 \to \overset{2}{A} + \overset{2}{B},\, x_1 \to \overset{1}{A}\right)$; by axiom (μ_1) we get

$$[\![f\left(\overset{2}{A} + \overset{2}{B}\right) - f\left(\overset{1}{A}\right) - \left(\overset{2}{A} + \overset{2}{B} - \overset{1}{A}\right) \frac{\delta f}{\delta x}\left(\overset{2}{A} + \overset{2}{B},\, \overset{1}{A}\right)]\!] = 0.$$

Hence, by axioms (μ_1), (μ_2) and Theorem 6.2, we get

$$[\![f(A+B)]\!] - [\![f(A)]\!] =$$

$$= [\![\left(\overset{3}{A} + \overset{3}{B}\right) \frac{\delta f}{\delta x}\left(\overset{3}{A} + \overset{3}{B},\, \overset{1}{A}\right) - \overset{1}{A} \frac{\delta f}{\delta x}\left(\overset{3}{A} + \overset{3}{B},\, \overset{1}{A}\right)]\!] =$$

$$= [\![\left(\overset{2}{A} + \overset{2}{B}\right) \frac{\delta f}{\delta x}\left(\overset{3}{A} + \overset{3}{B},\, \overset{1}{A}\right)]\!] - [\![\overset{2}{A} \frac{\delta f}{\delta x}\left(\overset{3}{A} + \overset{3}{B},\, \overset{1}{A}\right)]\!].$$

Applying Theorem 6.2 once more, we get the equation

$$[\![f(A+B)]\!] - [\![f(A)]\!] = [\![\overset{2}{B} \frac{\delta f}{\delta x}\left(\overset{3}{A} + \overset{3}{B},\, \overset{1}{A}\right)]\!].$$

Formula (6.9) is proved. Now applying successively formula (6.9) to the last term of the right-hand side of (6.8), we obtain the proof by induction. The theorem is proved.

We shall define the spectrum of pairs of operators of an algebra with the μ-structure along the same lines as we defined it in the case of convergent power series of ordered operators.

Definition. Let A, $B \in M$. *Let K be such a set of symbols $f(x_1, x_2)$ that*

$$(f(x_1, x_2) \in K) \Leftrightarrow \left([\![f\left(\overset{1}{A}, \overset{2}{B}\right)]\!] = 0\right),$$

i.e., the symbol transformed into zero by the operation μ: $(x_1 \to \overset{1}{A},\ x_2 \to \overset{2}{B})$. *The spectrum σ of a pair $\overset{1}{A}$, $\overset{2}{B}$ is an intersection of sets of zeroes of functions belonging to K:*

$$((x_1, x_2) \in \sigma) \Leftrightarrow (f(x_1, x_2) = 0 \quad \textit{for any} \quad f \in K).$$

Theorem 6.7. *The set K is an ideal of an algebra of symbols of rank* 2.

Proof. Let $f(x_1, x_2) \in K$. Prove that $g(x_1, x_2) f(x_1, x_2) \in K$ for any $g \in \mathscr{S}^\infty(\mathbf{R}^n)$, i.e., it must be proved that if $[\![f\left(\overset{1}{A}, \overset{2}{B}\right)]\!] = 0$, then $[\![g\left(\overset{1}{A}, \overset{2}{B}\right) f\left(\overset{1}{A}, \overset{2}{B}\right)]\!] = 0$. There exists an equality

$$[\![g\left(\overset{1}{A}, \overset{2}{B}\right) f\left(\overset{1}{A}, \overset{2}{B}\right)]\!] = [\![g\left(\overset{1}{A}, \overset{4}{B}\right) f\left(\overset{2}{A}, \overset{3}{B}\right)]\!] \tag{6.11}$$

by the axiom of shifting of indices. We see that

$$[\![f\left(\overset{2}{A}, \overset{3}{B}\right)]\!] = [\![f\left(\overset{1}{A}, \overset{2}{B}\right)]\!] = 0$$

by the same axiom. Thus the right-hand side of (6.11) is zero by the zero axiom, Q.E.D.

We derive a number of corollaries from the last theorem.

We shall consider the two most important corollaries. First reconsider the following version of axiom (μ_4).

(μ_4'). Let $\{f_j(x_1, \ldots, x_k)\}$ be such a sequence of symbols that there is only a finite number of non-zero terms of the series $\sum_{j=1}^{\infty} f_j(x_1, \ldots, x_k)$ at any fixed point $(x_1^0, \ldots, x_k^0)$ and only a finite number of operators $[\![f_j\left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}\right)]\!]$ is not equal to zero; then

$$[\![\left(\sum_{j=1}^{\infty} f_j\right)\left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}\right)]\!] = \sum_{j=1}^{\infty} [\![f_j\left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}\right)]\!].$$

Theorem 6.8. *If axiom (μ_4') is true, then the spectrum of any pair of operators belonging to M is not vacant.*

Proof. Assume the contrary: let $\sigma = \varnothing$. Then for any fixed point $(x_1^0, x_2^0) \in \mathbf{R}^2$ there exists such a function $f(x_1, x_2) \in K$ that $f(x_1, x_2) > 0$ in neighborhood U of the point (x_1^0, x_2^0). Any compact set in $\mathbf{R}^2$ can be covered by a finite number of such neighborhoods, hence there exists a function belonging to K strictly positive at points of the set. Cover the plane $\mathbf{R}^2$ by squares q_{ij}

$$q_{ij} = \{(x_1, x_2) \in \mathbf{R}^2; -\varepsilon + i \leqslant x_1 \leqslant i + 1 + \varepsilon, \quad j - \varepsilon \leqslant x_2 \leqslant j + 1 + \varepsilon\},$$

$i, j = 0, \pm 1, \pm 2, \ldots, \varepsilon > 0$ is a fixed number.

For every square q_{ij} consider a function $f_{ij}(x_1, x_2) \in K$, strictly positive in q_{ij}. Let $\varphi(x)$, $x \in \mathbf{R}$ be an infinitely differentiable nonnegative function such that $\varphi(x) = 1$ when $0 \leqslant x \leqslant 1$ and $\varphi(x) = 0$ when $x < -\frac{\varepsilon}{2}$ and when $x > 1 + \frac{\varepsilon}{2}$. Then a function

$$f(x_1, x_2) = \sum_{i, j=-\infty}^{\infty} f_{ij}(x_1, x_2)\,\varphi(x_1 - i)\,\varphi(x_2 - j)$$

is everywhere positive and $f_{ij}(x_1, x_2)/f(x_1, x_2) \in K$ by virtue of Theorem 6.7. Since $\sum f_{ij}(x_1, x_2)/f(x_1, x_2) = 1$, we have by axioms (μ_3), (μ_4')

$$0 = [\![\sum \frac{1}{f\left(\overset{1}{A}, \overset{2}{B}\right)} f_{ij}\left(\overset{1}{A}, \overset{2}{B}\right)]\!] = [\![1\left(\overset{1}{A}, \overset{2}{B}\right)]\!] = 1,$$

where $1(x_1, x_2)$ is a function equal to 1, and 1 in the right-hand side is a unit operator. This is a contradiction but we thus obtain the statement of the theorem.

Definition. *A complement of the spectrum* σ *is a resolvent set of a pair* $\overset{1}{A}$, $\overset{2}{B}$; *it is denoted by* $\rho\left(\overset{1}{A}, \overset{2}{B}\right)$.

We shall now indicate a criterium which will enable us to determine the spectrum using only the functions of operators of one argument.

Theorem 6.9. *A point* (λ, μ) *belongs to* $\rho\left(\overset{1}{A}, \overset{2}{B}\right)$ *if and only if there exist positive numbers* ε *and* δ *and finite functions* $p_\lambda^\varepsilon(x_1)$, $p_\mu^\delta(x_2)$ *positive when* $x_1 = \lambda$, $x_2 = \mu$ *and equal to zero when* $|x_1 - \lambda| > \varepsilon$, $|x_2 - \mu| > \delta$ *(respectively) such that* $p_\mu^\delta(B)\, p_\lambda^\varepsilon(A) = 0$.

Proof. The test of sufficiency. We have

$$p_\mu^\delta(B)\, p_\lambda^\varepsilon(A) = [\![p_\mu^\delta\left(\overset{2}{B}\right) p_\lambda^\varepsilon\left(\overset{1}{A}\right)]\!]$$

by axiom (μ_5). Hence $p_\mu^\delta(x_2)\, p_\lambda^\varepsilon(x_1) \in K$ and the function does not vanish at the point (λ, μ). Therefore, $(\lambda, \mu) \notin \sigma$ by the definition of the spectrum.

The test of necessity. Let $(\lambda, \mu) \in \rho\left(\overset{1}{A}, \overset{2}{B}\right)$; then there exists a function $\varphi \in K$ positive at the point (λ, μ) and, therefore, positive in its $(2\varepsilon, 2\delta)$-neighborhood. Let p_λ, p_μ be functions from $\mathcal{S}^\infty(\mathbf{R})$ equal to zero outside an ε-neighborhood of the point λ and a δ-neighborhood of the point μ, respectively.

Since $\varphi(A, B) = 0$, we have

$$[\![\varphi\left(\overset{1}{A}, \overset{2}{B}\right) p_\mu\left(\overset{2}{B}\right) p_\lambda\left(\overset{1}{A}\right)]\!] = p_\mu(B)\, [\![\varphi\left(\overset{1}{A}, \overset{2}{B}\right)]\!]\, p_\lambda(A) = 0.$$

Let $\psi(x_1, x_2) \in \mathcal{S}^\infty(\mathbf{R}^2)$ be a non-negative function equal to zero in an (ε, δ)-neighborhood of the point (λ, μ) and equal to $1 - \varphi(x_1, x_2)$ outside a $(2\varepsilon, 2\delta)$-neighborhood of the point (λ, μ). It is obvious that

$$[\psi(x_1, x_2) + \varphi(x_1, x_2)]\, p_\mu(x_2)\, p_\lambda(x_1) = \varphi(x_1, x_2)\, p_\mu(x_1)\, p_\lambda(x_2)$$

and

$$[\psi(x_1, x_2) + \varphi(x_1, x_2)]^{-1} \in \mathcal{S}^\infty(\mathbf{R}^2).$$

Thus

$$[\![\left[\psi\left(\overset{1}{A}, \overset{2}{B}\right) + \varphi\left(\overset{1}{A}, \overset{2}{B}\right)\right] p_\mu\left(\overset{2}{B}\right) p_\lambda\left(\overset{1}{A}\right)]\!] = 0$$

and by axioms (μ_2), (μ_6)

$$[\![\left[\psi\left(\overset{1}{A}, \overset{2}{B}\right) + \varphi\left(\overset{1}{A}, \overset{2}{B}\right)\right]^{-1} \left[\varphi\left(\overset{1}{A}, \overset{2}{B}\right) + \right.$$

$$\left. + \psi\left(\overset{1}{A}, \overset{2}{B}\right)\right] p_\mu\left(\overset{2}{B}\right) p_\lambda\left(\overset{1}{A}\right)]\!] = 0.$$

Therefore

$$[\![p_\mu \left(\overset{2}{B}\right) p_\lambda \left(\overset{1}{A}\right)]\!] = 0$$

and, finally, by axiom (μ_5), $p_\mu(B)\,p_\lambda(A) = 0$, Q.E.D.

Define the spectrum $\sigma(A)$ and a resolvent set $\rho(A)$ of one operator A in the same way. It is obvious that there are similar theorems in the case of the spectrum of one operator. Indeed, the following theorem is a corollary of Theorem 6.9.

Theorem 6.10. *There exists an inclusion*

$$\sigma\left(\overset{1}{A}, \overset{2}{B}\right) \subset \sigma(A) \times \sigma(B).$$

More comprehensive results are obtained in Chapters I and II; we need the concept of normed algebra to the effect.

Note. We can extend the concepts of the μ-algebra and the algebra of symbols into the domain of topological algebras modifying axiom (μ_6) conveniently with regard to the emerging topological structures (cf. Ch. II).

Example of a μ-structure. Let $C_0^\infty(\mathbf{R}^n)$ be a vector space of infinitely differentiable functions $f(x_1, \ldots, x_n)$ (defined on $\mathbf{R}^n$) equal to zero outside a ball (defined for every function separately) in $\mathbf{R}^n$. Let D_j and x_j be operators acting in $C_0^\infty(\mathbf{R}^n)$ by the formulas

$$D_j f(x) = -i\frac{\partial f(x)}{\partial x_j}, \quad \hat{x}_j f(x) = x_j f(x).$$

We shall denote D_j by $-i\frac{\partial}{\partial x_j}$ and $\hat{x}_j$ by x_j; the last notation will be used when it is impossible to mistake $\hat{x}_j$ for the independent variable x_j.

Let $P(x_1, \ldots, x_n; p_1, \ldots, p_n)$ be a polynomial. Consider a differential operator $P\left(\overset{2}{x_1}, \ldots, \overset{2}{x_n}; \overset{1}{D_1}, \ldots, \overset{1}{D_n}\right)$. The operator is written easily by the Fourier transform, which is defined by the formula

$$\tilde{f}(p) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbf{R}^n} e^{-ip\cdot x} f(x)\,dx, \quad x = (x_1, \ldots, x_n), \quad p = (p_1, \ldots, p_n),$$

where f is a function, $f(x_1, \ldots, x_n)$.

There exists an inverse transformation

$$f(x) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbf{R}^n} e^{ip\cdot x} \tilde{f}(p)\,dp.$$

It is easy to prove that $\widetilde{D_j f(x)} = p_j \tilde{f}(p)$. Hence

$$P\left(\overset{2}{x}_1, \ldots, \overset{2}{x}_n, \overset{1}{D}_1, \ldots, \overset{1}{D}_n\right) f(x) \overset{\text{def}}{=} P\left(\overset{2}{x}, \overset{1}{D}\right) f(x) =$$

$$= (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ip\cdot x} P(x, p)\, dp \int_{\mathbf{R}^n} e^{-ip\xi} f(\xi)\, d\xi. \tag{6.15}$$

The right-hand side of (6.15) is also valid when P are continuous functions (i.e., not necessarily polynomials) of sufficiently slow growth as $|p| \to \infty$, for example as $|p|$ in some power. We can state the following definition.

Definition. *A pseudodifferential operator with a symbol* $H(x, p)$ *is an operator* $H\left(\overset{2}{x}, \overset{1}{D}\right)$ *defined by the formula*

$$H\left(\overset{2}{x}, \overset{1}{D}\right) f(x) \overset{\text{def}}{=} (2\pi)^{-n} \iint_{\mathbf{R}^{2n}} e^{ip\cdot(x-\xi)} H(x, p) f(\xi)\, d\xi. \tag{6.16}$$

Note. The same formula defines $H\left(\overset{1}{x}, \overset{2}{D}\right)$ for the distributions (cf. Ch. II).

In the same way we define an operator $H\left(\overset{1}{x}, \overset{2}{D}\right)$

$$H\left(\overset{1}{x}, \overset{2}{D}\right) f(x) \overset{\text{def}}{=} (2\pi)^{-n} \iint_{\mathbf{R}^{2n}} e^{ip\cdot(x-\xi)} H(\xi, p) f(\xi)\, d\xi. \tag{6.17}$$

The structure of commutative algebra $\mathfrak{U}_p, \left(\overset{2}{x}, \overset{1}{D}\right)$ is now defined in the set of operators of the form $H\left(\overset{2}{x}, \overset{1}{D}\right)$. Thus

$$H\left(\overset{2}{x}, \overset{1}{D}\right) L\left(\overset{2}{x}, \overset{1}{D}\right) f(x) = (2\pi)^{-n} \iint_{\mathbf{R}^{2n}} e^{ip\cdot(x-\xi)} H(x, p) L(x, p) f(\xi)\, d\xi.$$

The operators $p = ih\frac{\partial}{\partial x}$, x and the noncommutative algebra generated by them are defined for $f(x, h) \in C_0^\infty(\mathbf{R}^n) \times \mathcal{S}^\infty(\mathbf{R})$ in the same way.

Now take M consisting of the set of operators having the form

$$A_k = ih\frac{\partial}{\partial x} + \frac{\partial S_k}{\partial x}, \qquad B_k = x;$$

$k = 1, 2, \ldots, S_k = S_k(x)$ is a set of functions belonging to $C_0^\infty(\mathbf{R}^n)$.

Note. The vector-operator $A_k = ih\frac{\partial}{\partial x} + \frac{\partial S_k(x)}{\partial x}$ has components $A_k^j = ih\frac{\partial}{\partial x_j} + \frac{\partial S_k}{\partial x_j}$. These components commute with each other, hence the vector-operator A_k^* may be indexed only by one number.

Define the μ-operation for $\Phi(x) \in \mathscr{S}^\infty(\mathbf{R}^n)$. Let

$$F_{x\to p} f(x) \overset{\text{def}}{=} h^{-\frac{n}{2}} \tilde{f}\left(\frac{p}{h}\right); \quad F^*_{p\to x} g(p) \overset{\text{def}}{=} h^{-\frac{n}{2}} \tilde{g}\left(-\frac{p}{h}\right).$$

Consider the case of two variables for the sake of simplicity. Take by definition, for $n_1 < m_1$, $\varphi \in C_0^\infty(\mathbf{R}^n)$,

$$\Phi\left(\overset{n_1}{A_1}, \overset{m_1}{B_1}\right) \varphi(x) = \Phi\left(\overset{1}{A_1}, \overset{2}{B_1}\right) \varphi(x) \overset{\text{def}}{=} e^{-\frac{i}{h} S_1(x)} \times$$
$$\times \Phi\left(\overset{1}{p}, \overset{2}{x}\right) e^{\frac{i}{h} S_1(x)} \varphi(x).$$

Now verify axioms (μ_1)-(μ_6).

All axioms except the second item of the shifting of indices axiom are obviously true by definition. The second item of the shifting of indices axiom is a corollary of the equation $F^*_{p\to x} F_{x\to p} = 1$. Concerning (μ_6) see Ch. II.

We have obtained the formula of commutation of Hamiltonian with exponential now by definition. The operation "prime" is defined similarly.

We shall indicate another generalization of the μ-structures as well; let symbols be functions defined on a m-dimensional torus M^m, i.e., a product of m circles. Consider infinitely differentiable functions on the torus valued in the space $\mathscr{S}^\infty(\mathbf{R}^n)$. We shall denote the functions by $f(x_1, \ldots, x_k, \alpha)$, where $\alpha \in M^m$ are angular coordinates on the torus and $x_1, \ldots, x_k$ are real numbers, k depends on f.

Consider the algebra $\mathscr{A}$ and its subset M as we did above. Consider, as well, a subset M_m consisting of m commuting elements of the set M; $B = (B_1, \ldots, B_m) \subset M_m$. The operation

$$\mu\colon \left(x_1 \to \overset{n_1}{A_1}, \ldots, x_k \to \overset{n_k}{A_k}, \alpha \to \overset{n_{k+1}}{B}\right)$$

is defined for any finite number of operators $A_1, \ldots, A_k \in M$ and any number of indices $n_1, n_2, \ldots, n_{k+1}$, satisfying the condition $n_i \neq n_j$ if A_i and A_j do not commute and $n_i \neq n_{k+1}$ if A_i and $B_s \in M_m$ do not commute. The operation substitutes an operator $A \in \mathscr{A}$ for a symbol $f(x_1, \ldots, x_n, \alpha)$ denoted by

$$A = [\![f\left(\overset{n_1}{A_1}, \ldots, \overset{n_k}{A_k}, \overset{n_{k+1}}{B}\right)]\!]$$

and satisfies all the preceding axioms (as if $\alpha \in \mathbf{R}^m$) except the correspondence axiom, which is true only with respect to the variable x and is supplemented by the following condition:

(μ_3') The condition

$$[\![f\left(\overset{n_k}{B}\right)]\!] = [\![f_*\left(\overset{1}{B}\right)]\!]$$

is true for any real infinitely differentiable function $f(\alpha)$, $\alpha \in M^m$ and the corresponding periodic function $f_*(x)$, $x \in \mathbf{R}^m$.

Problem. Consider an algebra M generated by elements x and p such that $[x, p] = ih$. Take a function defined on a circle of radius 1 valued in $\mathcal{S}^\infty$ as a symbol $f(x_1, \alpha)$.

Define a μ-structure with axiom (μ_3') using as the set M_1 the operator x, such that the elements of $\mathcal{A}$ might be operators acting on the functions on $\mathbf{R}$ and the following equation might be fulfilled

$$f\left(\overset{1}{p}, \overset{2}{x}\right) \varphi(\alpha) = \frac{1}{2\pi} \sum_{n=-\infty}^{n=\infty} f(nh, \alpha)\, e^{in\alpha} \int_{-\pi}^{+\pi} e^{-iny} \varphi(y)\, dy.$$

Symbols of the class $\mathcal{S}^\infty$ are primarily important for the definition of a μ-structure. We shall see later that the class $\mathcal{S}^\infty$ is too narrow for our aims. An extension of the class in the axiomatics of μ-structures can lead to a situation when an algebra containing $i\dfrac{\partial}{\partial x}$ and x has no μ-structures.

Consider an example to the effect.

Let a fixed class of symbols contain functions of the form $f(x_1, x_2) = e^{\pm ix_1x_2}$ and let A, B be such operators belonging to M that $[A, B] = -i$ $\left(\text{for example, } A = -i\dfrac{\partial}{\partial x},\ B = x\right)$. Apply the formula of commutation (6.5) to the left-hand side of the equation

$$e^{-i\left(\overset{3}{A}-\overset{4}{A}\right)\overset{2}{B}} = 1$$

and substitute the operator $\overset{3}{A}$ into the first place. Thus we get

$$1 = [\![e^{-i\left(\overset{1}{A}-\overset{4}{A}\right)\overset{2}{B}}]\!] - [\![\psi\left(\overset{1}{A}, \overset{3}{A}; \overset{2}{B}; \overset{4}{A}\right)]\!],$$

where a symbol of the operator $\psi\left(\overset{1}{A}, \overset{3}{A}; \overset{2}{B}; \overset{4}{A}\right)$ has the form

$$\psi(p_1, p_2; x; y) = i\frac{\partial}{\partial x}\frac{\delta}{\delta p} e^{-i(p-y)x} =$$

$$= \frac{(p_1-y)\, e^{-i(p_1-y)x} - (p_2-y)\, e^{-i(p_2-y)x}}{p_1-p_2} =$$

$$= e^{-i(p_1-y)x} + (p_2-y)\frac{e^{-i(p_1-y)x} - e^{-i(p_2-y)x}}{p_1-p_2}.$$

Hence by the axioms of zero and correspondence, we get

$$1=[\![e^{-i\left(\overset{1}{A}-\overset{4}{A}\right)\overset{2}{B}}]\!]-[\![e^{-i\left(\overset{1}{A}-\overset{4}{A}\right)\overset{2}{B}}]\!]-[\![\left(\overset{3}{A}-\overset{4}{A}\right)\frac{e^{-i\left(\overset{1}{A}-\overset{4}{A}\right)\overset{2}{B}}-1}{\overset{1}{A}-\overset{3}{A}}]\!]=0.$$

We see that for the given A, $B \in M$ and the class of symbols containing $e^{\pm ix_1x_2}$ there does not exist any μ-structure.

Therefore it is necessary to be very careful defining μ-structures for symbols of the class wider than $\mathcal{S}^\infty$, since this leads to wrong results even for the operators with symbols of the class $\mathcal{S}^\infty$ (cf. the details in Ch. II).

Now we shall define a generalization of μ-structures using an extended class of symbols.

Consider a two-sided module $\mathcal{L}$ over an algebra $\mathcal{A}$, i.e., $\mathcal{L}$ is such a linear space containing $\mathcal{A}$, that for any pair $a \in \mathcal{A}$, $l \in \mathcal{L}$ a product $al \in \mathcal{L}$ and $la \in \mathcal{L}$, linear in l, is defined.

Let X be a set of $n+m$ elements $A_1, \ldots, A_n, B$ of algebra $\mathcal{A}$ (n, m are fixed) indexed by $k_i < k_{i+1}$ (or $k_i > k_{i+1}$) and let $C^\infty_{\mathcal{L}}$ be the $\mathcal{S}^\infty$-module generated by the space of functions $\varphi(x, \alpha)$, $x \in \mathbf{R}^n$, $\alpha \in M^m$ of the rate of growth not greater than a power of x and such that

$$\left|\frac{\partial^l}{\partial\alpha^l}\frac{\partial^s}{\partial x^s}\varphi(x, \alpha)\right| \leqslant C_{l,s}|x|^{2l+l_0}+C^1_{l,s} \Leftrightarrow |\varphi| \overset{\text{def}}{=} O_{\mathcal{L}}(|x|^{l_0}).$$

A μ-structure is defined in $\mathcal{L}$, if for any set $\overset{k_1}{A_1}, \ldots, \overset{k_n}{A_n}, \overset{k_{n+1}}{B}$, $\overset{k_{n+2}}{C_1}, \ldots, \overset{k_{n+l+2}}{C_l}$, where $C_j \in M$, an operation is defined*

$$\mu: x_1 \to A_1, \ldots, x_n \to A_n, y_1 \to C_1, \ldots, y_l \to C_l, \alpha \to B,$$

which substitutes an element $L \in \mathcal{L}$ for a function $f(x, y, \alpha) \in C^\infty_{\mathcal{L}}$. The element L is denoted by

$$L=[\![f\left(\overset{k_1}{A_1}, \ldots, \overset{k_n}{A_n}, \overset{k_{n+1}}{B}, \overset{k_{n+2}}{C_1}, \ldots, \overset{k_{n+l+2}}{C_l}\right)]\!].$$

It must satisfy axioms (μ_1)-(μ_6), and be compatible with the μ-structure on $\mathcal{A}$.

A module $\mathcal{L}$ with the given operation μ is defined as a module over $\mathcal{A}$ with a μ-structure.

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* Of course, we suppose $k_i \neq k_j$ whenever the two operators corresponding to k_i and k_j, respectively, do not commute.

Sec. 7. An Example of a Solution of a Differential Equation

Consider the differential equation

$$\frac{\partial^2 u}{\partial t^2} - x^4\frac{\partial^2 u}{\partial x^2} = 0. \tag{7.1}$$

Denote by $D = -i\frac{\partial}{\partial x}$, $D_0 = -i\frac{\partial}{\partial t}$. Find $u(x, t)$ in the form

$$u(x, t) = e^{i\overset{1}{D}S\left(\overset{2}{x},\, \overset{2}{t}\right)}\varphi\left(\overset{2}{x},\, \overset{2}{t}\right)v(x), \tag{7.2}$$

where

$$S(x, 0) = 0, \quad \varphi(x, 0) = 1, \quad v \in C_0^\infty(\mathbf{R}).$$

Substitute function (7.2) into the left-hand side of (7.1) applying the formula of commutation of Hamiltonian and exponential:

$$\left(\frac{\partial^2}{\partial t^2} - x^4\frac{\partial^2}{\partial x^2}\right)[\![e^{i\overset{1}{D}S\left(\overset{2}{x},\, \overset{2}{t}\right)}\varphi\left(\overset{2}{x},\, \overset{2}{t}\right)]\!]\, v(x) =$$

$$= e^{i\overset{1}{D}S\left(\overset{4}{x},\, \overset{4}{t}\right)}\overset{3}{[\![}\, \overset{4'}{x^4}\left(D + \overset{1'}{D}\frac{\partial S}{\partial x}(x, t)\right)^2 -$$

$$-\left(D_0 + \overset{1'}{D}\frac{\partial S}{\partial t}(x, t)\right)^2]\!]\,\varphi\left(\overset{2}{x},\, \overset{2}{t}\right)v(x) = [\![e^{i\overset{1}{D}S\left(\overset{4}{x},\, \overset{4}{t}\right)}L]\!]\, v(x).$$

It should be noted that L is a part of the operational equation, but it is not an operator.

Let us transform the expression of L. Applying Theorem 6.4 and taking into account that $\delta^2 f$ is a unit operator in this case and

$$\left[i\frac{\partial}{\partial x},\ \frac{\partial S}{\partial x}\right] = i\frac{\partial^2 S}{\partial x^2}$$

we may put L in the form

$$L = \left[\overset{4}{x^4}\left(\overset{3}{D} + \overset{1}{D}\frac{\partial S}{\partial x}\left(\overset{4}{x},\, \overset{4}{t}\right)\right)^2 - \left(\overset{3}{D_0} + \overset{1}{D}\frac{\partial S}{\partial t}\left(\overset{4}{x},\, \overset{4}{t}\right)\right)^2\right]\varphi\left(\overset{2}{x},\, \overset{2}{t}\right) -$$

$$- i\overset{1}{D}\overset{2}{x^4}\frac{\partial^2 S}{\partial x^2}\left(\overset{2}{x},\, \overset{2}{t}\right)\varphi\left(\overset{2}{x},\, \overset{2}{t}\right) + i\overset{1}{D}\frac{\partial^2 S}{\partial t^2}\left(\overset{2}{x},\, \overset{2}{t}\right).$$

Then changing in the first summand the order of action of the operators $\overset{3}{D}$, $\overset{3}{D_0}$ and $\overset{2}{x}$, $\overset{2}{t}$ by virtue of Theorem 6.4, we obtain

$$L = \left[\overset{2}{x^4}\left(\overset{1}{D} + \overset{1}{D}\frac{\partial S}{\partial x}\left(\overset{2}{x},\, \overset{2}{t}\right)\right)^2 - \left(\overset{1}{D_0} + \overset{1}{D}\frac{\partial S}{\partial t}\left(\overset{2}{x},\, \overset{2}{t}\right)\right)^2\right]\varphi\left(\overset{2}{x},\, \overset{2}{t}\right) -$$

$$- i\left[\overset{2}{x^4}\left(\overset{1}{D} + \overset{3}{D}\right) + 2\overset{2}{x^4}\overset{1}{D}\frac{\partial S}{\partial x}\left(\overset{2}{x},\, \overset{2}{t}\right)\right]\frac{\partial\varphi}{\partial t}\left(\overset{2}{x},\, \overset{2}{t}\right) +$$

$$+ i\left[\left(\overset{1}{D_0} + \overset{3}{D_0}\right) + 2\overset{1}{D}\frac{\partial S}{\partial t}\left(\overset{2}{x},\, \overset{2}{t}\right)\right]\frac{\partial\varphi}{\partial t}\left(\overset{2}{x},\, \overset{2}{t}\right) -$$

$$- i\overset{1}{D}\overset{2}{x^4}\frac{\partial^2 S}{\partial x^2}\left(\overset{2}{x},\, \overset{2}{t}\right)\varphi\left(\overset{2}{x},\, \overset{2}{t}\right) + i\overset{1}{D}\frac{\partial^2 S}{\partial t^2}\left(\overset{2}{x},\, \overset{2}{t}\right)\varphi\left(\overset{2}{x},\, \overset{2}{t}\right).$$

Changing again the order of action of these operators, we obtain

$$L=[\![x^4\left(1+\frac{\partial S}{\partial x}\right)^2\varphi(x,t)D^2-[\![\left(\overset{1}{D}_0+\overset{1}{D}\frac{\partial S}{\partial t}\left(\overset{2}{x},\overset{2}{t}\right)\right)^2\varphi\left(\overset{2}{x},\overset{2}{t}\right)]\!]+$$
$$+i\left[-2x^4\frac{\partial\varphi}{\partial x}\left(1+\frac{\partial S}{\partial x}D\right)+2\frac{\partial\varphi}{\partial t}\frac{\partial S}{\partial t}D+2\frac{\partial\varphi}{\partial t}D_0-\right.$$
$$\left.-x^4\frac{\partial^2S}{\partial x^2}\varphi D+\frac{\partial^2S}{\partial t^2}\varphi D\right]+L_0]\!], \tag{7.3}$$

where $L_0=\frac{\partial^2\varphi}{\partial t^2}-x^4\frac{\partial^2\varphi}{\partial x^2}$. Assuming that the first two terms do not contain the derivatives with respect to x we get an equation for S:

$$x^4\left(1+\frac{\partial S}{\partial x}\right)^2-\left(\frac{\partial S}{\partial t}\right)^2=0$$

which is called an equation of *D-characteristics* of equation (7.1). The equation splits into two branches:

$$\frac{\partial S}{\partial t}=\pm x^2\left(1+\frac{\partial S}{\partial t}\right).$$

We solve the equation with the sign "—" in the right-hand side. Considering the initial condition $S(x, 0) = 0$, we obtain

$$S(x,t)=-\frac{x^2t}{1+xt}. \tag{7.4}$$

Assuming that the last but one term in (7.3) does not contain differentiation with respect to x and applying (7.4) we get an equation for φ, i.e. *the transfer equation*

$$-x^2\frac{\partial\varphi}{\partial x}-\frac{\partial\varphi}{\partial t}+x\varphi=0.$$

A solution of the equation with the initial value $\varphi(x, 0) = 1$ is of the form

$$\varphi(x, t) = (1 + xt).$$

Thus we obtain $L_0 = 0$ and

$$u(x,t)=e^{-i\overset{1}{D}\frac{\overset{2}{x}^2\overset{2}{t}}{1+\overset{2}{x}\overset{2}{t}}}\left(1+\overset{2}{x}\overset{2}{t}\right)v(x).$$

Note that

$$e^{i\overset{1}{D}f\left(\overset{2}{x}\right)}v_x=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{ipx}e^{ipf(x)}\tilde{v}(p)\,dp=v(x+f(x)).$$

Therefore we finally obtain

$$u(x,t)=(1+xt)\,v\left(x-\frac{x^2t}{1+xt}\right)=(1+xt)\,v\left(\frac{x}{1+xt}\right).$$

Sec. 8. Passage of the Equation of Oscillations of a Crystal Lattice into a Wave Equation

We shall apply the developed operational calculus to a physical problem. Indeed, consider the equation of oscillations of a lattice and study the question as to which of its solutions turn into the solutions of the wave equation when the step of the lattice tends to zero by passing to the limit.

This familiar problem is a classical one. It is hard to expect the operational method to produce anything new. But, nonetheless, the operational method permits us find unexpected physical and mathematical corolaries. We shall find a critical solution of the lattice oscillation equation by the operational method, which has no limit, as the step of the lattice tends to zero. These critical solutions are cores of physical phenomena discussed in this section, some of which had not been known before.

An example under consideration in the section indicates that the operational calculus is to some extent adequate for the physical problem. Indeed, we shall see that the equation of characteristics for a function of ordered operators not only defines the fundamental phenomena like the effects of the Cherenkov type (a tail of oscillations behind a shock wave) but indicates the boundary of Cherenkov's oscillation domain as well. These effects are illustrated fairly well by computer calculations.

We shall cancel out an elementary study of the obtained formulas and the computer calculations in the other parts of the book, but here this simple classical example is studied thoroughly.

(1) Transformation of the equation of lattice oscillations into a wave equation. We shall consider a system of ordinary differential equations of lattice oscillations and derive the wave equation.

Consider $2N$ atoms of mass m, lying on the circle of radius 1 at the same distance $h = \pi/N$ from each other. Assume that the atoms interact only with neighboring atoms. Let $u_i(t)$ be a displacement of the ith atom at a moment of time t from the equilibrium position: here $i = 0, \pm 1, \ldots, \pm N$, $u_{-N} = u_N$. The system of Newton's equations for such a lattice in the linear approximation has the form

$$\ddot{u}_n = c^2 \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2}, \quad n = 0, \pm 1, \ldots, \pm N, \tag{8.1}$$

where $u_N = u_{-N}$, $u_{N+1} \stackrel{\text{def}}{=} u_{-N+1}$, $u_{-N-1} \stackrel{\text{def}}{=} u_{N-1}$; $c = \sqrt{\frac{\gamma}{m}}$ is a velocity of sound.

Assume that N is a very great number. Consider a smooth function $u(x, t)$ which takes the values u_j at the lattice knots, i.e., $u(jh, t) = u_j(t)$. System (8.1) assumes the form of a differential-difference system on a circle or, equivalently, a differential-difference system

with periodic coefficients

$$\frac{\partial^2 u(x, t)}{\partial t^2} = \frac{c^2}{h^2}[u(x+h, t) - 2u(x, h) + u(x-h, t)], \ t > 0,$$
$$u(x+2\pi, t) = u(x, t). \tag{8.2}$$

We assume that, generally, the sound velocity c depends on x and $c(x) \in C^\infty$ is a 2π-periodic function, $c(x) \neq 0$.

It is natural to consider the smoothest class of functions generated by the values u_j at the knots, since only the values of $u(x, t)$ at the knots are important. Let M_N be a vector space generated by the vectors e^{ikx}, $k = 0, \pm 1, \ldots, \pm(N-1), +N$. It is easy to see that for any set $\{u_j, j = 0, \ldots, \pm N, u_N = u_{-N}\}$ there exists a function $u \in M_N$ taking on the values u_j at the knots of the lattice. Therefore the initial values for problem (8.2) may be put in the form

$$u(x, 0) = v_1(x) \in M_N, \ \frac{\partial u}{\partial t}(x, 0) = v_2(x) \in M_N. \tag{8.3}$$

Further, applying the equation

$$e^{i\left(-ih\frac{\partial}{\partial x}\right)} u(x, t) = u(x+h, t),$$

we rewrite problems (8.2), (8.3) in the form of pseudodifferential equation

$$L_h u \stackrel{\text{def}}{=\!=} h^2 \frac{\partial^2 u}{\partial t^2} + 4c^2\left(\overset{2}{x}\right) \sin^2\left(-\frac{ih}{2}\overset{1}{\frac{\partial}{\partial x}}\right) u = 0, \qquad t > 0, \tag{8.4}$$
$$u(x, 0) = v_1(x), \ \frac{\partial u}{\partial t}(x, 0) = v_2(x), \ u(x+2\pi, t) = u(x, t).$$

Assume

$$\|f\| = \left(\int_{-\pi}^{\pi} |f(x)|^2 \, dx\right)^{1/2}$$

and take $\varphi \in C_{h, t}^{(s)}$ if the function $\varphi = \varphi(x, t, h)$ has s derivatives with respect to x, t, 2π-periodic in x, and there is

$$\sup_{0<h<1} \left\| \frac{\partial^k \varphi(x, t; h)}{\partial x^k} \right\| \leqslant c_k < \infty, \qquad k = 0, 1, \ldots, s.$$

First consider a problem with a smooth (uniformly in h) right-hand side

$$L_h u = h^2 \frac{\partial^2 u}{\partial t^2} + 4c^2\left(\overset{2}{x}\right) \sin^2\left(-\frac{ih}{2}\overset{1}{\frac{\partial}{\partial x}}\right) u = f, \qquad f \in C_{h, t}^{(s)},$$
$$u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0, \ u(x+2\pi, t) = u(x, t). \tag{8.5}$$

Lemma 8.1. *Let* $\varphi \in C_{h,t}^{(m+2)}$, *then*

$$4\sin^2\left(-\frac{ih}{2}\frac{\partial}{\partial x}\right)\varphi = -h^2\frac{\partial^2\varphi}{\partial x^2} + \sum_{k=4}^{m+1}\alpha_k(-ih)^k\frac{\partial^k\varphi}{\partial x^k} + h^{m+2}Q_m\varphi,$$

where $\|Q_m\varphi\| \leqslant c_m \left\|\frac{\partial^{m+2}\varphi}{\partial x^{m+2}}\right\|$, α_k *are some fixed numbers.*

Proof. We have, by the Taylor formula,

$$4\sin^2\left(\frac{z}{2}\right) = z^2 + \sum_{k=4}^{m+1}\alpha_k z^k - 2z^{m+2}\int_0^1 \frac{(1-\tau)^{m+1}}{(m+1)!}\cos^{(m+2)}(\tau z)\,d\tau.$$

Hence

$$Q_m\varphi = -\frac{2(-i)^{m-2}}{(m+1)!}\left[\int_0^1 (1-\tau)^{m+1} \times \right.$$
$$\left. \times \cos^{(m+2)}\left(-ih\tau\frac{\partial}{\partial x}\right)\left(\frac{\partial^{m+2}\varphi}{\partial x^{m+2}}\right)d\tau\right].$$

The function in the brackets may be obviously put in the form of a linear combination of the functions

$$\int_0^1 (1-\tau)^{m+1} e^{\pm i\tau\left(-ih\frac{\partial}{\partial x}\right)}\psi(x)\,d\tau = \int_0^1 (1-\tau)^{m+1}\psi(x\pm\tau h)\,d\tau,$$

where $\psi = \frac{\partial^{m+2}\varphi}{\partial x^{m+2}}$.

The following inequality is obvious:

$$\left\|\int_0^1 (1-\tau)^{m+1}\psi(x\pm\tau h)\,d\tau\right\| \leqslant C\|\psi\|.$$

The sought-for inequality for $Q_m\varphi$ follows from this. The lemma is proved.

The first approximation for the operator L_h as $h\to 0$ in problem (8.5) is defined as a *wave operator* and is denoted by $\square_c$ or L. Thus we have by Lemma 8.1

$$L = \square_c \stackrel{\text{def}}{=\!=} \frac{\partial^2}{\partial t^2} - c^2(x)\frac{\partial^2}{\partial x^2}.$$

The equation $\square_c u = f$ is called *the wave equation*. Therefore we shall say that the system of ordinary differential equations of lattice oscillations (8.1) turns into the wave equation as $h\to 0$. But we still know nothing as to the behaviour of the solutions of the system. Do they turn into the solutions of the wave equation as $h\to 0$?

First of all, it is necessary to verify that the solutions of problem (8.2) and the corresponding wave equation exist.

We shall prove the theorems of existence and uniqueness for the wave equation and equation (8.4) only for sufficiently small t and, what is important, indicate formulas of the solutions as well.

(2) The existence and uniqueness of a solution of a wave equation. We shall prove the following theorem here.

Theorem 8.1. *Let $f \in C_{h,t}^{(s)}$ (s is sufficiently large). Then for $0 < t \leqslant T$, where T is a number, there exists a unique solution $u = R_0(f) \in C_{h,t}^{(s)}$ of the problem*

$$\square_c u = f, \tag{8.6}$$

$$u(x+2\pi, t) = u(x, t), \quad u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0 \tag{8.6'}$$

and at the same time,

$$\left\| \frac{\partial^l}{\partial x^l} R_0(f) \right\| \leqslant \sum_{k=0}^{l} c_{l,k} \left\| \frac{\partial^k f}{\partial x^k} \right\|, \quad l = 0, 1, 2, \ldots, s,$$

and $R_0(f) = R(x, t)$, where $R(x, t)$ is defined by (8.8).

We shall prove this theorem (as many others of the same type) with the help of the construction of *a regularizer of problem* (8.6), (8.6′).

Definition. *Let $A = -i\frac{\partial}{\partial x} \in M$, $B = x \in M^1$, M^1 be a circle, $G(t, x_1, \alpha, s) \in C_{\mathcal{L}}^{\infty}$, $s = 1, 2, \ldots$ be a sequence of one-parameter families of symbols such that $G(0, x_1, \alpha, s) = 1$, $G_t'(0, x_1, \alpha, s) = 0$ and the following inequality is satisfied*

$$|f_1(t, x, \alpha, s)| \leqslant C(x_1^2+1)^{-s/2} \tag{8.7}$$

for a symbol $f_1 \in C_{\mathcal{L}}^{\infty}$ of an operator

$$f_1(t, \overset{1}{A}, \overset{2}{B}, s) \stackrel{\text{def}}{=} \left(\frac{\partial^2}{\partial t^2} + [\![c^2(B) A^2]\!]\right) G\left(t, \overset{1}{A}, \overset{2}{B}, s\right). \tag{8.7'}$$

Then $G(t, \overset{1}{A}, \overset{2}{B}, s)$ is defined as a regularizer of problem (8.6), (8.6′) (we shall cancel out the argument s in the sequel).

Note. The regularizer is obviously not unique. For example, we may add a function finite in $\overset{1}{A}$ to G and thus obtain a new regularizer. This is to be followed henceforth, so that the support of the new regularizer might be equal to zero in the ε-neighborhood of $x_1 = 0$ and hence we could multiply it by a polynomial in x_1^{-1} and expand it by the powers of x_1^{-1}, defining the symbol x_1^{-1} as a function $x_1^{-1}\varphi(x_1)$, where $\varphi(x_1)$ has a support outside an ε-neighborhood of the point $x_1 = 0$, $\varphi(x_1) = 1$ when $x_1 > 2\varepsilon$.

Suppose that such a regularizer is constructed. Now we shall verify, that the theorem can be proved with such a regularizer. Substituting function

$$R(x, t) = \int_0^t \int_0^{t-\tau} G\left(\mu, \overset{1}{A}, \overset{2}{B}\right) \Phi(x, \tau)\, d\mu\, d\tau \tag{8.8}$$

into equation (8.6) we obtain

$$\frac{\partial^2 R}{\partial t^2} - c^2(x) \frac{\partial^2 R}{\partial x^2} = \Phi(x, t) + \int_0^t \int_0^{t-\tau} f_1\left(\mu, \overset{1}{A}, \overset{2}{B}\right) \Phi(x, \tau)\, d\mu\, d\tau. \tag{8.8$'$}$$

Thus, we have reduced problem (8.6), (8.6′) to a problem of finding Φ from the equation

$$\Phi(x, t) + \int_0^t f_2\left(t-\tau, \overset{1}{A}, \overset{2}{B}\right) \Phi(x, \tau)\, d\tau = f(x, t), \tag{8.8$''$}$$

where

$$f_2\left(t, \overset{1}{A}, \overset{2}{B}\right) \overset{\text{def}}{=\!=} \int_0^t f_1\left(\tau, \overset{1}{A}, \overset{2}{B}\right) d\tau$$

(the initial value conditions and periodicity conditions (8.6′) are obviously fulfilled for the function $R(x, t)$).

Equation (8.8″) is an integral equation of the Volterra type and is solved by the iteration method. Indeed, we take

$$\Phi = \sum_{k=0}^{\infty} \Phi_k,$$

where

$$\Phi_0(x, t) = f(x, t),$$

$$\Phi_1(x, t) = -\int_0^t f_2\left(t-\tau, \overset{1}{A}, \overset{2}{B}\right) f(x, \tau)\, d\tau$$

. . . .

$$\Phi_k(x_1, t) = (-1)^k \int_0^t f_2\left(t-\tau, \overset{2k-1}{A}, \overset{2k}{B}\right) \int_0^{\tau_1} f_2\left(\tau_1 - \right.$$

$$\left. - \tau_2, \overset{2k-3}{A}, \overset{2k-2}{B}\right) \dots \int_0^{\tau_{k-1}} f_2\left(\tau_{k-1} - \tau_k, \overset{1}{A}, \overset{2}{B}\right) \times$$

$$\times f(x, \tau_k)\, d\tau_k \dots d\tau_1, \quad k = 2, 3, \dots .$$

We have an inequality for any $\psi(x, t) \in C^{s}_{h,t}$, $0 \leqslant \tau \leqslant t$, $t \leqslant T$, and sufficiently large s

$$\left\| f_2\left(t-\tau, \overset{1}{A}, \overset{2}{B}\right) \psi(x, \tau) \right\| = \left\| \int_0^{t-\tau} f_1\left(\mu, \overset{1}{A}, \overset{2}{B}\right) \psi(x, \tau)\, d\mu \right\| \leqslant$$

$$\leqslant \int_0^{t-\tau} \left\| f_1\left(\mu, \overset{1}{A}, \overset{2}{B}\right) \psi(x, \tau) \right\| d\mu \leqslant M_0 T \left\| \psi(x, \tau) \right\|,$$

where $M_0 =$ const, by (8.7).

Hence for $0 \leqslant t \leqslant T$ we have

$$\|\Phi_k(x, t)\| \leqslant \frac{(M_0 T^2)^k}{k!} \max_{0 \leqslant t \leqslant T} \|f(x, t)\|.$$

Hence, the series $\sum_{k=0}^{\infty} \Phi_k$ is convergent. Indeed, it follows from similar inequalities for the derivatives of Φ_k that the series of derivatives of Φ_k converges uniformly (and very rapidly at that), the sum $\Phi = \sum_{k=0}^{\infty} \Phi_k$ belongs to $C^{(s)}_{h,t}$ and the inequalities of theorem 8.1 are true for $R_0(f) \equiv R(x, t)$.

It is also obvious that equation (8.8″) and, hence, problem (8.6), (8.6′), have a unique solution.

Thus, what we need is to construct the regularizer of problem (8.6), (8.6′) (together with its estimates) to prove the theorem. We shall proceed by the familiar operational method. Find the regularizer G in the form

$$G\left(t, \overset{1}{A}, \overset{2}{x}\right) = e^{i \overset{1}{A} S\left(\overset{2}{x}, t\right)} \varphi\left(\overset{2}{x}, t, \overset{1}{A}\right), \tag{8.9}$$

where $A = -i \frac{\partial}{\partial x}$, the functions S and φ are to be found (here S, φ are 2π-periodic in x; S is real).

Substituting operator (8.9) into the wave equation and applying the commutation formula, we obtain

$$\frac{\partial^2 G}{\partial t^2} - c^2\left(\overset{3}{x}\right) \frac{\overset{2}{\partial^2}}{\partial x^2} \overset{1}{G} = e^{i \overset{1}{A} S\left(\overset{2}{x}, t\right)} \left\{ \left(-\overset{1}{A}{}^2\right) \left[\left(\frac{\partial S}{\partial t}\right)^2 - \right.\right.$$

$$\left. - c^2\left(\overset{2}{x}\right) \left(\frac{\partial S}{\partial x} + 1\right)^2 \right] \varphi + i\overset{1}{A} \left[\left(\frac{\partial^2 S}{\partial t^2}\right) \varphi - c^2\left(\overset{2}{x}\right) \left(\frac{\partial^2 S}{\partial x^2}\right) \varphi + \right.$$

$$\left. + 2 \frac{\partial S}{\partial t} \cdot \frac{\partial \varphi}{\partial t} - 2c^2\left(\overset{2}{x}\right) \left(\frac{\partial S}{\partial x} + 1\right) \frac{\partial \varphi}{\partial x} \right] +$$

$$\left. + \left[\frac{\partial^2 \varphi}{\partial t^2} - c^2\left(\overset{2}{x}\right) \frac{\partial^2 \varphi}{\partial x^2} \right] \right\}, \tag{8.10}$$

where the arguments $\left(\overset{2}{x}, t\right)$, $\left(\overset{2}{x}, t, \overset{1}{A}\right)$ of the functions S and φ are cancelled out, respectively.

We want the right-hand side of (8.10) (indeed, the expression in braces) to be equal to zero. It is impossible to have exactly zero (in the case of the variable $c(x)$), but it is sufficient to get a smooth operator f_1 on the right for the construction of the regularizer.

The regularizer G is considered as an operator in a space of 2π-periodic sufficiently smooth functions. Such functions may be put in the form of a Fourier series

$$v(x) = a_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} a_n e^{inx} \equiv a_0 + v_1(x).$$

Note, that $f\left(\overset{1}{A}, \overset{2}{B}\right) v(x) = \sum\limits_{n=-\infty}^{\infty} a_n f(n, x) e^{inx}$ by definition. Then $G\left(t, \overset{1}{A}, \overset{2}{x}\right) v = G_0(x, t) + G\left(t, \overset{1}{A}, \overset{2}{x}\right) v_1$, where $G_0(x, t) \overset{\text{def}}{=\!=} = G\left(t, \overset{1}{A}, \overset{2}{x}\right) a_0$. The function G_0 is a solution of the problem

$$\Box_c G_0 = f_1\left(t, \overset{1}{A}, \overset{2}{x}\right) a_0,$$

$$G_0(x, 0) = a_0, \frac{\partial G_0}{\partial t}(x, 0) = 0,$$

$$G_0(x + 2\pi, t) = G_0(x, t).$$

Without changing the regularizer we can obviously take the constant a_0 instead of G_0, which satisfies the written out problem, but has a zero right-hand side.

Thus it is sufficient to consider the regularizer G only for the functions of the form $v_1(x) = \sum\limits_{\substack{n=-\infty \\ n\neq 0}}^{\infty} a_n e^{inx}$. But for such functions the operator A^{-1} is defined:

$$A^{-1} v_1 \overset{\text{def}}{=\!=} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a_n}{n} e^{inx}$$

and all its powers A^{-k}, $k = 1, 2, \ldots$.

Hence the function φ in (8.9) may be written in the form

$$\varphi = \sum_{k=0}^{s} \left(-i\overset{1}{A}{}^{-1}\right)^k \varphi_k \left(\overset{2}{x}, t\right).$$

Substituting this formula into (8.10) and equating to zero all coefficients of the powers $(-iA^{-1})^k$, $k = -2, -1, 0, \ldots, s-1$ in

the braces (8.10), we obtain the system of equations

$$\begin{aligned}
&\left(\frac{\partial S}{\partial t}\right)^2 - c^2(x)\left(\frac{\partial S}{\partial x}+1\right)^2 = 0,\\
&U_1\varphi_0 = 0,\\
&U_1\varphi_1 + \Box_c\varphi_0 = 0,\\
&\dots\dots\dots\dots\dots\dots\\
&U_1\varphi_s + \Box_c\varphi_{s-1} = 0,
\end{aligned} \tag{8.11}$$

where

$$U_1 = 2\frac{\partial S}{\partial t}(x, t)\frac{\partial}{\partial t} - 2c^2(x)\left(\frac{\partial S}{\partial x}(x, t)+1\right)\frac{\partial}{\partial x} + \Box_c S(x, t).$$

The operator G satisfies the wave equation with a right-hand side (cf. (8.10)):

$$f_1\left(t, \overset{1}{A}, \overset{2}{x}\right) = e^{i\overset{1}{A}S\left(\overset{2}{x},\, t\right)}(\Box_c\varphi_s)\left(\overset{2}{x}, t\right)\left(-i\overset{1}{A}{}^{-1}\right)^s.$$

We have by definition

$$f_1\left(t, \overset{1}{A}, \overset{2}{x}\right)v_1 = (-i)^s\,\Box_c\varphi_s(x, t)\sum_{n=-\infty}^{\infty}\frac{a_n}{n^s}e^{in(x+S(x,\, t))}.$$

Hence we see that f_1 satisfies the estimates (8.7). Thus we need only to solve system (8.11) with the initial conditions

$$G\left(0, \overset{1}{A}, \overset{2}{x}\right) = 1,\quad \frac{\partial G}{\partial t}\left(0, \overset{1}{A}, \overset{2}{x}\right) = 0. \tag{8.12}$$

The first equation (8.11) is the Hamilton-Jacobi equation for the function $S(x, t)$:

$$\frac{\partial S}{\partial t} = \pm c(x)\left(\frac{\partial S}{\partial x}+1\right). \tag{8.13}$$

Substitute the initial condition

$$S(x, 0) = 0. \tag{8.13$'$}$$

Consider the corresponding system of bicharacteristics, i.e., the Hamiltonian system, for the solution of the Cauchy problem, (8.13), (8.13′)

$$\begin{aligned}
&\frac{dX^{\pm}}{dt} = \mp c(X^{\pm}),\quad X^{\pm}|_{t=0} = x_0,\\
&\frac{dP^{\pm}}{dt} = \pm\frac{\partial c}{\partial x}(X^{\pm})(1+P^{\pm}),\quad P^{\pm}|_{t=0} = 0.
\end{aligned} \tag{8.14}$$

This system is easy to integrate. Let $X^{\pm}(x_0, \tau)$, $P^{\pm}(x_0, t)$ be its solution. Since $\left.\frac{\partial X^{\pm}}{\partial x_0}\right|_{t=0} = 1$, for any t we have

$\frac{\partial X^{\pm}}{\partial x_0}(x_0, t) \neq 0$ and, therefore, the equation

$$x = X^{\pm}(x_0, t)$$

has a solution, i.e., there exists a smooth solution $x_0 = x_0^{\pm}(x, t)$.

Then problem (8.13), (8.13′) has the following 2π-periodic in x solutions (corresponding to the signs $\pm$ in (8.13)):

$$S_{\pm}(x, t) = \int_0^t \Big[P^{\pm}(x_0, \tau)\frac{dX^{\pm}}{d\tau}(x_0, \tau) - H_{\pm}(P^{\pm}(x_0, \tau),$$

$$X^{\pm}(x_0, \tau))\Big]\, d\tau \,\Big|_{x_0 = x_0^{\pm}(x,\, t)} = x_0^{\pm}(x, t) - x.$$

Note that $\frac{\partial S_{\pm}}{\partial x}(X^{\pm}, t) = P^{\pm}$.

We now turn to the second equation in (8.11). The operator U_1 splits into two operators: U_1^+, U_1^- corresponding to two functions $S_{\pm}$. The operator $U_1^{\pm}$ may obviously be transformed into the form

$$U_1^{\pm} = -2H_{\pm}\left(\frac{d}{dt_{\pm}} + \frac{1}{2}\frac{\partial^2 H_{\pm}}{\partial p^2}\frac{\partial^2 S_{\pm}}{\partial x^2} - \frac{1}{2H_{\pm}}\frac{\partial H_{\pm}}{\partial p}\frac{\partial H_{\pm}}{\partial x}\right), \tag{8.15}$$

where $H_{\pm}(x, p) \overset{\text{def}}{=\!=} \mp c(x)(1+p)$, the variables $(X^{\pm}, t)$ $(X^{\pm}, P^{\pm})$ are cancelled out in the functions $S_{\pm}$, $H_{\pm}$, respectively, $\frac{d}{dt_{\pm}}$ is a derivative along the trajectories (8.14), i.e.,

$$\frac{d}{dt_{\pm}} = \frac{\partial}{\partial t} + \frac{dX^{\pm}}{dt}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} \mp c(X^{\pm})\frac{\partial}{\partial x}.$$

By (8.14)

$$\frac{\partial}{\partial t}[H_{\pm}(X_{\pm}(x_0, t),\ P_{\pm}(x_0, t))] = 0. \tag{8.16}$$

Hence, $H_{\pm}(X_{\pm}, P_{\pm}) = H_{\pm}(x_0, 0) = \mp c(x_0)$.

Let $J_{\pm}^0(x_0, t) = \frac{\partial X^{\pm}(x_0, t)}{\partial x_0}$. Differentiating the equation for $X^{\pm}$ (see (8.14))

$$\frac{dX^{\pm}}{dt} = \frac{\partial H^{\pm}}{\partial p}\left(X^{\pm}, \frac{\partial S_{\pm}}{\partial x}(X^{\pm}, t)\right)$$

with respect to x_0, we get

$$\frac{\partial}{\partial t}J_{\pm}^0 = J_{\pm}^0\left(\frac{\partial^2 H_{\pm}}{\partial p^2}\frac{\partial^2 S_{\pm}}{\partial x^2} + \frac{\partial^2 H_{\pm}}{\partial p\,\partial x}\right).$$

On comparing this formula with (8.16) for the Jacobian

$$J_{\pm}(x, t) \overset{\text{def}}{=\!=} 2H_{\pm}\left(x, \frac{\partial S_{\pm}}{\partial x}(x, t)\right) J_{\pm}^0(x_0^{\pm}(x, t), t)$$

we obtain an equation

$$\frac{d}{dt_\pm}\frac{1}{\sqrt{|J_\pm|}}=-\frac{1}{2\sqrt{|J_\pm|}}\left(\frac{\partial^2 H_\pm}{\partial p^2}\frac{\partial^2 S_\pm}{\partial x^2}+\frac{\partial^2 H_\pm}{\partial x\,\partial p}\right).$$

Hence, for the operator (8.15) we have

$$U_1^\pm\left(\frac{\psi(x,t)}{\sqrt{|J_\pm|}}\right)=$$

$$=-\frac{1}{\sqrt{|J_\pm|}}\left(\frac{d}{dt_\pm}-\frac{1}{2}\frac{\partial^2 H_\pm}{\partial x\,\partial p}-\frac{1}{2H_\pm}\frac{\partial H_\pm}{\partial p}\frac{\partial H_\pm}{\partial x}\right)\psi(x,t), \tag{8.17}$$

where the variables $\left(x, \frac{\partial S_\pm}{\partial x}(x, t)\right)$ of the function $H_\pm$ are cancelled out (ψ is any).

We have made all calculations making use of the most general notation, since they will be repeated in the sequel. In our case formula (8.17) for the operator $U_1^\pm$ is of the form:

$$U_1^\pm\left(\frac{\psi(x,t)}{\sqrt{|J_\pm|}}\right)=-\frac{1}{\sqrt{|J_\pm|}}\left[\frac{d}{dt_\pm}\pm\frac{\partial c(x)}{\partial x}\right]\psi(x,t).$$

Therefore the second equation of (8.11) has two solutions

$$\varphi_0^\pm(x,t)=\frac{\psi_0^\pm(x_0^\pm(x,t))\,c(x)}{\sqrt{|J_\pm(x,t)|}}, \tag{8.18}$$

where $\psi_0^\pm(x)$ are some initial conditions which ought to be defined.

In the same way all other equations of (8.11) are solved by (8.17) and 2π-periodic functions $\varphi_k^\pm$, $k = 0, 1, 2, \ldots, s$ are defined by some integrals along the trajectories $X^\pm$, $P^\pm$.

Finally the regularizer G is of the form

$$G\left(t, \overset{1}{A}, \overset{2}{x}\right)=\sum_{\pm}\sum_{k=0}^{s} e^{i\overset{1}{A}S_\pm\left(\overset{2}{x},\, t\right)}\varphi_k^\pm\left(\overset{2}{x}, t\right)\left(-i\overset{1}{A}{}^{-1}\right)^k, \tag{8.19}$$

where the functions $S_\pm$, $\varphi_k^\pm$ have been defined above and

$$\sum f_\pm \overset{\text{def}}{=\!=} f_+ + f_-.$$

Now we need only to choose the initial conditions for $\varphi_k^\pm$ such that (8.12) are verified. Take

$$\varphi_0^\pm(x, 0)=1/2. \tag{8.20}$$

Since $J_\pm(x, 0) = 2H_\pm(x, 0) = \mp 2c(x)$, we have the following equation for $\psi_0^\pm$:

$$\psi_0^\pm(x)=\frac{1}{\sqrt{2c(x)}}.$$

Take the conditions

$$\varphi_k^{\pm}(x, 0) = 0, \quad k = 1, 2, \ldots, s \tag{8.21}$$

for all other functions $\varphi_k^{\pm}$, $k \geqslant 1$.

By (8.19) at the initial moment of time $t = 0$, and, taking into account (8.20), we have

$$G\left(0, \overset{1}{A}, \overset{2}{x}\right) = \sum_{\pm} \sum_{k=0}^{s} \varphi_k^{\pm}\left(\overset{2}{x}, 0\right)\left(-i\overset{1}{A}{}^{-1}\right)^k =$$

$$= \sum_{\pm} \varphi_0^{\pm}(x, 0) = \frac{1}{2} + \frac{1}{2} = 1.$$

The derivative $\partial G/\partial t$ is of the form (by (8.19))

$$\frac{\partial G}{\partial t}\left(0, \overset{1}{A}, \overset{2}{x}\right) = \sum_{\pm} \sum_{k=0}^{s} \left[i\overset{1}{A} \frac{\partial S_{\pm}}{\partial t}\left(\overset{2}{x}, 0\right) \varphi_k^{\pm}\left(\overset{2}{x}, 0\right) + \right.$$

$$\left. + \frac{\partial \varphi_k^{\pm}}{\partial t}\left(\overset{2}{x}, 0\right) \right] \left(-i\overset{1}{A}{}^{-1}\right)^k. \tag{8.22}$$

The first terms in (8.22) are cancelled out by (8.20) and the conditions $\frac{\partial S_{\pm}}{\partial t}(x, 0) = -H_{\pm}(x, 0) = \pm c(x)$ and we have

$$\frac{\partial G}{\partial t}\left(0, \overset{1}{A}, \overset{2}{x}\right) = \sum_{\pm} \sum_{k=0}^{s} \frac{\partial \varphi_k^{\pm}}{\partial t}\left(\overset{2}{x}, 0\right)\left(-i\overset{1}{A}{}^{-1}\right)^k.$$

We find the derivatives $\frac{\partial \varphi_k^{\pm}}{\partial t}$ by (8.11). It is easy to see that they satisfy the condition

$$\frac{\partial \varphi_k^{+}}{\partial t}(x, 0) = -\frac{\partial \varphi_k^{-}}{\partial t}(x, 0)$$

at the initial moment. Therefore, the operator $G\left(t, \overset{1}{A}, \overset{2}{x}\right)$ satisfies the second initial condition of (8.12) as well. Thus the regularizer for (8.6), (8.6′) is constructed, Q.E.D.

(3) Asymptotic solutions of the system of lattice oscillations. We shall distinguish between the two problems: (1) finding an asymptotics of an exact solution; (2) finding an asymptotic solution, i.e., the construction of a function which, substituted into the right-hand side, is equal asymptotically to the given one.

We shall study problem (1), after we have constructed an asymptotic solution (i.e., after problem (2)). There is the following almost obvious lemma.

Lemma 8.2. *Let $m > 1$, $k \geqslant 2$ be integer numbers and $f \in C^{(p)}_{h, t}$ be a function, where $p = p(m, k)$ is sufficiently great. Then, there exist such functions $\psi_j \in C^{(k)}_{h, t}$, $j = 0, \ldots, m-2$ that a linear combination*

$$U_m = \frac{1}{h^2} R_0(f) + \sum_{j=0}^{m-2} h^j \psi_j$$

is an asymptotic solution of problem (8.5), *i.e.*,

$$\left[h^2 \frac{\partial^2}{\partial t^2} + 4c^2 \left(\overset{2}{x}\right) \sin^2 \left(-\frac{ih}{2} \overset{1}{\frac{\partial}{\partial x}}\right)\right] U_m = f + h^m F_m f,$$

$$U_m(x + 2\pi, t) = U_m(x, t),$$

$$U_m(x, 0) = \frac{\partial U_m}{\partial t}(x, 0) = 0,$$

and $F_m f$ satisfies an estimate

$$\| F_m f \| \leqslant \sum_{j=1}^{p} c_j \left\| \frac{\partial^j f}{\partial x^j} \right\|.$$

Proof. Substitute the function U_m into the equation and apply Lemma 8.1. On equating to zero the coefficients of the powers of h we obtain equations for the functions ψ_j, $j = 0, \ldots, m-2$. For example, assuming that the coefficient of h^2 equals zero, we obtain an equation for ψ_0:

$$\Box_c \psi_0 + \alpha_4 c^2(x) \left(-i \frac{\partial}{\partial x}\right)^4 R_0(f) = 0.$$

Hence we find by Theorem 8.1

$$\psi_0 = -\alpha_4 R_0 \left[c^2 \left(-i \frac{\partial}{\partial x}\right)^4 R_0(f)\right],$$

and the function ψ_0 satisfies the initial conditions with the right-hand sides equal to zero and is 2π-periodic in x.

In the same way we get all other functions ψ_j. The estimate of $F_m f$ is obtained from the corresponding estimates of Lemma 8.1 and Theorem 8.1. The lemma is proved.

Thus we see that the solution of the wave equation is an asymptotic solution of the system of equations of lattice oscillations. But, firstly, it is an asymptotic solution for a sufficiently smooth right-hand side, and, secondly, it is necessary to prove that it is an asymptotics of the sought-for solution. We shall apply the method of the regularizer, studied before in connection with the wave equation, to find a general asymptotic solution of the equation of lattice oscillations and to prove that it is an asymptotics of the sought solution.

In this case the regularizer is a usual operator $G\left(t, \overset{1}{A}, \overset{2}{B}\right)$ depending also on h, the definition is the same as before but in addition to (8.7) we demand that

$$| f_1(x_1, \alpha, t, s, h) | \leqslant c_s h^{s+2} (x_1^2 + 1)^{-s/2}. \tag{8.23}$$

We put the operator

$$h^2 \frac{\partial^2}{\partial t^2} + [\![4c^2 \left(\overset{2}{x}\right) \sin^2 \left(-\frac{ih}{2} \overset{1}{\frac{\partial}{\partial x}} \right)]\!]$$

into the right-hand side of (8.7′).

If the regularizer is constructed, we may obtain an integral equation of the Volterra type similarly to item 2, and prove that its solution exists and obtain as well an estimate for the solution uniform in h as $h \to 0$.

Note that it is sufficient to demand only condition (8.7) for f_1.

Indeed, if such a weakened regularizer is constructed, we can apply the results of this item, i.e. Lemma 8.2, which enables us to obtain a necessary regularizer correct to $h^N F$, where $F \in C_{h,t}^{(s)}$, i.e. to satisfy (8.23).

Thus we have reduced the problem to the construction of such a weakened regularizer, that the corresponding right-hand side f_1 satisfies condition (8.7). We shall fulfil this task in the present item with the help of ordered operators.

Construct the regularizer for the system of equations of lattice oscillations. Consider problem (8.4) for any arbitrary initial conditions $v_1, v_2 \in M_N$. It is obviously sufficient to consider the case $v_2(x) \equiv 0$. Next, put the function v_1 in the form

$$v_1(x) = a_0 + \sum_{n=-N+1}^{N} a_n e^{inx}.$$

The constant a_0 obviously satisfies problem (8.2). Therefore it is sufficient to consider the initial conditions

$$u(x, 0) = v_0(x) \equiv \sum_{n=\pm 1, \ldots, \pm(N-1), +N} a_n e^{inx}; \quad \frac{\partial u}{\partial t}(x, 0) = 0. \tag{8.24}$$

Let $A = -i\frac{\partial}{\partial x}$. Define the operator A^{-1} (as it was done before) for the function v_0 by the formula

$$A^{-1} v_0 \equiv A^{-1} \left(\sum_{n \neq 0} a_n e^{inx} \right) \overset{\text{def}}{=\!=} \sum_{n \neq 0} \frac{a_n}{n} e^{inx} \in M_N.$$

The action of A^{-1} transforms functions of this type into functions of the same type, therefore all the powers A^{-k}, $k = 1, 2, \ldots$ are defined for v_0. Besides, an estimate

$$\| A^{-1} v_0 \| \leqslant \| v_0 \|$$

is obviously true. Then $\frac{\partial}{\partial x}(A^{-1}v_0) = iv_0(x, h)$. Therefore

$$\left\| \frac{\partial^s}{\partial x^s}(A^{-k}v_0) \right\| \leqslant C_s \| v_0 \|, \quad s = 0, 1, \ldots, k.$$

Hence, we see that it is sufficient to solve problem (8.4) correct within functions of the type $(A^{-k}v_0)$.

We shall find the regularizer in the form (h is cancelled out)

$$G\left(t, \overset{1}{\omega}, \overset{2}{x}\right) v_0 = e^{i\overset{1}{A}S\left(\overset{2}{x}, t, \overset{1}{\omega}\right)} \varphi\left(\overset{2}{x}, t, \overset{1}{\omega}\right) v_0, \tag{8.25}$$

where $\omega = hA$, $h = \pi/N$, S, φ are infinitely differentiable 2π-periodic in x functions, S is a real function.

It is to be noted that an operator of the form $F\left(\overset{2}{x}, \overset{1}{\omega}\right)$ for a function $v \in M_N$ is defined by the formula

$$F\left(\overset{2}{x}, \overset{1}{\omega}\right) v = \frac{1}{2\pi} \sum_{k=-N+1}^{N} e^{ikx} F(x, kh) \int_{-\pi}^{+\pi} e^{-iky} v(y)\, dy. \tag{8.26}$$

The operator ω is obviously bounded on M_N

$$\| \omega v \| \leqslant \pi \| v \|, \qquad v \in M_N. \tag{8.27}$$

Substitute formula (8.25) into equation (8.4) and make use of the formula of commutation with exponential. Then we obtain that $G\left(t, \overset{1}{\omega}, \overset{2}{x}\right)$ satisfies equation (8.4), if the equation

$$\left\{ h^2 \frac{\partial^2 \varphi}{\partial t^2} - \overset{1}{\omega}{}^2 \left(\frac{\partial S}{\partial t}\right)^2 \varphi + 2i\overset{1}{\omega} h \frac{\partial S}{\partial t} \frac{\partial \varphi}{\partial t} + i\overset{1}{\omega} h \frac{\partial^2 S}{\partial t^2} \varphi + \right.$$
$$\left. + 4 \overset{4}{[\![} c^2(x)]\!] \sin^2 \overset{3}{[\![} - \frac{ih}{2} \frac{\partial}{\partial x} + \frac{1}{2} \overset{1'}{\omega} \frac{\partial S}{\partial x}\left(x, t, \overset{1'}{\omega}\right)]\!] \varphi \right\} v_0 = 0, \tag{8.28}$$

where the arguments $\left(\overset{2}{x}, t, \overset{1}{\omega}\right)$ of S, φ are cancelled out, is true. The last term can be transformed by the K-formula and the commutation formula

$$\sin^2 \overset{3}{[\![} - \frac{ih}{2} \frac{\partial}{\partial x} + \frac{1}{2} \overset{1'}{\omega} \frac{\partial S}{\partial x}\left(x, t, \overset{1'}{\omega}\right)]\!] \varphi\left(\overset{2}{x}, t, \overset{1}{\omega}\right) =$$
$$= \sin^2 \left(\frac{\overset{1}{\omega}}{2}\left(1 + \frac{\partial S}{\partial x}\right)\right) \varphi + \frac{1}{2}(-ih)\left[\sin\left(\overset{1}{\omega}\left(1 + \frac{\partial S}{\partial x}\right)\right) \frac{\partial \varphi}{\partial x} + \right.$$
$$\left. + \frac{1}{2} \overset{1}{\omega} \cos\left(\overset{1}{\omega}\left(1 + \frac{\partial S}{\partial x}\right)\right) \varphi \right] + \sum_{k=2}^{m-1} (-ih)^k \times$$
$$\times (R_k \varphi)\left(\overset{2}{x}, t, \overset{1}{\omega}\right) + h^m Q_m(h), \tag{8.29}$$

where the arguments $\left(\overset{2}{x},\ t,\ \overset{1}{\omega}\right)$ of the functions S, φ are omitted.

Here R_k are some differential operators of order k acting on the function $\varphi(x, t, \omega)$ in the argument x (it is possible to write out all operators R_k explicitly). The remainder Q_m and all its commutators $I_s = \overset{2}{Q}_m \left(\overset{1}{\frac{\partial}{\partial x}} - \overset{3}{\frac{\partial}{\partial x}}\right)^s$, $s = 0, 1, 2, \ldots$ satisfy by (8.27) the estimates

$$\| I_s v \| \leqslant c_s \| v \|, \qquad v \in M_N. \tag{8.30}$$

Note, that $hv_0 = A^{-1}\omega v_0$. Hence by (8.28), (8.29) we obtain the equation

$$\begin{aligned}
&\Big\{\Big[\Big(\overset{1}{\omega}\frac{\partial S}{\partial t}\Big)^2 - 4\,\overset{2}{[\![c^2(x)]\!]}\sin^2\Big(\frac{\overset{1}{\omega}}{2}\Big(1+\frac{\partial S}{\partial x}\Big)\Big)\Big]\varphi + \\
&\quad + \Big(-i\overset{1}{A^{-1}}\Big)\Big[2\overset{1}{\omega^2}\frac{\partial S}{\partial t}\frac{\partial \varphi}{\partial t} - 2\overset{1}{\omega}\overset{2}{[\![c^2(x)]\!]}\times \\
&\quad \times \sin\Big(\overset{1}{\omega}\Big(1+\frac{\partial S}{\partial x}\Big)\Big)\frac{\partial\varphi}{\partial x} + \\
&\quad + \overset{1}{\omega^2}\Big(\frac{\partial^2 S}{\partial t^2} - \overset{2}{[\![c^2(x)]\!]}\cos\Big(\overset{1}{\omega}\Big(1+\frac{\partial S}{\partial x}\Big)\Big)\frac{\partial^2 S}{\partial x^2}\Big)\varphi\Big] - \\
&\quad - \sum_{k=2}^{m-1}\Big(-i\overset{1}{A^{-1}}\Big)^k \overset{1}{\omega^k}(R_k\varphi)\Big(\overset{2}{x},\ t,\ \overset{1}{\omega}\Big)\overset{2}{[\![c^2(x)]\!]} - \\
&\quad - 4\overset{3}{[\![c^2(x)]\!]}\,\overset{1}{A^{-m}}\overset{1}{\omega^w}\overset{1}{Q}_m\Big\}\, v_0 = 0,
\end{aligned} \tag{8.31}$$

where the arguments $\left(\overset{2}{x},\ t,\ \overset{1}{\omega}\right)$ of the functions φ, S are omitted.

We shall solve (8.31) by expanding φ in powers of A^{-1}, i.e.,

$$\varphi = \sum_{k=0}^{m-2}\Big(-i\overset{1}{A^{-1}}\Big)^k \varphi_k\Big(\overset{2}{x},\ t,\ \overset{1}{\omega}\Big).$$

Substituting this formula into (8.31) and equating to zero all coefficients of the powers $(-iA^{-1})^k$, $k = 0, \ldots, m-1$, we obtain a system of equations

$$\begin{aligned}
&\Big(\omega\frac{\partial S}{\partial t}\Big)^2 - 4C^2(x)\sin^2\frac{\omega}{2}\Big(1+\frac{\partial S}{\partial x}\Big) = 0,\\
&R_1'\varphi_0 = 0,\\
&R_1'\varphi_1 + R_2'\varphi_0 = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&R_1'\varphi_k + R_2'\varphi_{k-1} + \ldots + R_{k+1}'\varphi_0 = 0, \qquad k = 1, 2, \ldots, m-2,
\end{aligned} \tag{8.32}$$

where

$$R_1' = 2\frac{\partial S}{\partial t}\frac{\partial}{\partial t} - 2\frac{C^2(x)}{\omega}\sin\left(\omega\left(1+\frac{\partial S}{\partial x}\right)\right)\frac{\partial}{\partial x} + \\ + \left(\frac{\partial^2 S}{\partial t^2} - C^2(x)\cos\left(\omega\left(1+\frac{\partial S}{\partial x}\right)\right)\frac{\partial^2 S}{\partial x^2}\right),$$

and the operators R_k', $k = 2, \ldots, m-1$ are given by the following formulas:

$$R_k' = -4\omega^{k-2}c^2(x)R_k.$$

Assume that S and φ_k are found from equation (8.32). Then we see that the function

$$V_m(x, t) = \sum_{k=0}^{m-2} e^{i\overset{1}{A}S\left(\overset{2}{x}, t, \overset{1}{\omega}\right)}\varphi_k\left(\overset{2}{x}, t, \overset{1}{\omega}\right)\left(-i\overset{1}{A}{}^{-1}\right)^k v_0 + a_0$$

satisfies equation (8.4) correct within the function

$$K_m(x, t) = e^{i\overset{1}{A}S\left(\overset{3}{x}, t, \overset{1}{\omega}\right)}\left[h^2\overset{2}{Q_m'}\overset{1}{A}{}^{-m+2}\right]v_0, \tag{8.33}$$

where the operator Q_m' is given by the following formula

$$Q_m' = 4\overset{3}{[\![c^2(x)]\!]}\left\{\sum_{j=2}^{m-1}\sum_{l=0}^{j-2}(-i)^{m-2}\overset{1}{\omega}{}^{j-2}\times\right. \\ \left.\times(R_j\varphi_{m-j+l})\left(\overset{2}{x}, t, \overset{1}{\omega}\right)\left(-i\overset{1}{A}{}^{-1}\right)^l + \overset{2}{Q}_m\overset{1}{\omega}{}^{m-2}\right\}. \tag{8.34}$$

We have, as well,

$$\left(-i\frac{\partial}{\partial x}\right)K_m(x, t) = \\ = \left\{\frac{\partial S}{\partial x}\left(\overset{3}{x}, t, \overset{1}{\omega}\right)e^{i\overset{1}{A}S\left(\overset{3}{x}, t, \overset{1}{\omega}\right)}h^2\overset{2}{Q_m'}\overset{1}{A}{}^{-m+3} - \right. \\ - e^{i\overset{1}{A}S\left(\overset{3}{x}, t, \overset{1}{\omega}\right)}h^2\left[Q_m', \overset{2}{-i}\frac{\partial}{\partial x}\right]^1 A^{-m+2} + \\ \left. + e^{i\overset{1}{A}S\left(\overset{3}{x}, t, \overset{1}{\omega}\right)}h^2\overset{2}{Q_m'}\overset{1}{A}{}^{-m+3}\right\}v_0.$$

We have from (8.34) and (8.30)

$$\left\|-i\frac{\partial}{\partial x}K_m(x, t)\right\| \leqslant c_1\|v_0\|,$$

where $c_1 = \text{const}$.

The estimates for all other derivatives are obtained similarly:

$$\left\|\frac{\partial^s}{\partial x^s}K_m(x, t)\right\| \leqslant c_s\|v_0\|, \quad s = 0, 1, 2, \ldots, m-2.$$

Therefore, if $\sup\limits_{0<h<1} \| v_1 \| \leqslant c$, then the function V_m satisfies equation (8.4) correct within an element from $C_{h,\, t}^{(m-2)}$:

$$\left[h^2 \frac{\partial^2}{\partial t^2} + 4c^2(x) \sin^2 \left(-\frac{ih}{2} \frac{\partial}{\partial x} \right) \right] V_m = K_m \in C_{h,\, t}^{(m-2)}.$$

We turn now to the solution of equation (8.32). Firstly, we have the equation of Hamilton-Jacobi for the function $S(x, t, \omega)$:

$$\frac{\partial S}{\partial t} = \pm 2 \frac{c(x)}{\omega} \sin \frac{\omega}{2} \left(1 + \frac{\partial S}{\partial x} \right). \tag{8.35}$$

The initial condition is

$$S(x, 0, \omega) = 0. \tag{8.35'}$$

Consider the corresponding system of characteristics for the solution of the Cauchy problem

$$\begin{cases} \dfrac{dX^\pm}{dt} = \mp c(X^\pm) \cos \left[\dfrac{\omega}{2} (1 + P^\pm) \right], & X^\pm |_{t=0} = x_0, \\ \dfrac{dP^\pm}{dt} = \pm \dfrac{2}{\omega} \sin \left[\dfrac{\omega}{2} (1 + P^\pm) \right] \dfrac{\partial c}{\partial x} (X^\pm), & P^\pm |_{t=0} = 0. \end{cases} \tag{8.36}$$

The system is easy to integrate. Let $X^\pm(x_0, \omega, t)$, $P^\pm(x_0, \omega, t)$ be its solution (the spectrum $\sigma(\omega) \in [-\pi, \pi]$). Since $\left. \frac{\partial X^\pm}{\partial x_0} \right|_{t=0} = 1$ the equation

$$x = X^\pm(x_0, \omega, t) \tag{8.36'}$$

can be solved for sufficiently small t, i.e., there exists a sufficiently smooth solution $x_0 = x_0^\pm(x, t, \omega)$. Then problem (8.35)-(8.35′) has the following 2π-periodic in x solution

$$\begin{aligned} S_\pm(x, t, \omega) = \int_0^t \Big\{ & P^\pm(x_0, \omega, \tau) \frac{dX^\pm}{d\tau}(x_0, \omega, \tau) \pm \\ & \pm 2 \frac{c(X^\pm(x_0, \omega, \tau))}{\omega} \sin \left[\frac{\omega}{2} (1 + P^\pm(x_0, \omega, \tau)) \right] \Big\} \times \\ & \times d\tau \Big|_{x_0 = x_0^\pm(x, t, \omega)}. \end{aligned}$$

Equations (8.32) are further solved just in the same way as in item 2 for the wave equation.

Thus the weakened regularizer of problem (8.4) is found in the form

$$G\left(t, \overset{1}{\omega}, \overset{2}{x}\right) v_0 = \sum_\pm \sum_{k=0}^{m-2} e^{\overset{1}{i} A S_\pm \left(\overset{2}{x}, t, \overset{1}{\omega}\right)} \varphi_k^\pm \left(\overset{2}{x}, t, \overset{1}{\omega}\right) \left(-i \overset{1}{A^{-1}} \right)^k v_0. \tag{8.37}$$

Now, considering the note made at the end of the last item, we obtain the following theorem.

Theorem 8.2. *For sufficiently small t there exists a unique solution of problem* (8.4) *which can be put into the form*

$$u(x, t) = G\left(t, \overset{1}{A}, \overset{2}{B}\right) v_0 - h^{-2} R_0 (K_m) + O(h),$$

by condition (8.24).

We shall indicate a few simple corollaries. For this purpose we must put formula (8.37) into the form of an integral, using a pseudo-differential operator on a circle, and to study an asymptotics for $h \to 0$ of the integral by the method of stationary phase (a simplified version of the saddle-point method for the real case).

First introduce a new version of the proof of the stationary phase method.

(4) The stationary phase method. Consider an integral

$$I(h) = \frac{1}{\sqrt{2\pi h}} \int e^{i\frac{\Phi(y)}{h}} \varphi(y)\, dy,$$

where $\Phi(y) \in C^\infty(\mathbf{R})$, $\varphi(y) \in C_0^\infty(\mathbf{R})$, $\frac{\partial^2 \Phi}{\partial y^2} \neq 0$ for $y \in \operatorname{supp} \varphi$ and the equation $\frac{\partial \Phi}{\partial y} = 0$ has a unique solution $y = y_0$ for $y \in \operatorname{supp} \varphi(y)$, i.e. a stationary point.

Lemma 8.3. *Under the above conditions for any $N \geqslant 1$ there is an expansion*

$$I(h) = e^{i\frac{\pi}{4}} e^{i\frac{\pi}{4}(1-\operatorname{sgn} \Phi''(y))} \frac{e^{\frac{i}{h}\Phi(y_0)}}{\sqrt{|\Phi''(y_0)|}} \left(\varphi(y_0) + \sum_{k=1}^{N} h^k \psi_k(y_0)\right) + O(h^{N+1}), \quad (8.38)$$

where

$$\psi_k(y_0) = \int_0^{\frac{\pi}{2}\operatorname{sgn} \Phi''(y_0)} \sqrt{J(y, t_1)} \frac{\partial^2}{\partial x^2} \frac{1}{\sqrt{J(y, t_1)}} \cdots$$

$$\cdots \int_0^{t_{k-1}} \sqrt{J(y, t_k)} \frac{\partial^2}{\partial x^2} \frac{1}{\sqrt{J(y, t_k)}} \varphi(y)\, dt_1 \ldots dt_k \big|_{x=0}$$

where $y = y(x, t)$ is a solution of equation (8.49) *and the Jacobian*

$$J(x, t) \overset{\text{def}}{=} \tilde{J}(y(x, t), t)$$

is defined by formula (8.48).

For the proof we shall apply the asymptotics for $h \to 0$ of the Cauchy problem *for the Schrödinger equation of the oscillator*

$$-ih\frac{\partial\psi}{\partial t} = -\frac{h^2}{2}\frac{\partial^2\psi}{\partial x^2} + \frac{x^2}{2}\psi, \quad \psi|_{t=0} = \varphi(x)\, e^{\frac{i}{h}\Phi(x)}. \tag{8.39}$$

This asymptotics is called *quasiclassical*. On substituting the following function into (8.39)

$$\psi(x, t) = \frac{e^{-i\frac{\pi}{4}} e^{-i\frac{\pi}{4}(1-\operatorname{sgn} t)}}{(2\pi h\,|\sin t\,|)^{1/2}} \times$$
$$\times \int_{-\infty}^{+\infty} e^{\frac{i}{2h\sin t}[\cos tx^2 - 2xy + \cos ty^2]}\, e^{\frac{i}{h}\Phi(y)}\, \varphi(y)\, dy \tag{8.40}$$

we see that it is the solution of Problem (8.39). On taking $x = 0$, $t = +\frac{\pi}{2}$ in formula (8.40), we obtain

$$I(h) = \psi\left(0, \frac{\pi}{2}\right) e^{\frac{+i\pi}{4}}, \tag{8.41}$$

On the other hand, taking $x = 0$, $t = -\frac{\pi}{2}$ in (8.40), we get

$$I(h) = e^{i\frac{3\pi}{4}}\psi(0, -\pi/2). \tag{8.42}$$

Thus, by virtue of (8.41), (8.42) the expansion of the function $I(h)$ in a series happens to be a quasiclassical asymptotics of the solution of the Cauchy problem (8.39). We shall find a solution of (8.39) in the form of an asymptotic series

$$\psi(x, t, h) = e^{\frac{i}{h}S(x,t)}[\varphi_0(x, t) + h\varphi_1(x, t) + \ldots$$
$$\ldots + h^N\varphi_N(x, t) + \ldots] \overset{\text{def}}{=\!=} e^{\frac{i}{h}S(x,t)}\varphi(x, t, h), \tag{8.43}$$

where $S(x, t) \in C^\infty$, $\varphi_i(x, t) \in C^\infty$ are the sought functions. On substituting the function $\psi(x, t, h)$ defined by equation (8.43) into equation (8.39), we obtain

$$\left(\frac{\partial S}{\partial t} - \frac{\left(\frac{\partial S}{\partial x}\right)^2 + x^2}{2}\right)\varphi +$$
$$+ ih\left(\frac{\partial\varphi}{\partial t} + \frac{\partial S}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{1}{2}\frac{\partial^2 S}{\partial x^2}\varphi\right) - h^2\frac{\partial^2\varphi}{\partial x^2} = 0. \tag{8.44}$$

On equating to zero the term not containing h, we obtain the *Hamilton-Jacobi equation of the oscillator*

$$\frac{\partial S}{\partial t} = \frac{1}{2}\left[\left(\frac{\partial S}{\partial x}\right)^2 + x^2\right]. \tag{8.45}$$

Let $S(x, 0) = \Phi(x)$.

Consider *the Hamiltonian system of the oscillator*

$$\dot{q} = p, \qquad q(0) = y,$$
$$\dot{p} = -q, \qquad p(0) = \frac{\partial \Phi}{\partial y}(y). \tag{8.46}$$

Its solutions are obviously the functions

$$q(t, y) = y \cos t + \frac{\partial \Phi}{\partial y} \sin t; \quad p(t, y) = \frac{\partial \Phi}{\partial y} \cos t - y \sin t. \tag{8.47}$$

The functions q and p are called *trajectories of the Hamiltonian system* (8.46). Denote by

$$\tilde{J}(y, t) \stackrel{\text{def}}{=\!=} \frac{\partial q}{\partial y}(t, y).$$

By (8.47) we obtain

$$\tilde{J}(y, t) = \cos t + \frac{\partial^2 \Phi}{\partial y^2} \sin t. \tag{8.48}$$

It follows from this formula that

(1) if $\frac{\partial^2 \Phi}{\partial y^2} > 0$ for $y \in \operatorname{supp} \varphi(y)$, then

$$\tilde{J}(y, t) \neq 0 \text{ when } y \in \operatorname{supp} \varphi(y), \ 0 \leqslant t \leqslant \frac{\pi}{2}.$$

(2) if $\frac{\partial^2 \Phi}{\partial y^2} < 0$ for $y \in \operatorname{supp} \varphi(y)$, then

$$\tilde{J}(y, t) \neq 0 \text{ when } y \in \operatorname{supp} \varphi(y), \ -\frac{\pi}{2} \leqslant t \leqslant 0.$$

Assume in the sequel that $\frac{\partial^2 \Phi}{\partial y^2} > 0$. All the arguments are the same in the case $\frac{\partial^2 \Phi}{\partial y^2} < 0$, if t is changed for $(-t)$.

Denote by $y = y(x, t)$ a solution of the equation

$$q(t, y) = x, \tag{8.49}$$

which exists by the theorem of the implicit function. By analogy with item 2 we obtain, that a solution of equation (8.45) is given by the formula:

$$S(x, t) = \Phi(y(x, t)) + \frac{1}{2} \int_0^t [p^2(\tau, y) - q^2(\tau, y)] \, d\tau \,|_{y=y(x, t)}. \tag{8.50}$$

Consider the remaining terms in equation (8.43). Introduce a vector field

$$\frac{\partial}{\partial t} + p(t, y(x, t)) \frac{\partial}{\partial x} \stackrel{\text{def}}{=\!=} \frac{d}{d\tau}.$$

We want to calculate the derivative $\frac{d}{d\tau}\widetilde{J}(y(x, t), t)$. Note first, that the vector field $d/d\tau$ corresponds to the differentiation along the projection on the plane (x, t) of the trajectories of the Hamiltonian system. We have

$$\frac{d}{d\tau}\widetilde{J}(y(x, t), t)\,|_{t=t_0}=\lim_{\Delta t\to 0}\frac{\widetilde{J}(y(x, t_0), t_0+\Delta t)-\widetilde{J}(y(x, t_0), t_0)}{\Delta t}=$$

$$=\widetilde{J}(y(x, t_0), t_0)\lim_{\Delta t\to 0}\frac{\widetilde{J}(x, \Delta t)-1}{\Delta t},$$

where $\widetilde{J}(x, \Delta t)$ corresponds to the Jacobian of the shift transformation along the trajectories of the following Hamiltonian system

$$\dot{\widetilde{q}}=\widetilde{p}, \qquad \widetilde{q}(0)=x \overset{\text{def}}{=\!=} q_0,$$

$$\dot{\widetilde{p}}=-\widetilde{q}, \quad \widetilde{p}(0)=p(t_0, y(x, t_0)) \overset{\text{def}}{=\!=} p_0.$$

In order to calculate $\widetilde{J}(x, \Delta t) \overset{\text{def}}{=\!=} J_{\Delta t}$, we note first that the Hamiltonian system provides the equation

$$\widetilde{q}(\Delta t)=x+p_0\cdot\Delta t+O(\Delta t^2)$$

and

$$J_{\Delta t}=\frac{\partial\widetilde{q}(\Delta t)}{\partial x}=1+\frac{\partial p_0}{\partial x}\Delta t+O(\Delta t)^2.$$

It is easy to see that the function $S(x, t)$ defined by (8.50) satisfies the condition $\frac{\partial S}{\partial x}=p$. Finally,

$$\frac{d}{d\tau}\widetilde{J}(y(x, t), t)\,|_{t=t_0}=\frac{\partial p_0}{\partial x}=\frac{\partial^2 S}{\partial x^2}(x, t_0)\,\widetilde{J}(y(x, t_0)\,t_0).$$

Denote now

$$J=J(x, t)=\widetilde{J}(y(x, t), t).$$

Hence for any $\theta(x, t)\in C^\infty$

$$\frac{d}{d\tau}\left(\frac{\theta(x, t)}{\sqrt{J}}\right)=\frac{1}{\sqrt{J}}\frac{d\theta}{d\tau}-\frac{1}{2}\frac{\theta}{\sqrt{J}}\frac{\partial^2 S}{\partial x^2}.$$

Now put in (8.43)

$$\varphi_h(x, t) \overset{\text{def}}{=\!=} \frac{\theta_h(x, t)}{\sqrt{J}}$$

and substitute the function $\psi(x, t, h)$ from equality (8.43) into equation (8.39); then equating coefficients of equal powers of h to each other and applying formulas (8.44), (8.51), we obtain a re-

current system

$$\frac{d\theta_0}{d\tau}=0, \quad \theta_0|_{t=0}=\varphi(y),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d\theta_k}{d\tau}=\sqrt{J}\,\frac{\partial^2}{\partial x^2}\,\frac{\theta_{k-1}}{\sqrt{J}}, \quad \theta_1|_{t=0}=\ldots=\theta_k|_{t=0}=0.$$

Hence it follows that $\theta_0(x, t)=\varphi(y(x, t))$ and θ_k, $k>1$ is defined by the formula

$$\theta_k(x, t)=\int_0^t \sqrt{J(t_1)}\,\frac{\partial^2}{\partial x^2}\,\frac{1}{\sqrt{J(t_1)}}\int_0^{t_1}\sqrt{J(t_2)}\,\frac{\partial^2}{\partial x^2}\,\frac{1}{\sqrt{J(t_2)}}\cdots$$

$$\cdots\int_0^{t_{k-1}}\sqrt{J}\,\frac{\partial^2}{\partial x^2}\,\frac{\varphi(y)}{\sqrt{J(t_k)}}\,dt_k\ldots dt_1, \tag{8.51}$$

where $J(t_j)\overset{\text{def}}{=}J(x, t_j)$ and the integration is carried along the trajectories of system (8.46).

Now we shall prove that the function $\psi(x, t, h)$ defined by equation (8.43), where the functions $\varphi_k(x, t)$ and $S(x, t)$ has been just found by us is indeed an asymptotic expansion of problem (8.39). Take

$$\psi_N(x, t, h)=\frac{1}{\sqrt{J(x, t)}}\,e^{\frac{iS(x, t)}{h}}\left[\varphi(y(x, t))+\sum_{k=1}^{N}h^k\theta_k(x, t)\right].$$

We must prove that the function $\psi_N(x, t, h)$ is insignificantly different from the function $\psi(x, t)$, i.e., the exact solution of problem (8.39).

Indeed, we shall prove that

$$\max_{(x, t)\in\mathbf{R}^n\times\left[0, \frac{\pi}{2}\right]}|\psi_N(x, t, h)-\psi(x, t)|=O\left(h^{N+\frac{3}{2}}\right). \tag{8.52}$$

In order to obtain estimate (8.52) we note that $\psi_N(x, t, h)$ satisfies the equation

$$-ih\frac{\partial\psi_N}{\partial t}=\frac{1}{2}\left(-h^2\,\frac{\partial^2\psi_N}{\partial x^2}+x^2\psi_N\right)-h^{N+2}\times$$

$$\times e^{\frac{i}{h}S(x, t)}\,\frac{\partial^2}{\partial x^2}\,\frac{\theta_N(x, t)}{\sqrt{J(x, t)}},$$

$$\psi_N\,|_{t=0}=\varphi e^{i\frac{\Phi}{h}}$$

by definition and therefore the difference $\psi_N-\psi=\widetilde{\psi}$ is a solution of the problem

$$-ih\frac{\partial\widetilde{\psi}}{\partial t}=\frac{1}{2}\left(-h^2\frac{\partial^2\widetilde{\psi}}{\partial x^2}+x^2\widetilde{\psi}\right)-$$
$$-h^{N+2}e^{\frac{i}{h}S(x,\,t)}\frac{\partial^2}{\partial x^2}\frac{\theta_N(x,\,t)}{\sqrt{J(x,\,t)}},\qquad \widetilde{\psi}\,|_{t=0}=0. \tag{8.53}$$

It is easy to prove by direct calculation that the solution of problem (8.53) is given by the formula

$$\widetilde{\psi}(x,\,t)=\int_0^t\frac{e^{-i\frac{\pi}{4}}\cdot e^{-i\frac{\pi}{2}(1-\operatorname{sgn}(t-\tau))}}{\sqrt{2\pi h\,|\sin(t-\tau)|}}\times$$
$$\times\int_{-\infty}^{+\infty}e^{\frac{i}{2h\sin(t-\tau)}[\cos(t-\tau)x^2-2xy+\cos(t-\tau)y^2]}\times$$
$$\times h^{N+2}e^{\frac{i}{h}S(y,\,t)}\frac{\partial^2}{\partial x^2}\frac{\theta_N(y,\,t)}{\sqrt{J(y,\,t)}}\,dy\,d\tau. \tag{8.54}$$

It follows from (8.51) that the functions $\theta_k(x,\,t)$, $k=0,\,\ldots$, and $\varphi(y)$ belong to $C_0^\infty(x,\,t)$. Taking into account that $1/\sqrt{\sin z}$ is a function integrable in the neighborhood of the point $z=0$, we obtain estimate (8.52) directly from equality (8.54).

Now substitute $x=0$, $t=\frac{\pi}{2}$ into our formulas. Note that $S\left(0,\,\frac{\pi}{2}\right)=\Phi\left(y\left(0,\,\frac{\pi}{2}\right)\right)$. This equality is easy to obtain from formula (8.50). We have from (8.47)

$$q\left(0,\,\frac{\pi}{2}\right)=0=\frac{\partial\Phi}{\partial y}.$$

By hypothesis, the equality $\frac{\partial\Phi}{\partial y}=0$ is possible only at the point $y=y_0$. Thus $y\left(0,\,\frac{\pi}{2}\right)=y_0$ and we obtain expansion (8.38).

Note 1. The method of stationary phase can be applied to integrals of the form

$$\int_a^b\varphi(y)\,e^{i\frac{\Phi(y)}{h}}\,dy.$$

Let the derivative $\frac{\partial\Phi}{\partial y}$ be different from zero in some neighborhoods of the points a and b. Consider a partition of unity $\{e_1(y),\ e_2(y),$

$e_3(y)\}$ on $[a, b]$ under the following conditions: $e_1(y)$, $e_2(y)$, $e_3(y) \in \in C_0^\infty$, $e_1+e_2+e_3=1$ on $[a, b]$; $e_2(a)=e_3(b)=1$, $\frac{\partial\Phi}{\partial y}\neq 0$, when $y \in \operatorname{supp} e_2 \cup \operatorname{supp} e_3$, $\operatorname{supp} e_1 \in (a, b)$. Then

$$\int_a^b \varphi(y)\, e^{i\frac{\Phi(y)}{h}}\, dy = \int_{-\infty}^{+\infty} \varphi(y)\, e_1(y)\, e^{i\frac{\Phi(y)}{h}}\, dy + \\ + \int_a^\infty e_2(y)\,\varphi(y)\, e^{i\frac{\Phi(y)}{h}}\, dy + \int_{-\infty}^b e_3(y)\,\varphi(y)\, e^{i\frac{\Phi(y)}{h}}\, dy.$$

The first integral has been calculated by us, the other two, as can easily be proved integrating by parts, are of the order $O(h)$.

Note 2. We shall also need sums of the form

$$\frac{1}{\sqrt{2\pi h}}\, h \sum_{n=-N-1}^{N} e^{\frac{i}{h}\Phi(nh)}\varphi(nh), \quad h=\frac{\pi}{N}.$$

They can be reduced to the previous case with the help of the equation

$$h \sum_{n=-N+1}^{N} e^{\frac{i}{h}\Phi(nh)}\varphi(nh) = \int_{-\pi}^{\pi} e^{i\frac{\Phi(y)}{h}}\varphi(y)\, dy + \\ + 2\sum_{k=1}^{\infty}\int_{-\pi}^{+\pi} e^{i\frac{\Phi(y)}{h}}\varphi(y)\cos(2kNy)\, dy, \tag{8.55}$$

which can be proved very easily by an expansion of the function $e^{i\frac{\Phi(y)}{h}}\varphi(y)$ on the segment $[-\pi, \pi]$ in a Fourier series and by subsequently using the formula of geometrical progression

$$h \sum_{n=-N+1}^{N} \cos knh = 2\pi\delta_{k,\,2mN}, \quad m \text{ is an integer.}$$

(5) The tail of oscillations. Solution (8.37) is a sum of functions of the form

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} v_1(\xi) \sum_{n=-N+1}^{N} e^{(i/h)nh[S_\pm(x,\,t,\,nh)+x-\xi]}\varphi^\pm(x, t, nh)\, d\xi.$$

Put by formula (8.55)

$$h \sum_{n=-N+1}^{N} e^{(i/h)nh[S_\pm(x,\,t,\,nh)+x-\xi]}\varphi^\pm(x, t, nh) = \\ = \int_{-\pi}^{\pi} \varphi(x, t\omega)\, e^{(i/h)\omega[S_\pm(x,\,t,\,\omega)+x-\xi]}\, d\omega + 2\sum_{k=1}^{\infty}\int_{-\pi}^{+\pi}\varphi(x, t, \omega)\times \\ \times e^{(i/h)\omega[S_\pm(x,\,t,\,\omega)+x-\xi]}\cos 2k\omega N\, d\omega.$$

The stationary point ω_0 of the integrals is defined by the equations

$$S_{\pm}(x, t, \omega_0) + x + \omega_0 \frac{\partial S_{\pm}}{\partial \omega}(x, t, \omega_0) = \xi + 2Nk. \tag{8.56}$$

For sufficiently small t this equation has a solution only for $k = 0$, since $S_{\pm}(x, t, \omega)\,|_{t=0} = 0$.

We shall estimate the kth ($k \neq 0$) term of the sum. We shall obtain the estimate of the form $O(h/k^2)$ integrating two times by parts the exponential (the first time putting the cosine under the differential sign and the second time expanding the cosine by Euler's formula). Hence it follows that a sum of the series from $k = 1$ to infinity is of the order $O(h)$.

Now turn to the solution of implicit equation (8.56). It turns out, that the left-hand side of the equation has a remarkable property. Its derivative along the integral curve of system of bicharacteristics (8.36) corresponding to a sign plus or minus equals zero. We leave the proof to the reader; in fact, it is of general nature.

Hence it follows that the left-hand side is constant along the corresponding bicharacteristic, and since

$$S_{\pm}\,|_{t=0} = \left.\frac{\partial S_{\pm}}{\partial \omega}\right|_{t=0} = 0,$$

it equals $x_0^{\pm}(x, t, \omega)$, i.e., a solution of equation (8.36): $x = X^{\pm}(x_0, \omega, t)$, for sufficiently small t. Therefore, equation (8.56) for the stationary point is equivalent to the equation

$$X^{\pm}(\xi, \omega_0, t) = x.$$

Thus the equation for the stationary point has a solution only for those x, which belong to closed domains

$$\Omega_{\pm} = \{X^{\pm}(\xi, \omega, t), \ |\omega| \leqslant \pi\},$$

i.e., the domains formed by the ends of the trajectories of the system of bicharacteristics (8.36) going out of the point ξ.

Let x lie inside a domain of the form $\Omega_{\pm}$. On differentiating system (8.36) with respect to ω, we obtain that $\frac{\partial X^{\pm}}{\partial \omega} \neq 0$ when $t \neq 0$ and therefore $\frac{\partial x_0^{\pm}}{\partial \omega} = -\frac{\partial X^{\pm}}{\partial \omega} \Big/ \frac{\partial X^{\pm}}{\partial x_0} \neq 0$ when $t \neq 0$, $\omega \neq 0$. Hence the derivative of the left-hand side of (8.56) with respect to ω is different from zero when $t \neq 0$, $\omega \neq 0$ and the method of stationary phase can be applied. Thus, the solution rapidly oscillates inside the domains $\Omega_{\pm}$, its derivatives are not bounded as $h \to 0$ and therefore they obviously do not turn into the derivatives of a solution of the wave equation.

Note. If x lies on characteristics of the limit wave equation (i.e., $x = X_0^{\pm}(\xi, t)$, where $\frac{dX_0^{\pm}}{dt} = \mp C(X_0^{\pm})$, $X_0^{\pm}(\xi, 0) = \xi$), then the stationary point $\omega_0 = 0$ is degenerate

$$\frac{\partial x_0^{\pm}}{\partial \omega}(x, t, 0) = 0.$$

Indeed, it means that the main shock wave propagates along the characteristics of the limit wave equation and the tail of rapid oscillations follows it. The support of the tail is defined by the characteristics $X^{\pm}$.

We see from this example that characteristics defined by Hamilton's equation dependent on ordered operators define exactly the support of the oscillation tail. The fact follows from the general theory for a large body of problems and the discussed equations are of the general type too.

Now we wish to compare the support of the tail with computer calculations.

We take $l = 10^{-5}$ cm for a measure of length, the length of the interval (a circle) $2l$, $h = 10^{-8}$ cm, the velocity of sound in a crystal $c = 10^5 \frac{\text{cm}}{\text{sec}}$ is taken for a measure of velocity. We take a periodic function of the form

$$u|_{t=0} = \frac{1}{2}(x - 2n - 1), \text{ when } 2n \leqslant x \leqslant 2n+2, \quad n = 0, \pm 1, \ldots$$

with a jump at the points $x = 2n$, $n = 0, \pm 1, \ldots$ for an initial condition.

In Fig. 1 there are graphs of the functions

$$\max_{y \in [x-5h,\, x+5h]} \frac{\partial u}{\partial x}(y), \quad \min_{y \in [x-5h,\, x+5h]} \frac{\partial u}{\partial x}(y),$$

which correspond to the upper and low boundaries of the domain. The exact solution is represented by a continuous line and the asymptotic solution is represented by a dotted line.

(6) **The Cherenkov effect.** In 1934 the Soviet physicist P. A. Cherenkov discovered the effect of non-damping radiation of light, produced in a media by moving charged particles. The effect was studied theoretically by I. E. Tamm and I. M. Frank in 1937. The extraordinarily wide scope of the effect, however, became clear only in the fifties. In 1958 P. A. Cherenkov, I. E. Tamm and I. M. Frank were awarded the Nobel Prize.

The effect deals with electrons. When an electron moves in a media with a velocity greater than a certain value depending on the media, fast oscillating electromagnetic waves, i.e., light waves, radiate in a cone behind it. These light waves are called the Cherenkov

radiation. The mathematical nature of the effect depends on the characteristics of ordered operators. The operational method makes it possible to find a similar (mathematical) effect for a broad class of pseudodifferential equations and, in particular, difference schemes (including systems of equations of a crystal). In the latter case it follows directly from the asymptotic formulas obtained above

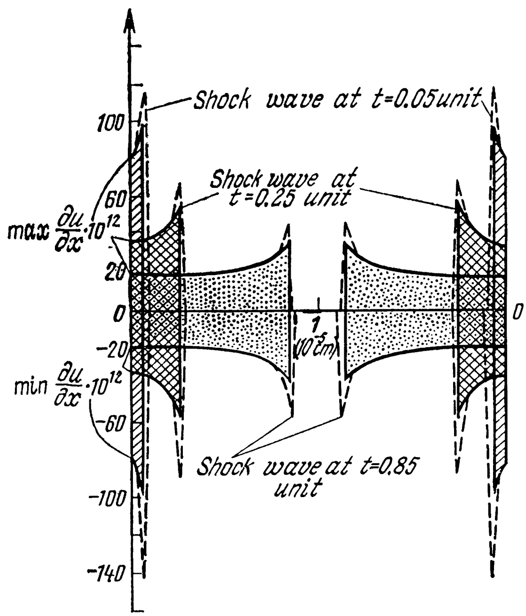

Fig. 1

A function of the δ-type at the point $x = vt$ is introduced into the right-hand side of the equation of oscillations (of the electromagnetic field for the Cherenkov effect, the crystal vibrations in our case) to define the movement of an electron with a velocity v. Mathematically, the Cherenkov radiation is a tail of fast oscillations of the solution, which follows the point $x = vt$. The effect is related to the one studied in the previous item. We shall study the tail theoretically and then compare the results with computer calculations. It will turn out just as in the previous example, that the tail (i.e., the domain of the Cherenkov radiation) is completely defined by the bicharacteristics.

Thus consider the problem

$$\begin{cases} L_h u \overset{\text{def}}{=} h^2 \dfrac{\partial^2 u}{\partial t^2} + 4c^2 \sin^2\left(-\dfrac{ih}{2}\dfrac{\partial}{\partial x}\right) u = \varphi(t)\,\delta_h(x - vt) \\ u(x, 0) = u_t'(x, 0) = 0, \quad u(x + 2\pi, t) = u(x, t), \end{cases} \tag{8.57}$$

where $\delta_h(x-vt)=\frac{1}{2\pi}\sum\limits_{k=-N+1}^{N} e^{ik(x-vt)}$ and $\varphi(t)$ is a smooth function. We shall solve the problem for sufficiently small t. The solution of problem (8.57) $L_h u=f$ is unique by Theorem 8.2: $u \overset{\text{def}}{=} L_h^{-1} f$. In analogy with (8.8')-(8.8'') we obtain that

$$R(x, t)=\frac{1}{h^2}\int\limits_0^t\int\limits_0^{t-\tau} G\left(\tau', \overset{1}{\omega}, \overset{2}{x}\right)\varphi(\tau)\,\delta_h(x-\tau v)\,d\tau'\,d\tau$$

satisfies the equation

$$L_h R=\varphi(t)\,\delta_h(x, t)+\int\limits_0^t\int\limits_0^{t-\tau} f_1\left(\tau', \overset{1}{\omega}, \overset{2}{x}\right)\varphi(\tau)\,\delta_h(x, \tau)\,d\tau'\,d\tau,$$

where $f_1\left(t, \overset{1}{\omega}, \overset{2}{x}\right)$ satisfies estimate (8.7). By Lemma 8.2 and Theorem 8.1 a solution of an equation

$$L_h R_1=\int\limits_0^t\int\limits_0^{t-\tau} f_1\left(\tau', \overset{1}{\omega}, \overset{2}{x}\right)\varphi(\tau)\,\delta_h(x, \tau)\,d\tau'\,d\tau,$$

$$R_1\Big|_{t=0}=\frac{\partial R_1}{\partial t}\Big|_{t=0}=0$$

belongs to $C_{t,h}^{k(m)}$, where $k(m)\to\infty$ as $m\to\infty$. Therefore, since $R\to R_1$ is a solution of initial problem (8.57) the whole non-smooth part of the solution of the problem is contained in R. We have come to the conclusion, that the main term of the asymptotics of R as $h\to 0$ having been integrated by parts is defined by an integral of the form (in analogy with the previous item):

$$\frac{1}{h^2\cdot 2\pi h}\int\limits_{-\pi}^{+\pi}\int\limits_0^{x/v}\left[\frac{e^{\frac{i\omega S_+(x,\, t-\xi/v,\,\omega)}{h}}\varphi_0^+\left(x, t-\frac{\xi}{v}, \omega\right)}{\frac{\partial S_+}{\partial t}\left(x, t-\frac{\xi}{v}, \omega\right)}-\right.$$

$$\left.-\frac{e^{\frac{i\omega S_-\left(x,\, t-\frac{\xi}{v},\,\omega\right)}{h}}\varphi_0^-\left(x, t-\frac{\xi}{v}, \omega\right)}{\frac{dS_-}{\partial t}\left(x, t-\frac{\xi}{v}, \omega\right)}\right]e^{i\omega(x-\xi)}\,\varphi\left(\frac{\xi}{v}\right)d\omega\,d\xi.$$

Applying successively the stationary phase method first in ω then in ξ, we obtain that the common stationary point ω_0, ξ_0

$(\omega_0 \neq 0)$ is defined by the system of equations

$$\begin{aligned} \xi_0 &= x_0^{\pm}\left(x,\ t-\frac{\xi}{v},\ \omega_0\right), \\ \omega_0 &= \pm\frac{c(x)}{v}\sin\left[\frac{\omega_0}{2}\left(1+\frac{\partial S_{\mp}}{\partial x}\left(x,\ t-\frac{\xi_0}{v},\ \omega_0\right)\right)\right]. \end{aligned} \tag{8.58}$$

It is easy to verify that if $\omega \neq 0$, $|\omega_0| < \pi$, $|\xi_0| < \pi$, then the second derivatives are different from zero and the stationary phase method can be applied.

For the sake of simplicity consider the case $c = \text{const}$. Then system (8.58) is of the form

$$\begin{aligned} \omega_0 &= \pm\frac{2c}{v}\sin\frac{\omega_0}{2}, \\ \xi_0 &= \frac{x \pm ct\cos\frac{\omega_0}{2}}{1\pm\frac{c}{v}\cos\frac{\omega_0}{2}}. \end{aligned} \tag{8.59}$$

We see on the graph that a solution of (8.59) exists, when $\omega_0 \neq$ $\neq 0$, if

$$c \geqslant v \geqslant \frac{2c}{\pi}, \tag{8.60}$$

since $|\omega| \leqslant \pi$. Inequality (8.60) being satisfied, the solution oscillates as $h \to 0$ in a domain consisting of the ends of all trajec-

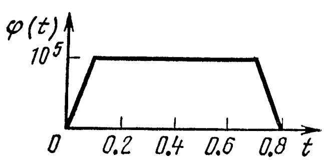

Fig. 2

tories coming out, when $t = 0$, of domains

$$\Omega_\xi^0 = \left\{X_0 = \xi\left(1 \pm \frac{c}{v}\cos\frac{\omega_0}{2}\right)\right\}, \tag{8.61}$$

where ω_0 is defined by (8.61). Then $\omega_0 = 0$ is a degenerate point and the solution is of a peculiar character ("resonance").

Compare the width of the oscillation tail for problem (8.57) calculated by formulas (8.61) with an exact solution of problem (8.58) obtained by means of a computer for various v. Take the length of a circle (in units of the previous item) equal to 2, $c = 1$, $h = 10^{-3}$.

The second derivative $\frac{\partial^2 u}{\partial t^2}$ may be approximated by the formula

$$-\frac{4}{\tau^2}\sin^2\frac{i\tau}{2}\frac{\partial}{\partial t}, \quad \tau = 1.5 \times 10^{-4}.$$

The graph of the function $\varphi(t)$ is shown in Fig. 2. For the graphs of the functions

$$\max_{[x-10h,\, x+10h]} \frac{\partial u}{\partial x}, \qquad \min_{[x-10h,\, x+10h]} \frac{\partial u}{\partial x}$$

in different moments of time see Figs. 3, 4. The interval between

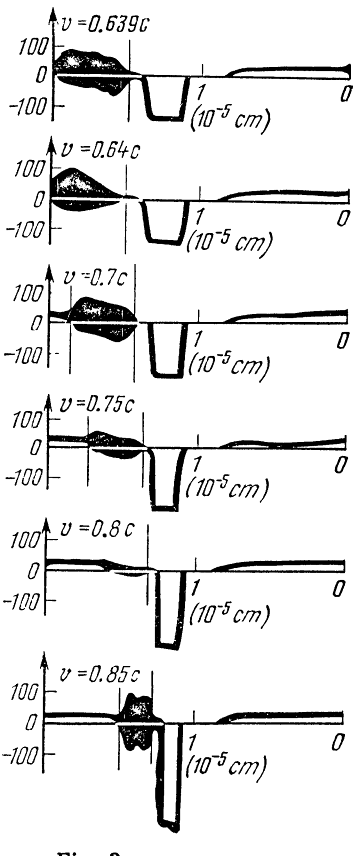

Fig. 3

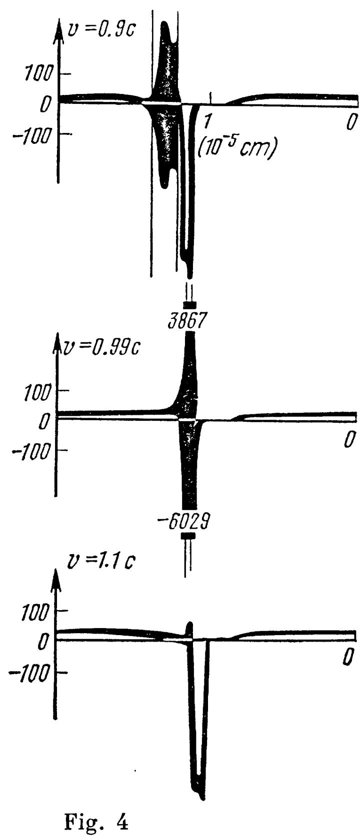

Fig. 4

the graphs of these functions is blackened. The boundary of the oscillation tail obtained from the asymptotic formulas is indicated by vertical lines.

(7) The focus in a crystal. Consider an example of a fast-oscillating initial condition

$$u|_{t=0} = \varepsilon\psi(x, h),$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = -\varepsilon\frac{2ic}{h}\sin\frac{\hat{\omega}}{2}\psi(x, h), \tag{8.62}$$

where $\psi(x, h)$ is a periodic extension from the segment $[0, 2\pi]$ of a function $\varphi(x)\cos\frac{S(x)}{h}$; $S, \varphi \in C^\infty$, $\operatorname{supp}\varphi \in (0, 2\pi)$, ε is a parameter.

The Fourier coefficients of the function ψ with a number n, $|n| > N$ are of order $O(h^\infty)$ by the stationary phase method, therefore the formulas obtained in the previous items may be applied in the case of initial conditions (8.62) as well.

Now we want to compare the results obtained by the formulas of item 3 and the corresponding computer calculations.

In the computer case we took: the length of a circle $3l$, $l = 10^{-5}$ cm, the moments of time $t_i = 2i\cdot 10^{-10}$ sec ($i = 0, 1, \ldots, 5$), in the units l

$$S(x) = x^2$$

$$\varphi(x) = \begin{cases} 4x & \text{when } x \in [0;\ 0.25] \\ 1 & \text{when } x \in [0.25;\ 2] \\ 4(2-x)+1 & \text{when } x \in [2;\ 2.25] \\ 0 & \text{when } x \in [2.25;\ 3]. \end{cases}$$

A solution of the crystal equation for $c = \text{const}$ may be calculated with the help of formulas (8.37) and (8.38) along the same lines as it was done in the two previous items. The term corresponding to the sign "+" has no stationary point and therefore may be cancelled out. The stationary phase method is obviously true for $ct < 1$. The condition $\partial^2\Phi/\partial y^2 \neq 0$ of Lemma 8.3 is violated at the point $ct = 1$, $x = \frac{\pi}{2}$. At this point, called focal, the crystal vibrations have an amplitude bigger than in the neighboring points.

Note that the focal point can be defined directly by the equations of bicharacteristics (similar to (8.36)) for $c = \text{const}$:

$$\frac{dx}{dt} = c\cdot\cos\frac{p}{2}, \quad \frac{dp}{dt} = 0. \tag{8.62'}$$

The initial conditions for the system of equations (8.62′) are

$$x(0) = x_0, \quad p(0) = p_0 = 2x_0$$

corresponding to initial conditions (8.62) with $S = x^2$. Consider a phase plane p, x and there the points $x = 2\pi n$. If we identify these points we shall obtain a cylinder. The line $p = 2x$, corres-

ponding to the initial condition $p_0 = 2x_0$, moves along trajectories of the system of bicharacteristics: indeed, we obtain a curve

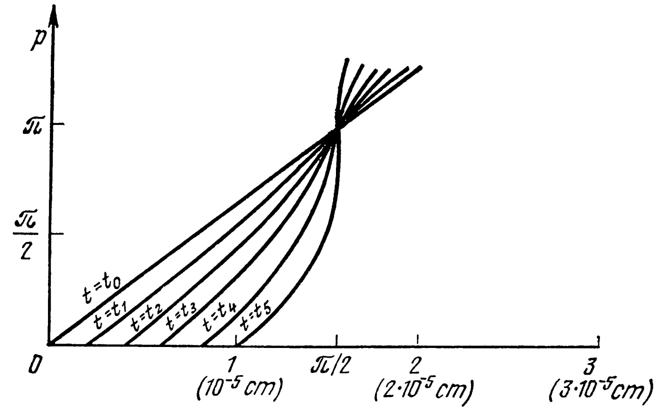

Fig. 5

$\{x(x_0, t), p(x_0, t)\}$ at the fixed moment t. This change of the curves at the moments of time $t = t_1, \ldots, t_5$ is shown in Fig. 5.

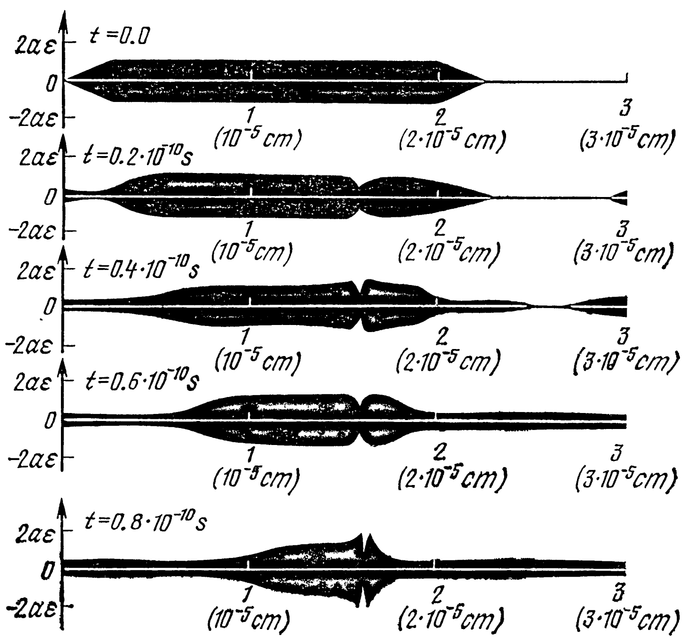

Fig. 6

The curve is projected on the axis p at the moment $t = t_5$ and when $p = \pi$ the tangent of the curve is parallel to the axis p. The

point at which the tangent is parallel to the axis p' is called a focal point.

The graphs max U and min U (with the domain between them blackened) for the exact solution of equation (8.4) with initial data

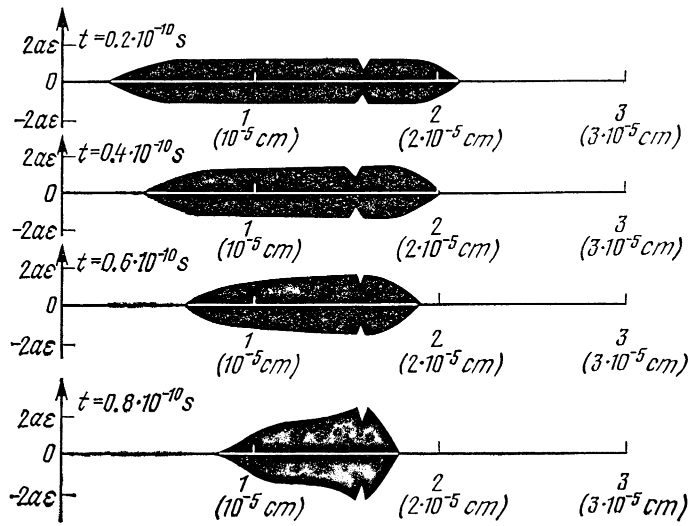

Fig. 7

(8.62) and $t = t_0, \ldots, t_4$ are shown in Fig. 6 and the corresponding graphs for the solution obtained with the asymptotic formulas ($a = 1$) are shown in Fig. 7.

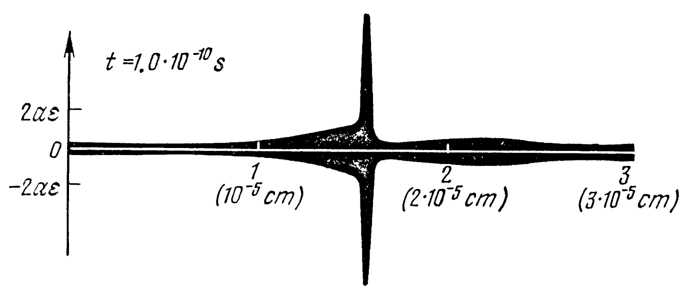

Fig 8

The graphs corresponding to $t = t_5$ are shown in Figs. 8 and 9 (Fig. 8 corresponds to the exact solution). We see that the amplitude of the solution grows rapidly at the focal point. We shall now consider a nonlinear equation of the lattice vibrations.

(8) **Nonlinear equation of a crystal.** We have confined ourselves to the harmonic approximation of the system of Newton's equations for the atoms of a lattice. In fact, we assumed the amplitude of the initial displacement to be small and ignored the non-linear

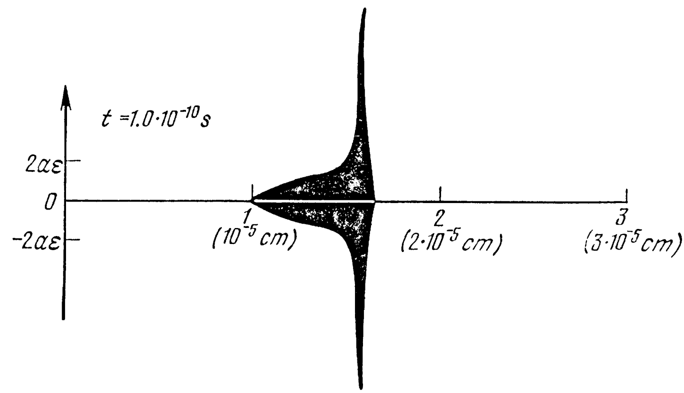

Fig. 9

terms. Now we shall take into consideration the amendment introduced into the equations of lattice oscillations by the quadratic terms. It means that a term

$$c^2\alpha\,[(u_{n+1}-u_n)^2-(u_n-u_{n-1})^2]$$

is added to equation (8.1), where α is a parameter.

On substituting U_n+W_n into this new equation, where U_n is a solution of the linear equation, we obtain the following non-homogeneous equation

$$\ddot{W}_n-c^2\frac{W_{n+1}-2W_n+W_{n-1}}{h^2}=c^2\alpha\,[(U_{n+1}-U_n)^2-(U_n-U_{n-1})^2]$$

for $|W_n-W_{n-1}|\ll|U_n-U_{n-1}|$. For the sake of simplicity we consider the case, where c and α are constants. We shall consider the general case in Appendix.

We assume that the dependence of the initial data $U_n(0)$ and $\dot{U}_n(0)$ on the parameter h satisfies a substantial condition. We shall introduce the following definition to the effect.

A system of functions $v_n(h)$, $-N+1\leqslant n\leqslant N$ corresponds to a limit distribution $g(x, p)$, $-\pi\leqslant p\leqslant\pi$, where g is an even distribution in p, if for any function $a\in C^\infty(-\pi, \pi)$ the following

limit condition is true

$$\lim_{h\to 0} h \sum_{n=-N+1}^{N} a(nh)\, v_n \frac{(v_{n+k}+v_{n-k})}{2} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} a(x)\, g(x, p) \cos kp\, dx\, dp, \tag{8.63}$$

where

$$v_{i+2N} \overset{\text{def}}{=} v_i.$$

The physical sense of this definition is especially clear when $k = 0$, since in this case it is necessary that the mean value $a(nh)$ should converge to a mean value $a(x)$ of distribution.

We shall consider, as we did before, a space of functions M_N instead of the functions on a lattice; moreover, we shall formulate the final theorem in terms of the space functions M_N. Note, that the arguments of this item can be extended to a wide class of non-linear pseudodifferential equations with non-linear terms considered small. The first approximation of the theory of perturbations in this case is a solution of equation (8.4) and the second one satisfies the equation

$$L_h u = -8ic^2\alpha h^2 \sin\frac{\omega}{2}\left(\left(\sin\frac{\omega}{2}\, u\right)^2\right), \quad \omega = \hat{\omega} = -ih\frac{\partial}{\partial x}, \tag{8.64}$$

where u satisfies the linear equation $L_h u = 0$. For the sake of simplicity we take one term of the sum $\sum_{\pm}$ of formula (8.37) corresponding to the minus sign as a solution u of the linear equation. We assume that the initial value $u_0 \in M_n$ corresponds to a condition

$$\lim_{h\to 0} \int_{-\pi}^{\pi} u_0 a(x) (\cos k\hat{\omega})\, u_0\, dx = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} a(x)\, g(x, \omega) \cos k\omega\, dx\, d\omega. \tag{8.65}$$

First note that a more natural for the continuous situation and a more general condition follows from condition (8.65). Denote

$$(f, \varphi) \overset{\text{def}}{=} \int_{-\pi}^{\pi} f(x)\overline{\varphi(x)}\, dx.$$

The following lemma is true.

Lemma 8.4. *Let $f(x, \omega) \in C^\infty$ be an even periodic function ω. If $\psi_h(x)$ corresponds to the limit distribution $g(x, \omega)$, then*

$$\lim_{h\to 0}\left(\psi_h, f\left(\overset{2}{x}, \overset{1}{\omega}\right)\psi_h\right) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x, \omega)\, g(x, \omega)\, dx\, d\omega.$$

Proof. Expand the function $f(x, \omega)$ in a Fourier series in ω. We have

$$f(x, \omega) = \sum_{n=-\infty}^{\infty} a_n(x) e^{-in\omega}, \tag{8.66}$$

where

$$a_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x, \omega) e^{in\omega} d\omega. \tag{8.67}$$

By the conditions of the lemma it follows that for any n there is an inequality

$$\max_x |a_n(x)| \leqslant \frac{C_k}{n^k}, \tag{8.68}$$

$k > 0$ is any integer, C_k = const. The estimate can be easily proved integrating equation (8.67) by parts.

We prove that

$$\lim_{k\to\infty} \left\| \left(f_k\left(\overset{2}{x}, \overset{1}{\omega}\right) - f\left(\overset{2}{x}, \overset{1}{\omega}\right)\right) \psi_n(x) \right\| = 0,$$

where we have denoted by $f_k(x, \omega)$ a partial sum of series (8.66)

$$f_k(x, \omega) = \sum_{|n| \leqslant k} a_n(x) e^{in\omega}.$$

Note that the function $f_k(x, \omega)$ uniformly converges to $f(x, \omega)$ as $k \to \infty$. We have for any smooth function $u(x)$

$$\left[f\left(\overset{2}{x}, \overset{1}{\omega}\right) - f_k\left(\overset{2}{x}, \overset{1}{\omega}\right)\right] u(x) =$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{n=\infty} \left[\int_{-\pi}^{\pi} u(y) e^{-iny} dy\right] e^{inx} \left[f(x, nh) - f_k(x, nh)\right] =$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-\pi}^{\pi} u(y) e^{-iny} dy\right] \sum_{|m|>k} a_m(x) e^{i(mnh+nx)}.$$

Finally, we obtain

$$\left[f\left(\overset{2}{x}, \overset{1}{\omega}\right) - f_k\left(\overset{2}{x}, \overset{1}{\omega}\right)\right] u(x) = \left(\sum_{|n|>k} a_n\left(\overset{2}{x}\right) e^{in\overset{1}{\omega}}\right) u(x). \tag{8.69}$$

Estimate the nth term in sum (8.69). The inequality is obviously true

$$\left\| a_n\left(\overset{2}{x}\right) e^{in\overset{1}{\omega}} u(x) \right\| \leqslant \max_x |a_n(x)| \; \|u\|,$$

or by (8.68)

$$\left\| a_n \left(\overset{2}{x} \right) e^{in\overset{1}{\omega}} u(x) \right\| \leqslant \frac{C_k}{n^k} \| u \|, \tag{8.70}$$

where $k > 0$ is an integer. We have by (8.69), (8.70)

$$\left\| \left[f\left(\overset{2}{x}, \overset{1}{\omega} \right) - f_k \left(\overset{2}{x}, \overset{1}{\omega} \right) \right] u(x) \right\| \leqslant \sum_{|n|>k} \frac{1}{n^l} C_l \| u \|, \ \forall l,$$

$$\lim_{k\to\infty} \left\| \left[f\left(\overset{2}{x}, \overset{1}{\omega} \right) - f_k \left(\overset{2}{x}, \overset{1}{\omega} \right) \right] u(x) \right\| = 0, \text{ as } k \to \infty.$$

Hence for any $\delta > 0$ there exists such a k_0, that

$$\left| \left(\psi_h, \left(f\left(\overset{2}{x}, \overset{1}{\omega} \right) - f_k \left(\overset{2}{x}, \overset{1}{\omega} \right) \right) \psi_h \right) \right| < \delta \text{ when } k \geqslant k_0,$$

$$\left| \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [f(x, \omega) - f_k(x, \omega)] g(x, \omega) \, dx \, d\omega \right| < \delta \text{ when } k \geqslant k_0,$$

finally by (8.63) we have

$$\left| \left(\psi_h, f_k \left(\overset{2}{x}, \overset{1}{\omega} \right) \psi_h \right) - \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f_k(x, \omega) g(x, \omega) \, dx \, d\omega \right| \leqslant c(k) \delta_h,$$

where $c(k) = \text{const}$, $\delta_h \to 0$ as $h \to 0$. Fix $k > k_0$ and let $h \to 0$, then for sufficiently small h the left-hand side of the last inequality is not greater than δ. The lemma follows from this statement and the previous inequalities.

Definition. *Let for any finite function $\varphi \in C^\infty(\mathbf{R}^n)$ the following condition be true*

$$\lim_{h\to 0} \int \varphi(x) F_h(x) \, dx = \int \varphi(x) r(x) \, dx, \quad x \in \mathbf{R}^n,$$

where $F_h(x) \in C^\infty$ is a one-parameter family of functions, h is sufficiently small. Then we shall say that $F_h(x)$ tends to $r(x)$ in the sense of the theory of distributions and write $F_h(x) \to r(x)$.

We find the limit of the right-hand side of equation (8.64) in the sense of the theory of distributions. Denote $F_h = -4 \left(\sin \frac{\hat{\omega}}{2} u \right)^2$. The following lemma is true.

Lemma 8.5.

$$F_h \to 4 \int_{-\pi}^{\pi} \sin^2 \frac{\omega}{2} g\left(x - ct \cos \frac{\omega}{2}, \ \omega \right) d\omega.$$

Proof. Denote by $F_{x\to p}$ and $F_{p\to x}$ the Fourier transforms

$$(F_{x\to p}f)(p)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-ipx}\,dx;\quad (F_{p\to x}f_p)(x)=\sum_{p=-\infty}^{\infty} f_p e^{ipx}.$$

It is obviously true, that

$$f(\hat{\omega})\,\psi=F_{p\to x}f(ph)\,F_{x\to p}\psi.$$

We shall obtain the following equality with the help of this formula

$$\int_{-\pi}^{\pi}\psi_2 f(\hat{\omega})\,\psi_1\,dx=\int_{-\pi}^{\pi}\psi_1\bar{f}(\hat{\omega})\,\psi_2\,dx,$$

where $\psi_1,\ \psi_2\in C^\infty$ are periodic functions. Applying this equation, we obtain the following equation

$$\int_{-\pi}^{\pi}\varphi(x)\,F_h(x,t)\,dx=-4\int_{-\pi}^{\pi}\varphi(x)\left(T\sin\frac{\hat{\omega}}{2}u_0\right)^2dx=$$

$$=4\int_{-\pi}^{\pi}\sin\frac{\hat{\omega}}{2}\,[T^*\varphi(x)\,T]\sin\frac{\hat{\omega}}{2}\,u_0u_0\,dx \tag{8.71}$$

for $\varphi\in C^\infty$, $u=Tu_0$ $(T:u_0\to u=e^{-\frac{2i}{h}ct\sin\frac{\hat{\omega}}{2}}a_0)$.

Apply Theorem 1.1 of Ch. III to the formula in the square brackets. We have $\varphi_{\bar{0}}=1$, $\varphi_{\bar{k}}=0$, $k\geqslant 1$ in (8.37) when $c=\text{const}$ and

$$S=-2ct\sin\frac{\omega}{2}.$$

We have

$$\frac{\delta S}{\delta\omega}=-2ct\,\frac{\sin\frac{\omega'}{2}-\sin\frac{\omega}{2}}{\omega'-\omega},$$

$$\tilde{x}=x-\frac{\delta S}{\delta\omega}=x+2ct\,\frac{\sin\frac{\omega'}{2}-\sin\frac{\omega}{2}}{\omega'-\omega}.$$

Hence

$$J\stackrel{\text{def}}{=}\frac{\partial\tilde{x}}{\partial x}=1.$$

We have by Theorem 1.1 of Ch. III

$$T^*\varphi(x)\,T=\varphi\left(\overset{2}{x}+2ct\,\frac{\sin\frac{\overset{1}{\omega}}{2}-\sin\frac{\overset{3}{\omega}}{2}}{\overset{1}{\omega}-\overset{3}{\omega}}\right).$$

Hence, by the commutation formula, equation (8.71) is different from the integral

$$4\int_{-\pi}^{\pi}\left(\varphi\left(\overset{2}{x}+ct\cos\frac{\omega}{2}\overset{1}{\right)}\sin^2\frac{\omega}{2}u_0\overset{1}{\right)}u_0\,dx$$

only by a term of order O (h).

The last integral tends to

$$+4\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\varphi\left(x+ct\cos\frac{\omega}{2}\right)\sin^2\frac{\omega}{2}\,g(x,\,\omega)\,dx\,d\omega$$

as $h\to 0$ by Lemma 8.4. Hence follows the statement of the lemma.

Next we shall prove the following theorem.

Theorem 8.3. *Let* a *equal to* $\frac{1}{h}$ *in equation* (8.64). *Then the solution* w *of problem* (8.64), *satisfying the zero initial conditions, converges in the distribution theory sense to a function*

$$w_1(x,\,t)=-2c\int_0^t\int_{-\pi}^{\pi}\sin^2\frac{\omega}{2}\left[g\left(x-c\tau\cos\frac{\omega}{2}+c\,(t-\tau),\,\omega\right)-\right.$$

$$\left.-g\left(x-c\tau\cos\frac{\omega}{2}-c\,(t-\tau),\,\omega\right)\right]d\omega\,d\tau. \tag{8.72}$$

Proof. Introduce the following notation:

$$r\,(x,\,t)=-4\int_{-\pi}^{\pi}\sin^2\frac{\omega}{2}\,g\left(x-ct\cos\frac{\omega}{2},\;\omega\right)d\omega.$$

It is easy to prove that $Lw_1(x,\,t)=c^2r'_x(x,\,t)$, where $L=\Box_c$ is a wave operator.

$$f_h=F_h-r\,(x,\,t);\;\; v_h=L_h^{-1}\left(\frac{2\sin\frac{\omega}{2}}{h}f_h\right);$$

$$L_h^*\overset{\text{def}}{=}h^2\frac{\partial^2}{\partial t^2}+4c^2\sin^2\frac{\omega}{2}c^2\,(x);$$

$$L^*\overset{\text{def}}{=}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}c^2\left(\frac{1}{x}\right).$$

Denote by $(L_h^*)^{-1}f$ and $(L^*)^{-1}f$ solutions of the problems $L_h^*u=f\,(x,\,t)$ and $L^*u=f\,(x,\,t)$ satisfying the zero initial conditions when $t=T$:

$$u\Big|_{t=T}=\frac{\partial u}{\partial t}\Big|_{t=T}=0.$$

We shall use the following facts for the proof:

1. An analogue of Lemma 8.2 is true for the operators L_h^* and L^*. In particular, if $\psi(x, t) \in C^\infty$ does not depend on h, then

$$h^2 (L_h^*)^{-1} \psi(x, t) - (L^*)^{-1} \psi(x, t) = O(h).$$

One can verify this following the proof of Lemma 8.2.

2. The following equality is true

$$h^2 \int_0^T \int_{-\pi}^{\pi} \psi(x, t) L_h^{-1} \frac{2i \sin \frac{\omega}{2}}{h} f_h \, dx \, dt =$$

$$= h^2 \int_0^T \int_{-\pi}^{\pi} \left[\frac{2i \sin \frac{\omega}{2}}{h} (L_h^{-1})^* \psi(x, t) \right] f_h \, dx \, dt, \qquad (8.73)$$

which can be proved with the help of the equality $\psi = L_h^* (L_h^*)^{-1} \psi$ and the integration by parts.

3. $\| F_h \| \leqslant c_1 h^{-1/2}$, where c_1 is a constant not depending on h. By (1)-(3), the right-hand side of the equation (8.73) is different from a function

$$\int_0^T \int \frac{\partial}{\partial x} (L^{*-1} \psi(x, t)) f_h(x, t) \, dx \, dt$$

by $O(h^{1/2})$ and this function tends to zero as $h \to 0$ by Lemma 8.5.

It follows from Lemma 8.2 that

$$\lim_{h \to 0} L_h^{-1} \left(\frac{2i \sin \frac{\omega}{2}}{h} \right) r(x, t) = \frac{w_1}{c^2}.$$

The theorem is proved.

Note, that the convergence in Theorem 8.3 in the distribution sense may be changed for the uniform convergence. But more estimates are necessary to the effect. A more general theorem was proved in Appendix 1 of the Russian edition.

Note. We have used only the fact, that the function $w_1(x, t)$ and not $g(x, \omega)$ is smooth when proving the theorem. The latter function could be, generally speaking, a distribution and the theorem is true for those t for which $w_1(x, t)$ remains smooth.

Note that the most interesting results are produced, when $g(x, \omega)$ is close to the δ-function. (The actual form of u_0 is of no importance.)

We shall consider the data of computer calculations in the case, when

$$g(x, \omega) = \frac{\varepsilon^2 \varphi^2(x)}{4} [\delta(\omega + 2x) + \delta(\omega - 2x)],$$

where $\varphi(x)$ is defined in the previous item.

We shall use the most common potential of interaction between atoms in a lattice leading to a system

$$\ddot{u}_n = \frac{c^2}{h^3}\left\{f\left(\frac{h^{-1}+u_n-u_{n-1}}{h^{-1}}\right) - f\left(\frac{h^{-1}+u_{n+1}-u_n}{h^{-1}}\right)\right\} \tag{8.74}$$

$$f(z) = -\frac{1}{8}\left\{\frac{1}{z^2}-\frac{1}{z^{10}}\right\},\quad h=10^{-3},\quad t\in[0,\,1]=[0,\,10^{-10}\text{s}],$$

$$c^2=1,\quad n=0,\ \pm 1,\ \ldots,\ \pm 3000,\quad u_h|_{t=0}=\varepsilon\varphi(hn)\cos\{hn^2\},$$

$$\dot{u}_n|_{t=0}=c\varepsilon h^{-1}\varphi\left(h\left(n-\frac{1}{2}\right)\right)\cos\left\{h\left(n-\frac{1}{2}\right)^2\right\}-$$

$$-c\varepsilon h^{-1}\varphi\left(h\left(n+\frac{1}{2}\right)\right)\cos\left\{h\left(n+\frac{1}{2}\right)^2\right\}.$$

We have taken the same initial conditions as at the beginning of item 7. The numerical solutions of the linear equation (see the corresponding graph in Fig. 6) and of equation (8.74) calculated for $\varepsilon = 10^{-1}$ are identical. The deformation of equilibrium points

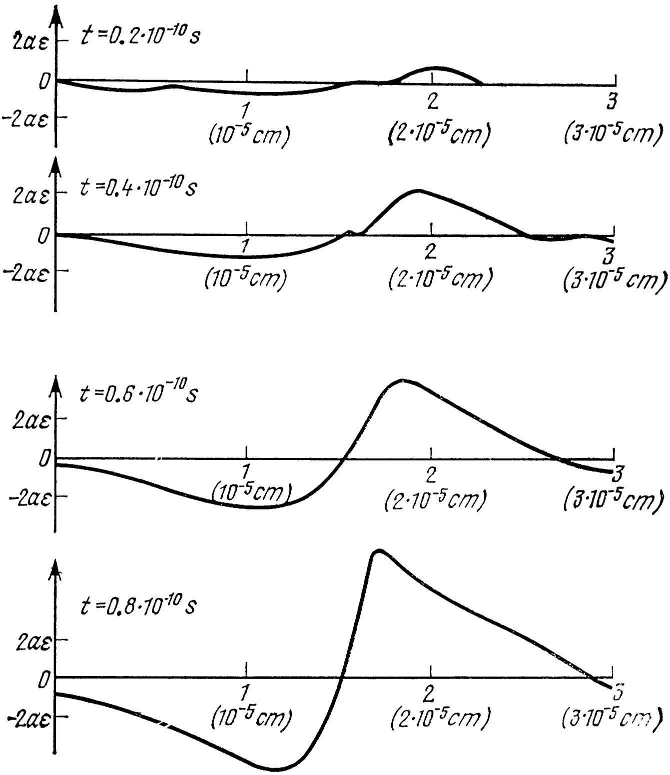

Fig. 10

of the numerical solution of equation (8.74) and the solution obtained by (8.72) are shown respectively in Figs. 10 and 11 for $\varepsilon = 2$, $a = 1$. The moments of time on the graphs are the same as in the linear

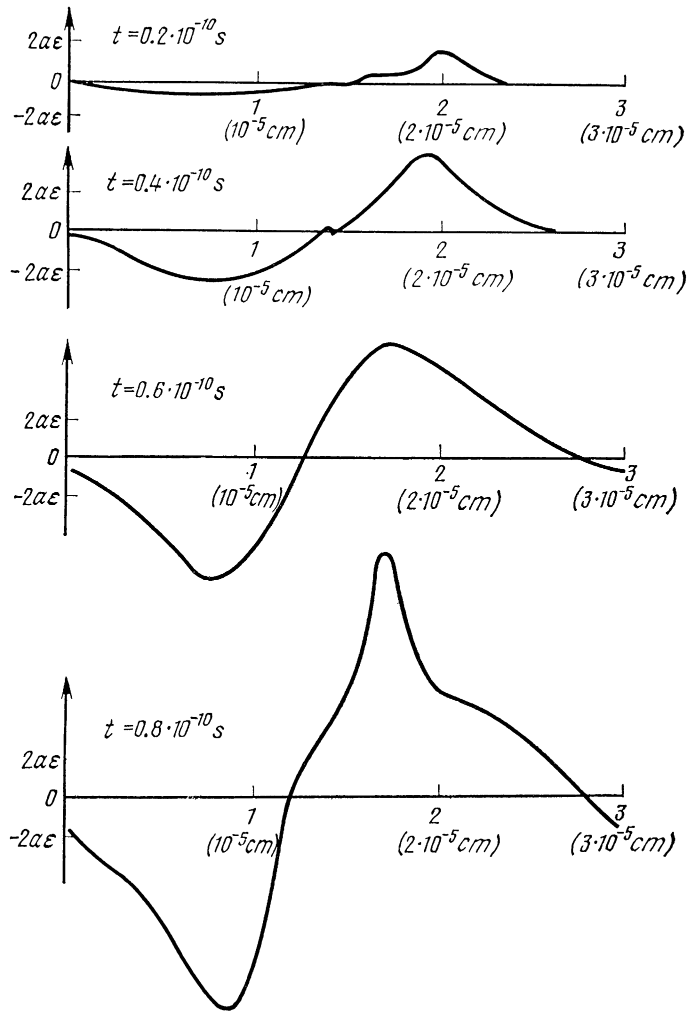

Fig. 11

case. The graph of the exact solution at the moment of generation of the focal point is indicated in Fig. 12. The amplitude of vibrations of the atoms of the lattice is blackened.

The newly found effect may be used to explain a number of physical phenomena. It is related to a well-known effect of the heat

expansion of a crystal, i.e., the transformation of heat vibrations of small amplitude (as if adding them) into a non-oscillating change of a crystal by a huge, in comparison with the amplitude of the heat vibrations, value. The obtained formula may be extended to general equations containing a small quadratic term. A similar

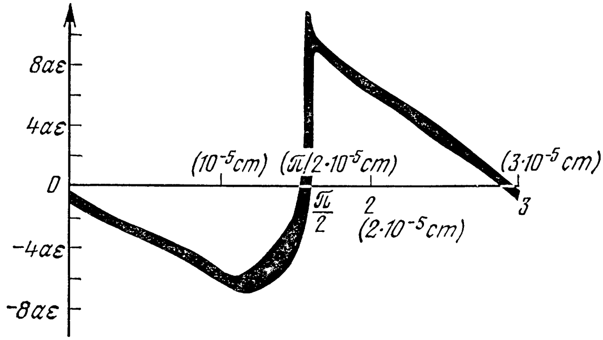

Fig. 12

effect (so-called rectification) of fast-oscillating initial conditions, is known in some branches of physics. Therefore we can say that by the combination of these two effects there emerges a mighty running wave of high energy concentration on the side of which the crysal is pressed, while on the other it is expanding.

Sec. 9. The Concept of a Quasi-Inverse Operator and Formulation of the Main Theorem

(1) The main problem. We have given examples of the application of the method of ordered operators similar to those in Sec. 1, where we gave examples of the solution of equations by Heaviside's operational method.

On the other hand, as we have seen in Sec. 3, the solution of systems of ordinary differential equations with constant coefficients is reduced to the purely algebraic problem. It turns out that it is possible to state and solve the problem, which, in essence, embraces both partial differential equations with variable coefficients and problems on the stability of difference schemes within the framework of algebras with μ-structures; i.e., it is possible to include the solution of all these problems into one algebraic theorem. Though the proof of the theorem is completed only in the last chapter, we shall try to elucidate its meaning and importance.

In the preceding examples the reduction of the given equation to an integral equation of the Volterra type with a smooth non-oscillating kernel was considered to be the solution of the problem.

Such reductions to integral equations (of the Volterra or Fredholm type) with a smooth kernel are the main problem in the theory of partial differential equations.

The concept of the quasi-inverse operator introduced below makes it possible to extend the method of solution of differential equations to the case of operators.

Definition. *Let $\mathcal{L}$ be a module over an algebra $\mathcal{A}$ with the given μ-structure, $A_1, A_2, \ldots, A_n, B \in X$. An element $f \in \mathcal{A}$ is called quasi-inverse, if there exist such sequences of elements $\mathcal{X}_k \in \mathcal{L}$, $\mathcal{X}'_k \in \mathcal{L}$ that the products $f\mathcal{X}_k$ and $\mathcal{X}'_k f$ can be expressed in the form*

$$\begin{aligned} f\mathcal{X}_k &= 1 + R_k\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right), \\ \mathcal{X}'_k f &= 1 + R'_k\left(\overset{n+1}{A_1}, \ldots, \overset{2}{A_n}, \overset{1}{B}\right), \end{aligned} \tag{9.1}$$

where the functions $R_k(x, \alpha)$ and $R'_k(x, \alpha)$ decrease when $|x| \to \infty$ faster than $|x|^{-k}$; $|R_k| = O_{\mathcal{L}}(|x|^{-k})$, $|R'_k| = O_{\mathcal{L}}(|x|^{-k})$.

We shall call the sequences $\{\mathcal{X}_k\}$ right quasi-inverse and $\{\mathcal{X}'_k\}$ — left quasi-inverse.

Note. In the case when the algebra $\mathcal{A}$ is an algebra of unbounded operators (see Chapter I), this definition can be replaced by the following one. *An operator $T \in \mathcal{A}$ is quasi-inverse if there exist such sequences $\{X_k\} \subset \mathcal{A}$, $\{X'_k\} \subset \mathcal{A}$ that $TX_k = 1 + S^+_k$, $X'_k T = 1 + S^-_k$, where operators S^+_k, S^-_k satisfy the conditions*

$$\Big\| \prod_{i=1}^{n} \overset{\cdot}{A_i^{j_i}} S^{\pm}_k \Big\| \leqslant c, \ \forall j_i\colon \sum j_i = k, \ c = \text{const}.$$

In this case the quasi-inverse sequence gives the asymptotic solution of the problem.

Everywhere below the elements f, $\mathcal{X}_k$, $\mathcal{X}'_k$ themselves are also represented in the form $f = f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)$, $\mathcal{X}_k = \mathcal{X}_k\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)$, $\mathcal{X}'_k = \mathcal{X}'_k\left(\overset{n+1}{A_1}, \ldots, \overset{2}{A_n}, \overset{1}{B}\right)$.

Our main problem is to find quasi-inverse sequences for an element $f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)$. We shall solve it for the operators $A_1, \ldots, A_n, B$ and the symbol $f(x, \alpha)$ under some assumptions.

Before formulating these conditions, it is natural to clear up the question of quasi-inverse elements in the trivial case of commutative algebra. In this case it is possible, for example, to consider operators $x_1, \ldots, x_n$ and α, the symbols coinciding with the operators satisfying them. If an operator $f = f(x_1, \ldots, x_n, \alpha)$ has

zeroes when $\alpha = \alpha^0$ and $|x| \to \infty$ (for example, $f = x_1^2 - x_2^2 + + ix_2^2 \sin^2 \alpha$), it is not quasi-inverse (since it is not possible to find such a smooth function $\mathscr{X}(x_1, x_2, \alpha)$ that $(x_1^2 - x_2^2 + ix_2^2 \sin^2 \alpha) \times \times \mathscr{X}(x_1, x_2, \alpha) = 1 + R(x_1, x_2, \alpha)$, where $R(x_1, x_2, \alpha) \to 0$ when $x_1^2 + x_2^2 \to \infty$, because the left-hand side of the equality becomes zero when $x_1 = x_2$, $\alpha = 0$). So it seems natural, at first glance, to demand that the symbol $f(x, \alpha)$ should not have any zeroes when $|x| \to \infty$. But this condition is too rigid: it would deprive us of the main applications. Indeed, all the difficulties of the construction of quasi-inverse sequences emerge when the symbol of the operator $f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)$ becomes zero when $|x| \to \infty$. In this respect the case of the commutative operators is a special one. Let us consider the example of a differential equation in order to comprehend this sudden phenomenon.

In this example, our statement within the framework of commutative algebras seems to be a paradox, since the problem surely has no solution in the simplest case of a differential equation with constant coefficients. It is clear that we would like to reduce the given problem (ignoring certain details and idealizing conditions somewhat) to an equation with constant coefficients. The crux of the matter is not only in the fact that these equations have been studied best of all, but also in the fact that the solution of equations with constant coefficients serve as a model for equations with variable coefficients. As we shall see from the main theorem, however, the case of constant coefficients proves to be very special and degenerate.

For example, consider the symbol $f_1(y_1, y_2) = y_1^2 - y_2^2$, $f_2(y_1, y_2, \alpha) = y_1^2 - y_2^2 - i\varphi(\alpha^2) y_2^2$, where the function $\varphi(\alpha^2) \geqslant 0$ equals zero when $\alpha^2 < T$ and is greater than zero when $2T + \delta \geqslant \geqslant \alpha^2 \geqslant 2T$. Both symbols for $\alpha^2 < T$ coincide and equal zero in the infinity when $y_1 = y_2$. Let $A_1 = x$, $A_2 = y$, $B = i \frac{\partial}{\partial x}$, hence A_1 and A_2 commute. As we have noticed, the operator $\overset{1}{A_1^2} - \overset{2}{A_2^2}$ corresponding to the symbol $f_1(y_1, y_2)$ is not quasi-inverse. But the operator $\overset{1}{A_1^2} - \overset{2}{A_2^2} - i\varphi\left(\overset{3}{B^2}\right) \overset{2}{A_2^2}$ proves to be quasi-inverse, which will follow from the main theorem. Suppose $A_1 = i \frac{\partial}{\partial t}$, $A_2 = i \frac{\partial}{\partial x}$, $B = t$. The operator corresponding to the symbol $f_1(y_1, y_2)$ will be the wave operator $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ (cf. Sec. 8). It is not quasi-inverse. The addition to it of the term $-i\varphi(t^2) \frac{\partial^2}{\partial x^2}$ corresponds to the symbol $f_2(y_1, y_2, \alpha)$ and reflects the phenomenon of the absorption of high-frequency waves for $t^2 > 2T$.

Note. Let the joint spectrum (taking into consideration axiom μ_1') of the operators $\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}$ belong to a domain $D \subset \subset \mathbf{R}^n \times M^m$ and let the function $P(x, \alpha) \in \mathcal{S}^\infty$ equal 1 for $x, \alpha \in \in D$ and zero outside some larger domain. Then, by the definition of the joint spectrum, we have $P\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right) = 1$ and consequently in this case we can substitute $P\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)$ for the unity in the formulas (9.1), since

$$\operatorname{supp}(1 - P(x, \alpha)) \cap \sigma\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right) = \varnothing .$$

Example. Consider the operators

$$A_1 = i\frac{\partial}{\partial x}, \quad A_2 = i\frac{\partial}{\partial y}, \quad A_3 = x, \quad A_4 = y,$$

and let $f(x_1, x_2, x_3, x_4)$ be a polynomial in x_1 and x_2 and $x_3, x_4 \in \in M^2$, i.e.,

$$f\left(\overset{1}{i\frac{\partial}{\partial x}}, \overset{1}{i\frac{\partial}{\partial y}}, \overset{2}{x}, \overset{2}{y}\right)$$

is a differential operator. The construction of the quasi-inverse operator reduces the problem of the differential equation

$$f\left(\overset{1}{i\frac{\partial}{\partial x}}, \overset{1}{i\frac{\partial}{\partial y}}, \overset{2}{x}, \overset{2}{y}\right) u(x, y) = F(x, y)$$

to the problem of the integral equation

$$[\![f\left(\overset{1}{i\frac{\partial}{\partial x}}, \overset{1}{i\frac{\partial}{\partial y}}, \overset{2}{x}, \overset{2}{y}\right)]\!]\, g_k\left(\overset{1}{i\frac{\partial}{\partial x}}, \overset{1}{i\frac{\partial}{\partial y}}, \overset{2}{x}, \overset{2}{y}\right) \varphi_k(x, y) =$$
$$= \left[1 + R_k\left(\overset{1}{i\frac{\partial}{\partial x}}, \overset{1}{i\frac{\partial}{\partial y}}, \overset{2}{x}, \overset{2}{y}\right)\right] \varphi_k(x, y) = F(x, y),$$

where $R_k(x_1, x_2, x_3, x_4)$ tends to zero faster than $(|x_1|^2 + + |x_2|^2)^{-k/2}$. It is not difficult to see (directly from the Fourier transform of the pseudodifferential operator), that the last equality is an integral equation of the Fredholm type of the second order with a smooth rapidly decreasing kernel. In this case the solution of the given problem can be represented in the form

$$u(x, y) = g_k\left(\overset{1}{i\frac{\partial}{\partial x}}, \overset{1}{i\frac{\partial}{\partial y}}, \overset{2}{x}, \overset{2}{y}\right) \varphi_k(x, y),$$

where $\varphi_k(x, y)$ is the solution of the integral equation

$$\varphi_k + R_k\varphi_k = F.$$

Note, that the first term of the iteration equals F, and the second one equals $R_k F$. Thus, the second term is smooth by virtue of the property of the operator R_k. Hence we may conclude that the term $g_k F$ contains the non-smooth part of the solution of the initial problem, i.e. the difference $g_k \varphi_k - g_k F$ is smooth.

(2) The rule for reducing the main problem. Let $A_1, \ldots, A_n$, $B \in X$, B be a vector operator, $B \in M^m$. Suppose, that the spectrum of the pair of the operators $\overset{1}{A}_i$, $\overset{3}{A}_i$ relative to $\overset{2}{A}_j$ is diagonal and has a finite multiplicity, i.e., for sufficiently great r

$$\left(\overset{1}{A}_i - \overset{3}{A}_i\right)^r \overset{2}{A}_j = 0, \quad i, j = 0, 1, 2, \ldots, n, \tag{9.2}$$

where $A_0 = B \in M_m$. Besides, suppose, that $\left(\overset{1}{A}_i - \overset{3}{A}_i\right) \overset{2}{A}_j = -iA_{k(i,\, j)}$ for such k that $\left(\overset{1}{A}_i - \overset{3}{A}_i\right)^k \overset{2}{A}_j \neq \text{const} \cdot 1$.

We shall call the operators $A_1, \ldots, A_n$, B *generative* operators and the algebra $\mathcal{N}$, generated by them, the *Lie nilpotent algebra.*

Note. It is easy to verify that conditions (9.2) are equivalent to the statement that the commutator of order n of A_i and A_j equals zero. Thus we may say that sufficiently high commutators of the elements $A_1, \ldots, A_n$, B equal zero. Besides, for any k, l $(1 \leqslant k, l \leqslant n)$ and any r $(1 \leqslant r \leqslant m)$ there exist the indices $j(k, l)$, $s(k, r)$ such that

$$[A_k, B_r] = -iA_{s(k,\, r)}, \quad [A_k, A_l] = -iA_{j(k,\, l)}.$$

Now we compare every operator A_k to a partial differential operator $L_k\left(\overset{2}{x}, -i\overset{1}{\frac{\partial}{\partial x}}, i\overset{1}{\frac{\partial}{\partial \alpha}}\right)$ acting in the space $C^\infty_{\mathscr{L}}(\mathbf{R}^n \times M^m)$ (here $\alpha = (\alpha_1, \ldots, \alpha_m)$ are local coordinates on M^m, $(x_1, \ldots, x_n) \in \mathbf{R}^n$).

Compare the vector operator B to an operator of multiplication by α.

We shall define the operators L_k, $k = 1, 2, \ldots, n$ with the help of the equality

$$(L_k \varphi)\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right) \overset{\text{def}}{=} A_k [\![\varphi\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right)]\!] \tag{9.3}$$

$(k = 1, 2, \ldots, n)$ for any symbol $\varphi(x, \alpha) \in C^\infty_{\mathscr{L}}(\mathbf{R}^n \times M^m)$. We shall call the operator L_k *a representation* of the operator A_k. Consider first the construction of the operators L_k. Take operators $A_1 \in M$, $B \in M_1$ and

$$A_2 = i[A_1, B], \quad A_3 = i[A_1, A_2], \quad [A_2, B] = 0,$$
$$[A_3, B] = 0, \quad [A_1, A_3] = 0, \quad [A_2, A_3] = 0.$$

Besides, let $A_2, A_3 \in M$. Then A_1, A_2, A_3, B satisfy the conditions given above.

The operator A_3 commutes with all the remaining operators and the operator A_2 commutes with B. Therefore their images L_3, L_2 are easily constructed

$$L_3 = x_3, \qquad L_2 = x_2.$$

Find the operator L_1. Consider $\overset{5}{A_1}\varphi\left(\overset{1}{A_1}, \overset{2}{A_2}, \overset{3}{A_3}, \overset{4}{B}\right)$. Expanding φ in the Taylor series in $\overset{4}{B} - \overset{6}{B}$ (see Sec. 5, Sec. 6), we obtain

$$\overset{5}{A_1}\varphi\left(\overset{1}{A_1}, \overset{2}{A_2}, \overset{3}{A_3}, \overset{4}{B}\right) = \overset{5}{A_1}\varphi\left(\overset{1}{A_1}, \overset{2}{A_2}, \overset{3}{A_3}, \overset{6}{B}\right) + \\ + \overset{5}{A_1}\left(\overset{4}{B} - \overset{6}{B}\right)\frac{\partial\varphi}{\partial\alpha}\left(\overset{1}{A_1}, \overset{2}{A_2}, \overset{3}{A_3}, \overset{6}{B}\right),$$

since it is given $\overset{5}{A_1}\left(\overset{4}{B} - \overset{6}{B}\right)^k = 0$ for $k \geqslant 2$.

Thus

$$A_1[\![\varphi\left(\overset{1}{A_1}, \overset{2}{A_2}, \overset{3}{A_3}, \overset{4}{B}\right)]\!] = \overset{3}{A_1}\varphi\left(\overset{1}{A_1}, \overset{2}{A_2}, \overset{5}{A_3}, \overset{6}{B}\right) + \\ + \left(-i\overset{2}{A_2}\right)\frac{\partial\varphi}{\partial\alpha}\left(\overset{1}{A_1}, \overset{2}{A_2}, \overset{3}{A_3}, \overset{4}{B}\right).$$

We shall expand the first summand on the right-hand side in the Taylor series in $\overset{2}{A_2} - \overset{4}{A_2}$. Considering $\overset{3}{A_1}\left(\overset{2}{A_2} - \overset{4}{A_2}\right)^k = 0$ when $k \geqslant 2$, we obtain

$$A_1[\![\varphi\left(\overset{1}{A_1}, \overset{2}{A_2}, \overset{3}{A_3}, \overset{4}{B}\right)]\!] = \overset{3}{A_1}\varphi\left(\overset{1}{A_1}, \overset{4}{A_2}, \overset{5}{A_3}, \overset{6}{B}\right) + \\ + \overset{3}{A_1}\left(\overset{3}{A_2} - \overset{4}{A_2}\right)\frac{\partial\varphi}{\partial x_2}\left(\overset{1}{A_1}, \overset{4}{A_2}, \overset{5}{A_3}, \overset{6}{B}\right) + \\ + \left(-i\overset{2}{A_2}\right)\frac{\partial\varphi}{\partial\alpha}\left(\overset{1}{A_1}, \overset{2}{A_2}, \overset{3}{A_3}, \overset{4}{B}\right).$$

Hence we see that we can obtain equality (9.3) (for $k = 1$) by getting

$$L_1 = x_1 - ix_3\frac{\partial}{\partial x_2} - ix_2\frac{\partial}{\partial\alpha}.$$

In the same way, for any set $A_1, A_2, \ldots, A_n, B$ of generators of the Lie nilpotent algebra there exists the representation L_k of the operator A_k satisfying condition (9.3). Indeed, to substitute A_k for the operator in the kth place in the operator $\overset{n+2}{A_2}\cdot\varphi\left(\overset{1}{A_1}, \ldots\right.$

$\ldots, \overset{n}{A}_n, \overset{n+1}{B}\Big)$ we tug A_k with the help of the Taylor expansion of φ computing a number of commutators $[A_k, B]$, $[[A_k, B], B]$, ... first with B, then with $A_n, A_{n-1}, \ldots$, etc. In the process many commutators emerge: A_k with B and with $A_n, A_{n-1}, \ldots, A_{k-1}$. By definition these commutators are some operators from the set $(A_1, A_2, \ldots, A_n)$. At the next stage we shall tug every of the commutators to its place with the Taylor expansion. Here again a set of commutators emerges, but they are all commutators from $A_1, A_2, \ldots, A_n$ and we shall tug them to their place. It is clear, the process results in obtaining large commutators which commute with all $A_1, A_2, \ldots, A_n$ (since $A_1, A_2, \ldots, A_n, B$ are the generators of the nilpotent algebra). These large commutators take the indicated places automatically. At last it is worth noting that every application of the Taylor expansion to φ produces a sum of several derivatives of φ. The order of these derivatives cannot exceed n_0, i.e., the maximal order of the commutator of $A_1, \ldots, A_n$. Therefore

$$A_k \left[\!\left[\varphi \left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B} \right) \right]\!\right] = \varphi_k \left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B} \right),$$

where $\varphi_k(x, \alpha)$ is the result of the application to the symbol φ of a differential operator $L_k\left(x, i\frac{\partial}{\partial x}, i\frac{\partial}{\partial \alpha}\right)$ of the order not greater than n_0. Let $T = \sum\limits_{j, \alpha} C_{j, \alpha} A_{j1}^{\alpha_{j1}} \cdot \ldots \cdot A_{jk}^{\alpha_{jk}}$. Take

$$\lambda(T) = \sum_{j, \alpha} C_{j, \alpha} (L_{j1})^{\alpha_{j1}} \cdot \ldots \cdot (L_{jk})^{\alpha_{jk}}.$$

If $T \in \mathcal{N}$, then for any symbol $\varphi \in C^\infty_{\mathcal{L}}(\mathbf{R}^n \times M^m)$

$$T \left[\!\left[\varphi \left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B} \right) \right]\!\right] = (\lambda(T)\varphi) \left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B} \right). \tag{9.4}$$

Indeed, take for example $T = A_k A_j$, then by (9.3) we get

$$A_k A_j \left[\!\left[\varphi \left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B} \right) \right]\!\right] = A_k \left[\!\left[(L_j \varphi) \left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B} \right) \right]\!\right] =$$
$$= (L_k L_j \varphi) \left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B} \right) \equiv$$
$$\equiv (\lambda(A_k A_j)\varphi) \left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B} \right).$$

Equality (9.4) for all elements T of the algebra $\mathcal{N}$ is verified in the similar way. Thus the operators L_j are themselves the generators of the Lie nilpotent algebra, which we shall denote by Π.

The constructed mapping λ: $\mathcal{N} \to \Pi$ is called *an ordered representation of the nilpotent algebra* $\mathcal{N}$.

It follows from (9.4) that the operators L_k, $k = 1, 2, \ldots, n$; α satisfy the same commutating conditions as the operators A_k, B. Particularly the operators $L_1, L_2, \ldots, L_n$, α are generators of the nilpotent algebra Π. Hence if $P(x, \alpha)$ is a polynomial in x with coefficients depending on α, then

$$[\![P\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)]\!]\cdot[\![G\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)]\!] = \\ = \psi\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right), \tag{9.5}$$

where $\psi(x, \alpha) = P\left(\overset{1}{L_1}, \ldots, \overset{n}{L_n}, \overset{n+1}{\alpha}\right) G(x, \alpha)$ for any $G(x, \alpha) \in C^\infty_{\mathscr{L}}$. We can ask, whether this equality holds for $P(x, \alpha) \in \mathscr{S}^\infty$ and $G(x, \alpha) \in C^\infty_{\mathscr{L}}$.

In all probability the answer is negative if we limit ourselves to axioms μ_1-μ_6 (without topological axioms). To give a positive answer to the question we shall formulate two additional algebraic axioms, which are useful and, in addition, permit to obtain an explicit formula for any function of non-commuting operators via a composition of functions of commuting operators.

Consider symbols $f(x, \alpha, t)$ depending on the parameters $t \in \mathbf{R}^k$ (k is not fixed); all the derivatives with respect to the parameters belong to $\mathscr{S}^\infty$.

Axiom (μ_7) (the parameter axiom). Let $f(x, \alpha, t) \in C^\infty_{\mathscr{L}}$, $t \in \mathbf{R}^k$ verify the equation

$$[\![f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}, t\right)]\!] = 0$$

for all $t \in \mathbf{R}^k$. Then for any function $\Phi(t) \in \mathscr{S}^\infty(\mathbf{R}^k)$ and symbols

$$h(x, \alpha, t) = \Phi\left(i\frac{\partial}{\partial t_1}, \ldots, i\frac{\partial}{\partial t_k}\right) f(x, \alpha, t)$$

the condition is satisfied

$$[\![h\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}, t\right)]\!] = 0.$$

If the condition of Axiom (μ_7) is satisfied for all $t \neq 0$, it is valid for $t = 0$ as well.

Axiom (μ_8) (the uniqueness axiom). Let the following conditions be verified

$$(\mu_8^a)\ [\![f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}, 0\right)]\!] = 0,$$

$$(\mu_8^b)\ [\![[f'_t]\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}, t\right)]\!] = i\overset{n+2}{A} f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}, t\right)]\!]$$

for $A \in M$ and symbols $f(x, \alpha, t) \in \mathscr{S}^\infty$, $t \in \mathbf{R}$.

Then

$$[\![f\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}, t\right)]\!] = 0$$

for all $t \in \mathbf{R}$.

Problem. Prove the following formula for an operator $f\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n\right)$

$$f\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n\right) = \left[f\left(i\frac{\partial}{\partial t_1}, \ldots, i\frac{\partial}{\partial t_n}\right) \times \right.$$

$$\left. \times \left(e^{-iA_nt_1}e^{-iA_{n-1}t_2}\ldots e^{-iA_1t_n}\right)\right]_{t=0}.$$

We shall prove in Chapter II, that the positive answer to the question posed above (see (9.5)) follows from these axioms.

The main point is to construct such a function $G_N(x, \alpha)$ for the given function $P(x, \alpha)$ that condition (9.5) is verified when

$$\psi(x, \alpha) = 1 + R_N(x, \alpha),$$

where the function $R_N(x, \alpha) = O_{\mathscr{L}}(|x|^{-N})$ for any N. It means that we should solve the problem (9.5); i.e., the main problem (9.1) is reduced to the following: find the sequence $g_N(x, \alpha)$ such that

$$f\left(\overset{1}{L}_1, \ldots, \overset{n}{L}_n, \overset{n+1}{\alpha}\right) g_N(x, \alpha) = 1 + R_N(x, \alpha), \tag{9.6}$$

where $R_N = O_{\mathscr{L}}(|x|^{-N})$ decreases faster than $|x|^{-N}$ when $|x| \to \infty$.

Suppose we have constructed the function

$$\psi_N(\alpha, \eta, x, t) \in C^\infty_{\mathscr{L}}(M^m \times \mathbf{R}^{2n}); \qquad t \in [0, \infty)$$

(for some numbers $\rho_k \geqslant 1$, $\mu \geqslant 1$, T, η_0 and a finite function $\rho(\eta)$ which equals unity when $|\eta| \leqslant \eta_0$) decreasing for $t = T$ when $x \to \infty$ faster than Λ^{-N}, where

$$\Lambda = \left(\sum_{k=1}^{n}(x_k)^{2/\rho_k} + 1\right)^{1/2}$$

and which satisfies the following conditions:

$$[\![i\overset{3}{\Lambda}{}^{\overset{1'}{\mu}} \frac{\partial}{\partial t} - f\left(\overset{1}{L}_1, \ldots, \overset{n}{L}_n, \overset{n+1}{\alpha}\right)]\!] \psi_N\left(\alpha, i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}, t\right) \doteq$$

$$\doteq B_N\left(\alpha, i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}, t\right), \tag{9.7}$$

$$\psi_N(\alpha, \eta, x, 0) = \rho(\eta),$$

where $B_N(\alpha, \eta, x, t)$ decreases faster than Λ^{-N} as $x \to \infty$ uniformly in t. Here and below $F \doteq \hat{f}$ denotes $F\ \varphi(x) = \hat{f}\varphi(x)$, $\forall\ \varphi \in C_0^\infty$.

We shall integrate equation (9.7) with respect to t from 0 to T and apply both sides of equality (9.7) to the unit function; taking into consideration that $\rho\left(i\frac{\partial}{\partial x}\right)1 = 1$ and denoting

$$\psi'_N(\alpha, x, t) = \psi_N\left(\alpha, i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}, t\right)\overset{1}{\Lambda}{}^{-\mu}\ 1,$$

$$B'_N(\alpha, x, t) = B_N\left(\alpha, i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}, t\right)\overset{1}{\Lambda}{}^{-\mu}\ 1, \tag{9.8}$$

$$g_N(x, \alpha) = +i\int_0^T \psi'_N(\alpha, x, t)\,dt,$$

we obtain the following equation:

$$f\left(\overset{1}{L_1}, \ldots, \overset{n}{L_n}, \overset{n+1}{\alpha}\right) g_N(x, \alpha) =$$
$$= 1 - \psi'_N(\alpha, x, T) - i\int_0^T B'_N(\alpha, x, t)\,dt \overset{\text{def}}{=} 1 + R_N(\alpha, x),$$

where R_N decreases faster than Λ^{-N} as $x \to \infty$ by the initial conditions and therefore we obtain a solution of the main problem.

Example. Consider the equation

$$-i\frac{\partial u}{\partial t} + i\frac{\partial u}{\partial y} - yu = 0, \quad u|_{t=0} = f(y), \quad f(y) \in C(R^1). \tag{9.9}$$

We shall use this equation to demonstrate the general method of constructing solutions up to the functions of the class C^N in y and decreasing faster than $|y|^{-N}$ as $|y| \to \infty$ ($N > 0$ is a given number). Let A_1, A_2, B be the operators

$$A_1 = -i\frac{\partial}{\partial y}, \quad A_2 = y, \quad B_1 = t, \quad B_2 = y. \tag{9.10}$$

The operators are evidently generators of the Lie nilpotent algebra. Calculate their representation. Consider an operator $\varphi\left(\overset{1}{A_1}, \overset{2}{A_2}, \overset{3}{B}\right)\overset{4}{A_1}$. We must transform in such a way that the operator A_1 should act first in the obtained operator; to this end we must pass the operator

A_1 through the operators $\overset{2}{A}_2$, $\overset{3}{B}_2$. We have

$$\overset{4}{A}_1\varphi\left(\overset{1}{A}_1, \overset{2}{A}_2, \overset{3}{B}\right) = \overset{4}{A}_1\varphi\left(\overset{1}{A}_1, \overset{2}{A}_2, \overset{5}{B}\right) + \\ + \overset{4}{A}_1\left(\overset{3}{B} - \overset{5}{B}\right)\frac{\partial\varphi}{\partial\alpha_1}\left(\overset{1}{A}_1, \overset{2}{A}_2, \overset{5}{B}\right) = \\ = \overset{4}{A}_1\varphi\left(\overset{1}{A}_1, \overset{2}{A}_2, \overset{5}{B}\right) - i\frac{\partial\varphi}{\partial\alpha_1}\left(\overset{1}{A}_1, \overset{2}{A}_2, \overset{3}{B}\right). \tag{9.11}$$

Commutate $\overset{4}{A}_1$ and $\overset{2}{A}_2$

$$\overset{4}{A}_1\varphi\left(\overset{1}{A}_1, \overset{2}{A}_2, \overset{5}{B}\right) = \overset{4}{A}_1\varphi\left(\overset{1}{A}_1, \overset{5}{A}_2, \overset{6}{B}\right) + \\ + \overset{4}{A}_1\left(\overset{2}{A}_2 - \overset{5}{A}_2\right)\frac{\partial\varphi}{\partial x_2}\left(\overset{1}{A}_1, \overset{5}{A}_2, \overset{6}{B}\right) = \\ = \overset{1}{A}_1\varphi\left(\overset{1}{A}_1, \overset{2}{A}_2, \overset{3}{B}\right) - i\frac{\partial\varphi}{\partial x_2}\left(\overset{1}{A}_1, \overset{2}{A}_2, \overset{3}{B}\right). \tag{9.12}$$

By Eq. (9.11) and (9.12) we have

$$L_1 = x_1 - i\frac{\partial}{\partial\alpha_1} - i\frac{\partial}{\partial x_2}.$$

The representation L_2 of the operator A_2 evidently has the form

$$L_2 = x_2.$$

Now we shall observe the following general fact. Let us find such an operator $g\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}, t\right)$ that

$$-i\frac{\partial}{\partial t}[\![g]\!] + [\![f]\!][\![g]\!] = R_N\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}, t\right), [\![g]\!]|_{t=0} = 1, \tag{9.13}$$

where the symbol $R_N(x, \alpha, t)$ satisfies an estimate $R_N = O_{\mathscr{L}}(|x|^{-N})$ uniformly with respect to α, t. The symbol $g(x, \alpha, t)$ may be determined by the equation

$$g(x, \alpha, t) = [\![g'\left(\overset{1}{x}, -i\overset{2}{\frac{\partial}{\partial x}}, \alpha, t\right)]\!]1(x), \tag{9.14}$$

where $1(x)$ is the unit function of x, and $g'\left(\overset{1}{x}, -i\overset{2}{\frac{\partial}{\partial x}}, \alpha, t\right)$ is a solution of the problem (cf. (9.7)):

$$-i\frac{\partial}{\partial t}g' + f\left(\overset{1}{L}_1, \ldots, \overset{n}{L}_n, \overset{n+1}{\alpha}, t\right)g' \doteq R'_N, \left(\overset{1}{x}, -i\overset{2}{\frac{\partial}{\partial x}}, \alpha, t\right), \tag{9.15}$$

$$g'(x, \eta, \alpha, 0) = \rho(\eta), \quad R'_N(x, \eta, \alpha, t) = O_{\mathscr{L}}(|x|^{-N}).$$

The proof of the statement follows from (9.5). Hence, to construct a solution of problem (9.9) we must solve the problem

$$-i\frac{\partial\Psi}{\partial t}-(L_1+L_2)\Psi\doteq 0,\quad \Psi=\Psi\left(\overset{1}{x},-i\overset{2}{\frac{\partial}{\partial x}},\alpha,t\right). \tag{9.16}$$

$$\Psi|_{t=0}=\rho\left(-i\frac{\partial}{\partial x}\right),\quad \rho(\eta)\in C_0^\infty(\mathbf{R}^n)$$

$$\rho(\eta)=1 \text{ as } |\eta|\leqslant\varepsilon.$$

We shall search the solution of (9.16) in the form

$$\Psi=e^{i\overset{1}{\Lambda}S\left(\overset{1}{x},\,-i\overset{2}{\frac{\partial}{\partial x}},\,\alpha,t\right)}\varphi\left(\overset{1}{x},\ -i\overset{2}{\frac{\partial}{\partial x}},\ \alpha,\ t\right), \tag{9.17}$$

where

$$S(x,\ \eta,\ \alpha,\ 0)=0,\qquad \varphi(x,\ \eta,\ \alpha,\ 0)=\rho(\eta),\quad \Lambda=|x|.$$

On substituting the function Ψ defined by (9.17) into (9.16) we obtain by the commutation formulas (we drop the arguments $\overset{1}{x},-i\overset{2}{\frac{\partial}{\partial x}}$, α, t of φ, S):

$$\overset{1}{\Lambda}\frac{\partial S}{\partial t}\varphi-i\frac{\partial\varphi}{\partial t}-\left[x_1-\overset{1}{\Lambda}\frac{\partial S}{\partial\eta_1}+\overset{1}{\Lambda}\frac{\partial S}{\partial\alpha_1}\right]\varphi+i\overset{2}{\frac{\partial}{\partial x}}\varphi-$$

$$-\left[\overset{1}{x_2}-\overset{1}{\Lambda}\frac{\partial S}{\partial\eta_2}\right]\varphi-i\frac{\partial\varphi}{\partial\eta_1}-i\frac{\partial\varphi}{\partial\eta_2}=0.$$

Let $x_1\Lambda^{-1}=\omega_1$, $x_2\Lambda^{-1}=\omega_2$. Using the symbols and equating to zero the coefficients at the powers of Λ we obtain

$$\left\{\begin{aligned}&\frac{\partial S}{\partial t}-\omega_1+\frac{\partial S}{\partial\eta_1}-\frac{\partial S}{\partial\alpha_1}-\omega_2+\frac{\partial S}{\partial\eta_2}=0, &&(9.18)\\&\frac{\partial\varphi}{\partial t}-i\eta_2\varphi+\frac{\partial\varphi}{\partial\eta_1}+\frac{\partial\varphi}{\partial\eta_2}=0. &&(9.19)\end{aligned}\right.$$

Equations (9.18), (9.19) have solutions of the form

$$S=(\omega_1+\omega_2)t$$

$$\varphi=e^{\frac{i\eta_2^2}{2}-\frac{i(\eta_2-t)^2}{2}}\rho(\eta_1-t,\ \eta_2-t),$$

thus

$$\Psi(x,\alpha,\eta,t)=e^{i(x_1+x_2)t}e^{i\frac{\eta_2^2}{2}-i\frac{(\eta_2-t)^2}{2}}\rho(\eta_1-t,\ \eta_2-t).$$

To construct the solution of the initial problem we must calculate the value of $\Psi\left(\overset{1}{x},\ -i\overset{2}{\frac{\partial}{\partial x}},\ \alpha,t\right)1(x)$.

We have

$$\Psi\left(\overset{1}{x},\ -i\overset{2}{\frac{\partial}{\partial x}},\ \overset{2}{\alpha},\ t\right)(1\,(x)) = e^{\frac{i}{2}\left(-i\frac{\partial}{\partial x_2}\right)^2-\frac{i}{2}\left(-i\frac{\partial}{\partial x_2}-t\right)^2}\times$$

$$\times\,\rho\left(-i\frac{\partial}{\partial x_1}-t,\ -i\frac{\partial}{\partial x_2}-t\right)e^{i\,(x_1+x_2)\,t}=$$

$$= e^{i\,(x_1+x_2)\,t}\,e^{\frac{it^2}{2}} = G\,(x,\ \alpha,\ t).$$

The solution of (9.9) has the form

$$u\,(y,\ t) = G\left(\overset{1}{A_1},\ \overset{2}{A_2},\ \overset{3}{B},\ t\right)f = e^{t\overset{1}{\frac{\partial}{\partial y}}}e^{i\overset{2}{y}t}\,e^{\frac{it^2}{2}}f = e^{iyt+\frac{it^2}{2}}f\,(y+t)$$

Thus an application of the operational method enables us to solve the problem exactly.

Now let $a\,(y)$ be a positive smooth function uniformly bounded with its derivatives in the region $y \in \mathbf{R}^1$, $0 < C_1 \leqslant a\,(y) \leqslant C_2$.

Consider the equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial y^2}-y^2 a\,(y)\,u = 0 \tag{9.20}$$

with the initial conditions

$$u\,|_{t=0} = u_1\,(y), \qquad u_t'\,|_{t=0} = 0, \tag{9.21}$$

where the function $u_1\,(y)$ belongs to the space $H_s\,(\mathbf{R})$ for some s. We shall investigate the same problem for (9.20) with initial conditions as for Eq. (9.9).

Let A_1, A_2, B be the operators defined by equation (9.10). Their ordered representation was calculated in the previous example.

Rewrite (9.20) in the form

$$\frac{\partial^2 u}{\partial t^2}-[\![A_1^2+a\,(B_2)\,A_2^2]\!]\,u = 0.$$

Similarly to the previous example it is sufficient for the solution of the given problem to construct an operator $g_N'\,(\overset{1}{x},\ \overset{2}{-i\,\partial/\partial x},\ \alpha,\ t)$ satisfying the conditions

$$\frac{\partial^2 g_N'}{\partial t^2}-[\![L_1^2+a\,(\alpha)\,L_2^2]\!]\,g_N' \doteq O_{\mathscr{L}}(|\,x\,|^{-N}),$$

$$g_N'\left(x,\ \eta,\ \alpha,\ 0\right) = \rho\,(\eta),\quad \frac{\partial g_N'}{\partial t}\left(x,\ \eta,\ \alpha,\ 0\right) = 0. \tag{9.22}$$

We shall search for the solution of (9.22) in the form

$$g'_N\left(\overset{1}{x},\ -i\overset{2}{\frac{\partial}{\partial x}},\ \alpha,\ t\right)=e^{\overset{1}{i\Lambda S}\left(\overset{1}{x},\ -i\overset{2}{\frac{\partial}{\partial x}},\ \alpha,\ t\right)}\times$$
$$\times\Psi_N\left(\overset{1}{x},\ -i\overset{2}{\frac{\partial}{\partial x}},\ \alpha,\ t\right),$$
$$\Lambda=(x_1^2+x_2^2)^{1/2}. \tag{9.23}$$

On substituting the function g'_N defined by (9.23) into (9.22) we obtain by the commutation formulas

$$e^{\overset{1}{i\Lambda S}\left(\overset{1}{x},\ -i\overset{4}{\frac{\partial}{\partial x}},\ \overset{4}{\alpha},\ t\right)}\left\{\overset{1}{\Lambda^2}\overset{3}{[\![}-\left(\frac{\partial S}{\partial t}-i\overset{1'}{\Lambda^{-1}}\frac{\partial}{\partial t}\right)^2+\right.$$
$$+\left(x_1\overset{1'}{\Lambda^{-1}}+\frac{\partial S}{\partial\eta_1}+\frac{\partial S}{\partial\alpha_1}-i\overset{1'}{\Lambda^{-1}}\frac{\partial}{\partial\alpha}-i\overset{1'}{\Lambda^{-i}}\frac{\partial}{\partial x_2}\right)^2+$$
$$\left.+a\left(\overset{2}{\alpha}\right)\left(x_2\overset{1'}{\Lambda^{-1}}+\frac{\partial S}{\partial\eta_2}\right)^2]\!]\,\Psi_N\right\}\doteq O_{\mathscr{L}}(|x|^{-N}). \tag{9.24}$$

The arguments $\overset{1'}{x}$, $-i\overset{2}{\frac{\partial}{\partial x}}$, α, t of S and $\overset{1}{x}$, $-i\overset{2}{\frac{\partial}{\partial x}}$, α, t of Ψ_N are omitted.

Apply the K-formula to the term in the braces in (9.24). We obtain that the symbol of the operator

$$\overset{3}{[\![}-\left(\frac{\partial S}{\partial t}-i\overset{1'}{\Lambda^{-1}}\frac{\partial}{\partial t}\right)^2+\left(x_1\overset{1'}{\Lambda^{-1}}+\frac{\partial S}{\partial\eta_1}+\frac{\partial S}{\partial\alpha_1}-i\overset{1'}{\Lambda^{-1}}\frac{\partial}{\partial\alpha}-\right.$$
$$\left.-i\overset{1'}{\Lambda^{-1}}\frac{\partial}{\partial x_2}\right)^2+a(\alpha)\left(x_2\overset{1'}{\Lambda^{-1}}+\frac{\partial S}{\partial\eta_2}\right)^2]\!]\,\Psi_N$$

is

$$\left\{-\left(\frac{\partial S}{\partial t}\right)^2+\left(\omega_1+\frac{\partial S}{\partial\eta_1}+\frac{\partial S}{\partial\alpha_1}\right)^2+a(\alpha)\left(\omega_2+\frac{\partial S}{\partial\eta_2}\right)^2+\right.$$
$$+(-i\Lambda^{-1})\left[-2\frac{\partial S}{\partial t}\frac{\partial\varphi}{\partial t}-\frac{\partial^2 S}{\partial t^2}+2a(\alpha)\left(\omega_2+\frac{\partial S}{\partial\eta_2}\right)\frac{\partial}{\partial\eta_2}+\right.$$
$$+a(\alpha)\frac{\partial^2 S}{\partial\eta_2^2}+2\eta_2\left(\omega_1+\frac{\partial S}{\partial\eta_1}+\frac{\partial S}{\partial\alpha_1}\right)+2\left(\omega_1+\frac{\partial S}{\partial\eta_1}+\right.$$
$$\left.\left.+\frac{\partial S}{\partial\alpha_2}\right)\left(\frac{\partial}{\partial\alpha_1}+\frac{\partial}{\partial\eta_1}\right)+\frac{\partial^2 S}{\partial\eta_1^2}+2\frac{\partial^2 S}{\partial\eta_1\,\partial\alpha_1}+\frac{\partial^2 S}{\partial\alpha_1^2}\right]+$$
$$\left.+\sum_{k=2}^{N}(-i\Lambda^{-1})^k R_k\right\}\Psi_N+\Lambda^{-N+1}R_N\Psi_N, \tag{9.25}$$

where $\omega_i=x_i\Lambda^{-1}$, $i=1,2$; R_k are differential operators of the order k with smooth coefficients, R_N is the remainder term in the commutation formula.

Note that the successive application of the K-formula enables us to determine an explicit form for all the operators.

Equation (9.22) is an easy corollary of the equations

$$\left(\frac{\partial S}{\partial t}\right)^2-\left(\omega_1+\frac{\partial S}{\partial \eta_1}+\frac{\partial S}{\partial \alpha_1}\right)^2-a(\alpha)\left(\omega_2+\frac{\partial S}{\partial \eta_2}\right)^2=0, \tag{9.26}$$

$$\left(\sum_{k=1}^{N}(-i\Lambda^{-1})^k R_k\right)\Psi_N=O(|x|^{-N}), \tag{9.27}$$

where R_1 is the operator in the square brackets on the right-hand side of (9.25) and $O(|x|^{-N})$ is a smooth function decreasing as $|x|^{-N}$ with its derivatives as $|x|\to\infty$.

Equation (9.26) is a Hamilton-Jacobi equation with the initial condition $S|_{t=0}=0$.

We shall search for the function Ψ_N in the form of the expansion

$$\Psi_N=\sum_{k=0}^{N-1}(-i\Lambda^{-1})^k\,\widetilde{\Psi}_k.$$

On substituting this expansion into (9.27) and equating to zero the coefficients of the powers Λ^{-k}, $k=0,\ldots,N$ we obtain the system

$$\begin{aligned}
&R_1\widetilde{\Psi}_0=0,\\
&R_1\widetilde{\Psi}_1+R_2\widetilde{\Psi}_0=0\\
&\cdots\cdots\cdots\\
&R_1\widetilde{\Psi}_{N-1}+\ldots+R_{N-1}\widetilde{\Psi}_0=0,\\
&\widetilde{\Psi}_0|_{t=0}=\rho(\eta);\quad \widetilde{\Psi}_k|_{t=0}=0,\quad k>0.
\end{aligned} \tag{9.28}$$

Equations (9.26), (9.28) are solved similarly to (8.32) of Sec. 8. Denote their solutions by $S^{\pm}$, $\widetilde{\Psi}_0^{\pm}$, $\ldots$, $\widetilde{\Psi}_{N-1}^{\pm}$. Let

$$g_N'\left(\overset{1}{x},\ -i\overset{2}{\frac{\partial}{\partial x}},\ \alpha,\ t\right)=\frac{1}{2}\sum_{\pm}\sum_{k=0}^{N-1}e^{i\overset{1}{\Lambda}S_{\pm}\left(\overset{1}{x},\ -i\overset{2}{\frac{\partial}{\partial x}},\ \alpha,\ t\right)}\times$$
$$\times(-i\overset{1}{\Lambda}{}^{-1})^k\,\widetilde{\Psi}_k^{\pm}\left(\overset{1}{x},\ -i\overset{2}{\frac{\partial}{\partial x}},\ \alpha,\ t\right),$$

then the asymptotic solution of (9.20)-(9.21) is a function

$$u^{(N)}(y,t)=[\![g_N\left(\overset{1}{A_1},\ \overset{2}{A_2},\ \overset{2}{B},\ t\right)]\!]\,u_1(y),$$

where $g_N\left(\overset{1}{A_1},\ \overset{2}{A_2},\ \overset{2}{B},\ t\right)$ is an operator with the symbol

$$g_N(x,\alpha,t)=[\![g_N'\left(\overset{1}{x},\ -i\overset{2}{\frac{\partial}{\partial x}},\ \alpha,\ t\right)]\!]\,1(x).$$

Sometimes it happens to be useful to reduce the solution of the main problem to an exact solution of a simpler equation. We have seen, that the asymptotic of the equations of crystal vibrations might not be found generally only with the help of bicharacteristics. We are able to reduce merely the solution of the asymptotic problem to a simpler one; i.e., an exact solution of the wave equation (see Sec. 8).

This section deals with the construction of a quasi-inverse operator of an operator $f\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right)$.

Choose a certain operator $r\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right) \in X$ as a standard one.

Definition 9.1′. *A right quasi-inverse sequence for an operator* $f\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right)$ *is such a sequence*

$$\varkappa_k\left(\overset{1}{[\![} r\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right)]\!], \overset{2}{A}_1, \ldots, \overset{n+1}{A}_n, \overset{n+2}{B}\right) \in \mathscr{L},$$

that the product $f\varkappa_k$ *can be put in the form*

$$f\varkappa_k = 1 + R_k\left(\overset{1}{r}, \overset{2}{A}_1, \ldots, \overset{n+1}{A}_n, \overset{n+2}{B}\right),$$

where the function $R_k(x_0, x_1, \ldots, x_i, x_{i+1}, \ldots, x_n, \alpha)$ *decreases faster than* $\left[\sum_{j=1}^{N} x_j^2\right]^{-k/2}$ *as* $|x| \to \infty$, *i.e.*,

$$R_k = O_{\mathscr{L}}\left(\left(\sum_{j=1}^{N} x_j^2\right)^{-k/2}\right).$$

The left quasi-inverse sequence for the operator $f\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right)$ is defined along the same lines (cf. 9.1).

We shall investigate the method of reduction of the main problem to a solution of the Cauchy problem for a function of differential operators.

Let $G_k\left(\overset{2}{\alpha}, \overset{2}{x}, -i\overset{1}{\frac{\partial}{\partial\alpha}}, -i\overset{1}{\frac{\partial}{\partial x}}\right)$ be a differential operator with partial derivatives defined in the space $C^{\infty}_{\mathscr{L}}(\mathrm{R}^n \times M^m)$. Denote this operator by G_k. We define G_k by the following equation:

$$(G_k\varphi)\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right) = [\![\varphi\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right)]\!] A_k,$$

Further, we define G_{n+1} as follows:

$$(G_{n+1}\varphi)\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right) = [\![\varphi\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right)]\!] B.$$

The operators G_k are generators of the Lie nilpotent algebra which will be denoted by Π'. The proof is the same as in Sec. 9. The mapping $\lambda' : \mathscr{N} \to \Pi'$ is called the (right) ordered representation of the algebra $\mathscr{N}$.

An analogue of (9.5) is true

$$[\![\varphi\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)]\!][\![P(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B})]\!]= =P\left(\overset{n+1}{G_1}, \ldots, \overset{2}{G_n}, \overset{1}{G_{n+1}}\right)\varphi(x, \alpha)\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right), \tag{9.29}$$

where $\varphi(x, \alpha)\in C^\infty_{\mathscr{L}}$, $P(x, \alpha)$ is a polynomial.

Equation (9.29) is verified for any $P(x, \alpha)\in\mathscr{S}^\infty(\mathbf{R}^n\times M^m)$, $\varphi(x, \alpha)\in C^\infty_{\mathscr{L}}$ if axioms (μ_7) and (μ_8) are satisfied. The proof is essentially the same as of (9.5).

Take $A_1, \ldots, A_n, B$, $r\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)\in X$,

$$f(x, \alpha)\in\mathscr{S}^\infty, \quad r(x, \alpha)\in\mathscr{S}^\infty.$$

Consider a symbol $g(x_0, x, \alpha)\in\mathscr{S}^\infty(\mathbf{R}^{n+1}\times M^m)$ and calculate the product

$$[\![f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)]\!][\![g\left(\overset{1}{[\![r\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)]\!]}, \overset{2}{A_1}, \ldots, \overset{n+1}{A_n}, \overset{n+1}{B}\right)]\!].$$

On applying (9.5) we get

$$[\![f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)]\!][\![g\left(\overset{1}{[\![r\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)]\!]}, \overset{2}{A_1}, \ldots, \overset{n+1}{A_n}, \overset{n+2}{B}\right)]\!]=$$

$$=f\left(\overset{1}{L_1}, \ldots, \overset{n}{L_n}, \overset{n+1}{\alpha}\right)g(x_0, x, \alpha)+$$

$$+[\![g\left(\overset{1}{[\![r\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)]\!]}, \overset{2}{A_1}, \ldots, \overset{n+1}{A_n}, \overset{n+2}{B}\right)]\!]\times$$

$$\times\left([\![r\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)]\!]-[\![r\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)]\!]\right),$$

where $x_0=\overset{1}{[\![r\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)]\!]}$, $x=\left(\overset{2}{A_1}, \ldots, \overset{n+1}{A_n}\right)$, $\alpha=\overset{n+2}{B}$.

On applying (9.29) we get

$$\left[\!\left[g\left(\left[\!\left[r\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)\right]\!\right], \overset{2}{A_1}, \ldots, \overset{n+1}{A_n}, \overset{n+2}{B}\right)\right]\!\right] \times$$
$$\times \left[\!\left[r\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)\right]\!\right] =$$
$$= \left(r\left(\overset{n+1}{G_1}, \ldots, \overset{2}{G_n}, \overset{1}{G_{n+1}}\right) g(x_0, x, \alpha)\right) \times$$
$$\times \left(\left[\!\left[r\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)\right]\!\right], \overset{2}{A_1}, \ldots, \overset{n+1}{A_n}, \overset{n+2}{B}\right).$$

In the same way

$$\left[\!\left[f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)\right]\!\right] \left[\!\left[g\left(\left[\!\left[r\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)\right]\!\right], \overset{2}{A_1}, \ldots\right.\right.$$
$$\left.\left.\ldots, \overset{n+1}{A_n}, \overset{n+2}{B}\right)\right]\!\right] =$$
$$= \Psi\left(\left[\!\left[r\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)\right]\!\right], \overset{2}{A_1}, \ldots, \overset{n+1}{A_n}, \overset{n+2}{B}\right),$$

where

$$\Psi(x_0, x, \alpha) = \left[f\left(\overset{1}{L_1}, \ldots, \overset{n}{L_n}, \overset{n+1}{\alpha}\right) - r\left(\overset{n+1}{G_1}, \ldots, \overset{2}{G_n}, \overset{1}{G_{n+1}}\right) +\right.$$
$$\left. + x_0\right] g(x_0, x, \alpha).$$

Thus the main problem is reduced to (9.6) with the left-hand side having the form

$$\left[f\left(\overset{1}{L_1}, \ldots, \overset{n}{L_n}, \overset{n+1}{\alpha}\right) - r\left(\overset{n+1}{G_1}, \ldots, \overset{2}{G_n}, \overset{1}{G_{n+1}}\right) + x_0\right] g(x_0, x, \alpha).$$

The remaining part of the proof of the reduction to the Cauchy problem is preserved unchanged. The operator

$$f\left(\overset{1}{L_1}, \ldots, \overset{n}{L_n}, \overset{n+1}{\alpha}\right) - r\left(\overset{n+1}{G_1}, \ldots, \overset{2}{G_n}, \overset{1}{G_{n+1}}\right) + x_0$$

is called the Hamiltonian of the operator $f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)$ with respect to the standard operator $r\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)$.

Example. Let $C_h^\infty(\mathbf{R}^n)$ be the space of functions satisfying the conditions

$$\|(1-h^2\Delta)^s \varphi(x, h)\|_{L_2} < C_s,$$

where $s = 0, 1, \ldots, C_s$ are constants depending on the function φ but not on h as $h \to 0$.

Let $\varphi(x, h) \in C_h^\infty(\mathbf{R}^n)$. We shall write

$$\varphi(x, h) = O_{C_h^\infty}(h^\beta) \text{ if } \varphi(x, h) \in C_h^\infty,$$

$$\|h^{-\beta}\varphi(x, h)\|_{L_2} \leqslant \text{const for } h \to +0.$$

Let $\varphi(x, h) = O_{C_h^\infty}(h^\beta)$. Let Ω be a subset of the space $\mathbf{R}^{2n}$ of the variables x and p and let Ω have the following property: the equation

$$f\left(\overset{2}{x}, -i\overset{1}{\frac{\partial}{\partial x}}\right)\varphi(x, h) = O_{C_h^\infty}(h^{\beta+1})$$

is true for any function $f(x, p) \in \mathcal{S}^\infty(\mathbf{R}^n \times \mathbf{R}^n)$, such that $\operatorname{Supp} f(x, p) \cap \Omega = \varnothing$.

The set Ω is called a support of singular points of $\varphi(x, h)$.

Consider an equation of the form

$$\begin{cases} -ih\dfrac{\partial\Psi}{\partial t} + H\left(\overset{2}{y}, -ih\overset{1}{\dfrac{\partial}{\partial y}}\right)\Psi = 0, \\ \Psi|_{t=0} = \Psi_0(y, h), \end{cases} \tag{9.30}$$

where $H(y, p) \in \mathcal{S}^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ is a real function, $\Psi_0(y, h) \in C_h^\infty(\mathbf{R}^n)$. Suppose that for any $\Psi_0(y, h) \in C_h^\infty(\mathbf{R}^n)$ a solution of problem (9.30) does exist and belongs to $C_h^\infty(\mathbf{R}^n)$. Let Ω_0 be a support of singular points $\Psi_0(y, h)$. Let Ω_t be a support of singular points of the solution $\Psi(y, t, h)$ of problem (9.30).

Theorem. *The set Ω_t is included in an image of the set Ω_0 by a canonical transformation g_H^t with the Hamiltonian function $H(q, p)$.*

Proof. We shall use the concept of the standard operator in the Hamiltonian formalism to prove the theorem.

Take the operators

$$A_1 = -ih\frac{\partial}{\partial y_1}, \ldots, A_n = -ih\frac{\partial}{\partial y_n}, \quad B = y.$$

These operators are evidently generators of the Lie nilpotent algebra. It is easy to verify that the left-ordered representation of these operators is constituted by the operators

$$L_1 = x_1 - ih\frac{\partial}{\partial \alpha_1}, \ldots, L_n = x - ih\frac{\partial}{\partial \alpha_n}, \quad L_0 = \alpha.$$

The right-ordered representation has the form

$$G_1 = x_1, \ \ldots, \ G_n = x_n,$$

$$G_n^{(i)} = \alpha_i - ih\frac{\partial}{\partial x_i}, \quad i = 1, \ \ldots, \ n.$$

Take the operator

$$r\left(\overset{2}{y}, \ -ih\overset{1}{\frac{\partial}{\partial y}}\right) = H\left(\overset{2}{y}, \ -ih\overset{1}{\frac{\partial}{\partial y}}\right) = r.$$

as the standard one.

To construct a solution of (9.30) we may construct an operator $g\left(\overset{1}{r}, \overset{2}{A_1}, \ldots, \overset{2}{A_n}, \overset{3}{B}, t, h\right)$ satisfying the condition

$$-ih\frac{\partial g}{\partial t}\left(\overset{1}{r}, \overset{2}{A_1}, \ldots, \overset{2}{A_n}, \overset{3}{B}, t, h\right) +$$

$$+ [\![H\left(\overset{2}{B}, \overset{1}{A_1}, \ldots, \overset{1}{A_n}\right)]\!] [\![g\left(\overset{1}{r}, \overset{2}{A_1}, \ldots, \overset{2}{A_n}, \overset{3}{B}, t, h\right)]\!] = 0,$$

$$g\left(\overset{1}{r}, \overset{2}{A_1}, \ldots, \overset{2}{A_n}, \overset{3}{B}, 0, h\right) = g_0\left(\overset{1}{A_1}, \ldots, \overset{1}{A_n}, \overset{2}{B}\right),$$

where $g_0(x, \alpha) \in \mathcal{S}^\infty$ is an arbitrary function.

By the previous construction we need a solution of the Cauchy problem

$$-ih\frac{\partial g}{\partial t}(x_0, x, \alpha, t, h) + [\![H\left(\alpha^2, \overset{1}{L_1}, \ldots, \overset{1}{L_n}\right) -$$

$$-H(\overset{1}{G}_{n+1}, \overset{2}{G_1}, \ldots, \overset{2}{G_n}) + x_0]\!] g(x_0, x, \alpha, t, h) = 0,$$

$$g(x_0, x, \alpha, 0, h) = g_0(x, \alpha) \tag{9.31}$$

to construct the symbol g.

If $g_0(x, \alpha) \equiv 1$ then a solution of (9.31) is the function

$$g(x_0, x, \alpha, t, h) = e^{-\frac{i}{h}tx_0}.$$

Hence we obtain the following formula for the solution of problem (9.30):

$$\Psi(y, t, h) = \exp\left\{[\![-\frac{it}{h}H\left(\overset{2}{y}, \ -ih\overset{1}{\frac{\partial}{\partial y}}\right)]\!]\right\}\Psi_0(y, h).$$

Now to prove the theorem it is sufficient to prove the following statement: if $F'(q, p) \in \mathcal{S}^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ is a function with $\operatorname{Supp} F'(q, p) \cap \Omega_\tau \neq 0$ (τ is any) then

$$F'\left(\overset{2}{y}, \ -ih\overset{1}{\frac{\partial}{\partial y}}\right)\Psi(y, t, h) = O_{C_h^\infty}(h^{\beta+1}),$$

whence $\Psi_0(y, h) = O_{C_h^\infty}(h^\beta)$. (9.32)

Consider the canonical transformation g_H^t corresponding to the Hamiltonian function $H(q, p)$. By definition, $g_H^{-\tau}(\Omega_\tau) = \Omega_0$. Let $(g_H^\tau)^*$ be an automorphism generated by g_H^τ of the ring of smooth function defined in $\mathbf{R}^{2n}$.

Let $F(q, p) \in \mathcal{S}^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ be such a function that $\operatorname{Supp} F(q, p) \cap \Omega_\tau = \varnothing$. Let $f(q, p) = (g_H^\tau)^* F(q, p)$. It is obvious that

$$f(q, p) \in \mathcal{S}^\infty(\mathbf{R}^n \times \mathbf{R}^n), \qquad \operatorname{Supp} f(q, p) \cap \Omega_0 = \varnothing.$$

Hence the estimate is true

$$f\left(\overset{2}{y}, -ih\overset{1}{\frac{\partial}{\partial y}}\right) \Psi_0(y, h) = O_{C_h^\infty}(h^{\beta+1}).$$

Let $G(x_0, x, \alpha, t, h)$ be a solution of problem (9.31) with initial values

$$g_0(x, \alpha) = f(x, \alpha). \tag{9.33}$$

We shall search the function $g(x_0, x, \alpha, t, h)$ in the form

$$G_N(x_0, x, \alpha, t, h) = e^{-\frac{itx_0}{h} + \frac{i}{h} S(x, \alpha, t)} \sum_{k=0}^{N} (-ih)^k \varphi_k(x, \alpha, t). \tag{9.34}$$

To satisfy the initial value conditions let

$$S(x, \alpha, 0) = 0, \quad \varphi_0(x, \alpha, 0) = f(x, \alpha)$$

$$\varphi_k(x, \alpha, 0) = 0, \quad k \geqslant 1, \tag{9.35}$$

then substituting the function G_N of (9.34) into (9.31) and equating to zero coefficients of the powers $h^0, h^1, \ldots, h^{N-1}$, we obtain a system of equations

$$\frac{\partial S}{\partial t} + H\left(\alpha, x + \frac{\partial S}{\partial \alpha}\right) - H\left(\alpha + \frac{\partial S}{\partial x}, x\right) = 0 \tag{9.36}$$

$$\frac{\partial \varphi_0}{\partial t} + \left\langle H_p\left(\alpha, x + \frac{\partial S}{\partial \alpha}\right), \frac{\partial \varphi_0}{\partial \alpha}\right\rangle - \left\langle H_q\left(\alpha + \frac{\partial S}{\partial x}, x\right), \frac{\partial \varphi_0}{\partial x}\right\rangle = 0 \tag{9.37}$$

$$R_1\varphi_k + R_2\varphi_{k-1} + \ldots + R_{k+1}\varphi_0 = 0, \quad k = 1, 2, \ldots, N-1, \tag{9.38}$$

where

$$R_1 = \frac{\partial}{\partial t} + \left\langle Hp\left(\alpha, x + \frac{\partial S}{\partial \alpha}\right), \frac{\partial}{\partial \alpha}\right\rangle - \left\langle Hq\left(\alpha + \frac{\partial S}{\partial x}, x\right), \frac{\partial}{\partial x}\right\rangle,$$

R_k, $1 < k < N$ are differential operators of order not higher than k with smooth coefficients.

Equation (9.36) is a Hamilton-Jacobi equation. Solving (9.36) and taking into account the initial condition (9.35) we obtain

$$S(x, \alpha, t) \equiv 0$$

Hence we may rewrite the operator R_1 in the form

$$R_1 = \frac{\partial}{\partial t} + \left\langle H_p(\alpha, x), \frac{\partial}{\partial \alpha} \right\rangle - \left\langle H_q(\alpha, x), \frac{\partial}{\partial x} \right\rangle = \frac{d}{dt},$$

where $\frac{d}{dt}$ is a Hamiltonian vector field corresponding to the one-parameter family g_H^t of canonical transformations of phase space. Therefore by (9.35) the function $\varphi_0(x, \alpha, t)$ is a solution of (9.37)

$$\varphi_0(x, \alpha, t) = (g_H^{-t})^* f(x, \alpha).$$

In particular, when $t = \tau$

$$\varphi_0(x, \alpha, \tau) = F(x, \alpha).$$

When $k = 2, \ldots, N - 1$ equations (9.38) have the form

$$\begin{cases} \frac{d}{dt} \varphi_k + F_k(\varphi_{k-1}, \ldots, \varphi_0) = 0, \\ \varphi_k |_{t=0} = 0, \end{cases}$$

where the alleged function $F_k(\varphi_{k-1}, \ldots; \varphi_0)$ depends on the functions $\varphi_{k-1}, \ldots, \varphi_0$ and its derivatives. Hence (9.38) is easy to solve.

On substituting the function $G_N(x_0, x, \alpha, t, h)$ into (9.31) the right-hand side is evidently obtained in the form $h^N R_N(x, \alpha, t, h)$, where the function $R_N(x, \alpha, t, h)$ is of the space $\mathscr{S}^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ for any t and $h \in [0, 1]$. Hence the asymptotic solution of problem (9.31) with initial condition (9.33) has to be of the form

$$G_N(x_0, x, \alpha, t, h) = e^{\frac{-it}{h} x_0} \{(g_H^{-t})^* f(x, \alpha) + h g_1(x, \alpha, t, h)\}, \tag{9.39}$$

where the function $g_1(x, \alpha, t, h)$ belongs to the space $\mathscr{S}^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ for any t and $h \in [0, 1]$.

Since the function G_N is a solution of (9.31) with initial condition (9.33), the following equation is true:

$$-ih \frac{\partial G_N}{\partial t} \left(\overset{1}{r}, \overset{2}{A_1}, \ldots, \overset{2}{A_n}, \overset{3}{B}, t, h \right) +$$

$$+ [\![H \left(\overset{2}{B}, \overset{1}{A_1}, \ldots, \overset{1}{A_n} \right)]\!] [\![G_N \left(\overset{1}{r}, \overset{2}{A_1}, \ldots, \overset{2}{A_n}, \overset{3}{B}, t, h \right)]\!] = O(h^N),$$

$$G_N \left(\overset{1}{r}, \overset{2}{A_1}, \ldots, \overset{2}{A_n}, \overset{3}{B}, 0, h \right) = f \left(\overset{2}{B}, \overset{1}{A_1}, \ldots, \overset{1}{A_n} \right).$$

Hence the function $\Psi'(y, t, h)$

$$\Psi'_N(y, t, h) = [\![G_N \left(\overset{1}{r}, \overset{2}{A}_1, \ldots, \overset{2}{A}_n, \overset{3}{B}, t, h \right)]\!] \Psi_0(y, h)$$

is a solution of the problem

$$-ih\frac{\partial \Psi'_N}{\partial t} + H\left(\overset{2}{y}, -ih\overset{1}{\frac{\partial}{\partial y}}\right)\Psi'_N = O_{C_h^\infty},$$

$$\Psi'_N|_{t=0} = f\left(\overset{2}{y}, -ih\overset{1}{\frac{\partial}{\partial y}}\right)\Psi_0(y, h) = O_{C_h^\infty}(h^{\beta+1}).$$

Hence the estimate

$$\Psi'_N(y, t, h) = O_{C_h^\infty}(h^{\beta+1})$$

is valid for any t.

From the last equation and (9.39) we have

$$((g_H^{-t})^* f)\left(\overset{2}{y}, -ih\overset{1}{\frac{\partial}{\partial y}}\right) \exp\left\{\frac{-it}{h}[\![H\left(\overset{2}{y}, -ih\overset{1}{\frac{\partial}{\partial y}}\right)]\!]\right\} \times$$
$$\times \Psi_0(y, h) = O_{C^\infty}(h^{\beta+1}).$$

Taking $t = \tau$ we finally get

$$[\![F\left(\overset{2}{y}, -ih\overset{1}{\frac{\partial}{\partial y}}\right)]\!] \Psi(y, t, h) = O_{C_h^\infty}(h^{\beta+1}), \qquad \text{Q.E.D.}$$

Example. Consider the pseudodifferential operator

$$A = -i\frac{\partial}{\partial t} + H\left(\overset{2}{y}, -i\overset{1}{\frac{\partial}{\partial y}}\right),$$

where $H(y, p) \in C^\infty(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$ is a real function positively homogeneous in p of the first order. Let $u_t(y)$ be a solution of the problem

$$Au_t = 0, \quad u_t|_{t=0} = u_0, \quad u_0 \in H_s.$$

Following Hörmander we call the set *

$$WF(v) \subset T^*\mathbf{R}^n$$

$$WF(v) = \bigcap_{Bu \in C^\infty} \operatorname{char}(B)$$

a wave front of the function $v \in H_s(\mathbf{R}^n)$, where $B = B\left(\overset{2}{y}, -i\overset{1}{\frac{\partial}{\partial y}}\right)$ is a pseudodifferential operator with a symbol $B(x, p) \in S^1(\mathbf{R}^n \times \mathbf{R}^n)$. In the phase space $T^*\mathbf{R}^n$ char (B) is a set such that $B(y, p) = 0$ when $(y, p) \in \operatorname{char}(B)$.

* For the notations see L. Hörmander. *Acta Math.*, 127, 1-2 (1971)

Let g_H^t be the canonical transformation of the phase space $T^*\mathbf{R}^n$ with the Hamiltonian function H.

Theorem. *There is an inclusion*

$$WF\,(u_t) \subset g_H^t WF\,(u_0).$$

Proof. Following the previous example we use the concept of the standard operator to prove the theorem.

It is evidently sufficient to prove that the relation

$$[\![B\left(\overset{2}{y},\ -i\frac{\overset{1}{\partial}}{\partial y}\right)]\!]\,u_0 \in C^\infty \Rightarrow [\![((g_H^{-t})^* B)\left(\overset{2}{y},\ -i\frac{\overset{1}{\partial}}{\partial y}\right)]\!]\,u_t \in C^\infty \tag{9.40}$$

is valid for any operator $B\left(\overset{2}{x},\ -i\frac{\overset{1}{\partial}}{\partial x}\right)$ with a symbol of $S^1(\mathbf{R}^n\times\mathbf{R}^n)$.

The left representation of the operators

$$A_1 = -i\frac{\partial}{\partial y_1},\ \ldots,\ A_n = -i\frac{\partial}{\partial y_n},\ B = y$$

has the form

$$\alpha_1 = x_1 - i\frac{\partial}{\partial\alpha_1},\ \ldots,\quad \alpha_n = x_n - i\frac{\partial}{\partial\alpha_n},\ \alpha$$

and the right representation of the same operators has the form

$$G_1 = x_1,\ \ldots,\ G_n = x_n,$$

$$G_{n+1}^{(i)} = \alpha_i - i\frac{\partial}{\partial x_i},\quad i = 1,\ \ldots,\ n.$$

Let

$$r\left(\overset{1}{A},\ \overset{2}{B}\right) = H\left(\overset{1}{A},\ \overset{2}{B}\right).$$

Consider the problem (cf. pr. Example)

$$\left[-i\frac{\partial}{\partial t} + H\left(\overset{2}{\alpha},\ x - i\frac{\overset{1}{\partial}}{\partial\alpha}\right) - H\left(\alpha - i\frac{\overset{1}{\partial}}{\partial x},\ \overset{2}{x}\right) + x_0\right]\times$$
$$\times\,\Psi(x_0,\ x,\ \alpha,\ t) = 0,$$
$$\Psi(x_0,\ x,\ \alpha,\ 0) = B(\alpha,\ x), \tag{9.41}$$

where $B(\alpha,\ x) \in S^1(\mathbf{R}^n\times\mathbf{R}^n)$, $B(\alpha,\ x) = 0$ when $|x| < 1$.

We shall seek for an asymptotic solution of problem (9.41) in the form

$$\Psi_N = e^{-ix_0t}\sum_{k=0}^{N}\varphi_k(\alpha,\ x,\ t), \tag{9.42}$$

where the functions $\varphi_k(\alpha, x, t)$ satisfy the estimates

$$|D_\alpha^\gamma D_x^\beta \varphi_k(x, \alpha, t)| \leqslant C_{\gamma\beta k}(1+|x|)^{-k-|\beta|+1},$$

where γ, β are indices and $C_{\alpha\beta k}$ are constants. To satisfy the initial conditions of problem (9.41) let

$$\varphi_0(\alpha, x, 0) = B(\alpha, x),$$
$$\varphi_k(\alpha, x, 0) = 0, \quad k \geqslant 1. \tag{9.43}$$

On substituting the function $\Psi_N(x_0, x, \alpha, t)$ of form (9.42) into (9.41) and equating to zero the terms belonging to the spaces $S^{-k}(\mathbf{R}^n \times \mathbf{R}^n)$, $k = 0, -1, \ldots$ we obtain

$$\frac{d\varphi_0}{dt} = 0, \tag{9.44}$$

$$\frac{d\varphi_1}{dt} + \left\{\frac{i}{2}\left\langle \overset{1}{\frac{\partial}{\partial\alpha}},\ H_{pp}\left(\overset{2}{\alpha}, x\right)\overset{1}{\frac{\partial}{\partial\alpha}}\right\rangle + \right.$$
$$\left. + \frac{i}{2}\left\langle \overset{1}{\frac{\partial}{\partial x}},\ H_{xx}\left(\alpha, \overset{2}{x}\right)\overset{1}{\frac{\partial}{\partial x}}\right\rangle\right\}\varphi_0 = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d\varphi_k}{dt} + R_1\varphi_{k-1} + \ldots + R_k\varphi_0 = 0, \quad k = 2, \ldots, N-1, \tag{9.45}$$

where R_k, $1 \leqslant k \leqslant N$ are differential operators of order not higher than k with coefficients smooth in the region $x \neq 0$.

From (9.44) we get

$$\varphi_0(\alpha, x, t) = (g_H^{-t})^* B(\alpha, x).$$

It is easy to see that the function $\varphi_0(\alpha, x, t)$ belongs to $S^1(\mathbf{R}^n \times \mathbf{R}^n)$ for any t. Hence the solutions of (9.45) φ_k belong to the spaces $S^{-k+1}(\mathbf{R}^n \times \mathbf{R}^n)$ for any t.

By construction the function $\Psi_N(x_0, x, \alpha, t)$ satisfies the condition

$$\left[-i\frac{\partial}{\partial t} + H\left(\overset{2}{\alpha},\ x - i\overset{1}{\frac{\partial}{\partial\alpha}}\right) - H\left(\alpha - i\overset{1}{\frac{\partial}{\partial x}},\ \overset{2}{x_0}\right) + x_0\right]\times$$
$$\times \Psi_N(x_0, x, \alpha, t) = f_N(x, \alpha, t),$$

$$\Psi_N(x_0, x, \alpha, 0) = B(\alpha, x),$$

where $f_N(x, \alpha, t) \in S^{-N}(\mathbf{R}^n \times \mathbf{R}^n)$ for any t

We infer from (9.5) and (9.29) that the operator $\Psi_N\left(\overset{1}{r}, \overset{2}{A}, \ldots\right.$ $\left.\ldots, \overset{2}{A}_n, \overset{3}{B}, t\right)$ satisfies the condition

$$-i\frac{\partial\Psi_N}{\partial t}\left(\overset{1}{r}, \overset{2}{A}_1, \ldots, \overset{2}{A}_n, \overset{3}{B}, t\right)+[\![H\left(\overset{2}{B}, \overset{1}{A}\right)]\!]\times$$
$$\times[\![\Psi_N\left(\overset{1}{r}, \overset{2}{A}_1, \ldots, \overset{2}{A}_n, \overset{3}{B}, t\right)]\!]=f_N\left(\overset{1}{A}_1, \ldots, \overset{1}{A}_n, \overset{2}{B}, t\right)$$
$$\Psi_N\left(\overset{1}{r}, \overset{2}{A}_1, \ldots, \overset{2}{A}_n, \overset{3}{B}, 0\right)=B\left(\overset{2}{y}, -i\overset{1}{\frac{\partial}{\partial y}}\right).$$

Hence the function

$$v_N(y, t)=[\![\Psi_N\left(\overset{1}{r}, \overset{2}{A}_1, \ldots, \overset{2}{A}_n, \overset{3}{B}, t\right)]\!]u_0(y)$$

satisfies the equation

$$\begin{aligned}&-i\frac{\partial v_N}{\partial t}+H\left(\overset{2}{y}, -i\overset{1}{\frac{\partial}{\partial y}}\right)v_N=R_N(y, t)\in H_{s+N}(R_y^n),\\ &v_N\Big|_{t=0}=[\![B\left(\overset{2}{y}, -i\overset{1}{\frac{\partial}{\partial y}}\right)]\!]u_0(y).\end{aligned}\tag{9.46}$$

By (9.42) the function $v_N(y, t)$ has the form

$$v_N(y, t)=((g_H^{-t})^*B)\left(\overset{2}{y}, -i\overset{1}{\frac{\partial}{\partial y}}\right)\exp\left\{-it[\![H\left(\overset{2}{y}, -i\overset{1}{\frac{\partial}{\partial y}}\right)]\!]\right\}\times$$
$$\times u_0+\varkappa\left(\overset{2}{y}; -i\overset{1}{\frac{\partial}{\partial y}}, t\right)u_t(y),\tag{9.47}$$
$$\varkappa(\alpha, x, t)\in S^0(\mathbf{R}^n\times\mathbf{R}^n).$$

Since $N>0$ is any, we infer from Eqs. (9.46), (9.47) that

$$((g_H^{-t})^*B)\left(\overset{2}{y}, -i\overset{1}{\frac{\partial}{\partial y}}\right)u_t(y)\in C^\infty$$

$$\text{if } B\left(\overset{2}{y}, -i\overset{1}{\frac{\partial}{\partial y}}\right)u_0(y)\in C^\infty,$$

Q.E.D.

Now we shall consider a still more general case. Let $f_i=$ $=f_i\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right)\in M$. Consider an operator

$$\hat{F}=F\left(\overset{1}{f_1}, \overset{2}{f_2}, \ldots, \overset{k}{f_k}\right).$$

In the same way we obtain that the symbol of the product

$$[\![F\left(\overset{1}{f_1}, \overset{2}{f_2}, \ldots, \overset{k}{f_k}\right)]\!][\![g\left(\overset{2}{A_1}, \overset{3}{A_2}, \ldots, \overset{n+1}{A_n}, \overset{n+2}{B}, \overset{1}{r}\right)]\!]$$

is equal to

$$F\left(\overset{1}{[\![f_1}\left(\overset{1}{L_1}, \ldots, \overset{n}{L_n}, \overset{n+1}{\alpha}\right) - r\left(\overset{n+1}{G_1}, \ldots, \overset{2}{G_n}, \overset{1}{G_{n+1}}\right) + x_0]\!],\right.$$
$$\overset{2}{[\![f_2}\left(\overset{1}{L_1}, \ldots, \overset{n}{L_n}, \overset{n+1}{\alpha}\right) - r\left(\overset{n+1}{G_1}, \ldots, \overset{2}{G_n}, \overset{1}{G_{n+1}}\right) + x_0]\!], \ldots$$
$$\ldots, \overset{k}{[\![f_k}\left(\overset{1}{L_1}, \ldots, \overset{n}{L_n}, \overset{n+1}{\alpha}\right) - r\left(\overset{n+1}{G_1}, \ldots, \overset{2}{G_n}, \overset{1}{G_{n+1}}\right) +$$
$$\left. + x_0]\!]\right) g(x_1, \ldots, x_n, \alpha, x_0)$$

We shall call the operator in the right-hand side applied to the function g the Hamiltonian of the operator $\hat{F}$.

Now let $r = 0$, $g(x_1, \ldots, x_n, \alpha, x_0) \equiv 1$. Then we obtain a formula of the reduction of a composite function to a simple function of the same ordered operators. Indeed, to define the symbol $\psi(x, \alpha)$ of the operator

$$\psi\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right) = F\left(\overset{1}{f_1}, \ldots, \overset{k}{f_k}\right)$$

we obtain the formula.

$$\psi(x, \alpha) = F\left(\overset{1}{[\![f_1}\left(\overset{1}{L_1}, \ldots, \overset{n}{L_n}, \overset{n+1}{\alpha}\right)]\!], \ldots\right.$$
$$\left.\ldots, \overset{k}{[\![f_k}\left(\overset{1}{L_1}, \ldots, \overset{n}{L_n}, \overset{n+1}{\alpha}\right)]\!]\right) 1, \tag{9.48}$$

where the composite function of differential operators in the right-hand side is applied to the unit element belonging to the space of symbols.

3. Bicharacteristics and absorption conditions. In the last examples we came across the concepts of bicharacteristics and characteristics. We have seen that in those examples a non-smooth and rapidly oscillating part of the solution is defined by solutions of equations with partial derivatives of the first order which we have called the equation of Hamilton-Jacobi (the characteristics equation) and the system of ordinary differential equations of Hamilton (the bicharacteristics equation).

We have considered an example of the equations of vibrations of a crystal lattice over a circle or, which is just the same, the equations with periodicity conditions. The corresponding solutions of the system of equations of bicharacteristics (8.36) $X(x^0, t)$, $P(x^0, t)$ were such that $X(x^0, t)$ could be considered as coordinates of a point moving in a circle and $P(x^0, t)$—in a line. The pair $X(x^0, t)$,

$P(x^0, t)$ as a whole defined a trajectory of the point on the cylinder, i.e., a product of a line by a circle. This cylinder, called a *phase space*, is denoted by p, x, where p is a coordinate on a line and x is a coordinate on a circle (or a q-coordinate). Henceforth, we shall consider its property more thoroughly and the property of the system of Hamilton equations defining a family of curves in this phase space. It is clear that for any n-dimensional torus M^n there exists a phase space T^*M^n of dimension $2n$, such that x-coordinates belong to M^n and p-coordinates belong to $\mathbf{R}^n$.

We have seen that the equation of Hamilton-Jacobi as well as the system of Hamilton is defined by a function $H(p, x)$, which is called a Hamiltonian function. This function has the form

$$H = \mp 2\frac{c(x)}{\omega}\sin\left[\frac{\omega}{2}(1+p)\right]$$

in the case of a crystal and depends on the parameter $|\omega| \leqslant \pi$. This parameter is a symbol of the operator of differentiation and substantially influences the property of the solution.

In the general case of a function of ordered generators of the Lie nilpotent algebra the symbol is a function of $m+n$ coordinates, where m is a dimension of the manifold M^m, i.e., the number of components of a vector operator B and n is a number of operators $A_1, A_2, \ldots, A_n$. Thus, the domain of the definition of the symbol is a manifold $M^n \times \mathbf{R}^m = M_1^{n+m}$.

Let $T^*M_1^{n+m}$ be the corresponding phase space. We want to discuss the problem, how to find a quasi-inverse operator. We have seen from previous examples that the sought quasi-inverse operator is connected with the system of bicharacteristics and, therefore, with the Hamiltonian function. Thus it is natural to ask, how to define the Hamiltonian function corresponding to the symbol $f\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right)$. We want to define bicharacteristics of this system. We consider the function f. On substituting the representations $L_1, \ldots, L_n, \alpha$ of the operators $A_1, \ldots, A_n, B$ for its arguments, we obtain the Hamiltonian $f\left(\overset{1}{L}_1, \ldots, \overset{n}{L}_n, \overset{n+1}{\alpha}\right)$.

Then we make a certain leading term of the obtained formula with respect to the variables $x_1, \ldots, x_n, i\dfrac{\partial}{\partial\alpha}$. This leading term defines the *Hamiltonian function*.

We shall introduce the definition of the leading term of a function.

(1) Let $f(y) \in \mathcal{S}^\infty(\mathbf{R}^s)$ and let there be such numbers $\rho_1, \ldots, \rho_s \geqslant 1$ and $r > 0$, that

$$f(\lambda^{\rho_1}y_1, \ldots, \lambda^{\rho_s}y_s) = \lambda^r f(y), \quad \forall\lambda > 0;$$

then the function is a ρ-*quasi-homogeneous function of degree* r.

(2) The function $\sigma(y)$ is called *subordinate to the function* $f(y)$, if

$$\left|\frac{\partial^{|k|}\sigma(y)}{\partial y_1^{k_1}\dots\partial y_s^{k_s}}\right|\leqslant c\left(\sqrt{y^{2/\rho_i}+1}\right)^{r-1-|k|}.$$

Let $F(y)=\sum_{0}^{N} f_i(y)+\sigma(y)$, where $f_i(y)$ are ρ-quasi-homogeneous ($\rho=\rho_1,\dots,\rho_s$) functions of orders r_i, $r_i\geqslant r_{i+1}+\delta$ for some $\delta>0$ and $\sigma(y)$ is subordinate to $f_0(y)$. Under these conditions $F(y)$ is called *asymptotically* ρ-*quasi-homogeneous* ($\rho=\rho_1,\dots,\rho_s$), $f_0(y)$ is called *the leading term of the function* $F(y)$ and $\sum_{0}^{N} f_i(y)$ is called *an essential part of the function* $F(y)$.

For the time being take $x_i=y_i$, $i\leqslant n$, $i\frac{\partial}{\partial\alpha_k}=y_{n+k}$, $-i\frac{\partial}{\partial x}=\eta$. Let the obtained function be asymptotically ρ-quasi-homogeneous in y and $\rho_{n+i}=1$, $0<i\leqslant m$ (the variables η, α are taken as parameters; η belongs to a small neighborhood of zero in $\mathbf{R}^n$, $\alpha\in M^m$).

The leading term of the function will be called the Hamiltonian function corresponding to the given Hamiltonian, and will be denoted by $\pi(y,\eta,\alpha)$.

Thus the construction of the Hamiltonian function of an operator $A\in\mathcal{A}$ consists of 4 steps: (1) (μ^{-1}) the choice of $A_1,\dots,A_n$, $B\in X$, $r=r(x,\alpha)$, $f(x,x_0,\alpha)\in\mathcal{S}^\infty$ such that $f(\overset{2}{A}_1,\dots,\overset{n+1}{A}_n,\overset{n+2}{B},\overset{1}{r})=A$;

(2) ($A\to L$) the construction of the Hamiltonian by the ordered representation;

(3) (ρ) the pointing out of the leading term $\pi(y,\eta)$ of the Hamiltonian;

(4) $\to(p,q)$ the determining of the Hamiltonian arguments in the function $\pi(y,\eta)$.

Thus

$$A\overset{\mu^{-1}}{\longrightarrow}f\left(\overset{2}{A}_1,\dots,\overset{n+1}{A}_n,\overset{n+2}{B},\overset{1}{r}\right)\xrightarrow{A\to L}$$

$$\to f\left(\overset{1}{L}_1,\dots,\overset{n}{L}_n,\overset{n+1}{\alpha}\right)-r\left(\overset{n+1}{G}_1,\dots,\overset{1}{G}_{n+1}\right)+x_0\overset{\rho}{\longrightarrow}$$

$$\to\pi(y,\eta,\alpha)\xrightarrow{\to p,q}H(p,q,\omega).$$

The operations (μ^{-1}) and (ρ) are defined not uniquely generally and do not exist occasionally for a given $A\in\mathcal{A}$. Now we pass over to the last step of the indicated construction.

For the sake of definiteness let ρ_i be equal to a unity for the first s arguments, $s\leqslant n$, $\rho_{n+i}=1$ for $0<i\leqslant k$.

Take new variables

$$y_1=\omega_1-p_{m+1},\ \dots,\ y_s=\omega_s-p_{m+s};\quad y_{s+1}=\omega_{s+1},\ \dots,\ y_n=\omega_n;$$
$$y_{n+1}=p_1,\ \dots,\ y_{n+k}=p_k,\quad y_{n+k+1}=\dots=y_{n+m}=0;$$
$$\eta_1=q_{m+1},\ \dots,\ \eta_n=q_{m+n};\quad \alpha_1=q_1,\ \dots,\ \alpha_m=q_m.$$

Denote by

$$\pi(y,\ \eta,\ \alpha)=\mathscr{H}(p,\ q,\ \omega),\quad H=\operatorname{Re}\mathscr{H},\quad \widetilde{H}=\operatorname{Im}\mathscr{H}.$$

Besides this, denote by $\mathscr{H}_0(p, q, \omega)$ the essential part of the symbol $f(L_1, \dots, L_n, \alpha)$. Assume that $\operatorname{Im}\mathscr{H}_0 \leqslant 0$ in this item.

Let Ω_ε be the manifold in the space of arguments p, q, ω defined by the conditions

$$\sum_{i=1}^{n}\omega_i^{2/\rho_i}=1,\quad \sum_{i=1}^{n}(q_{m+i})^2<\varepsilon$$
$$(q_1,\ \dots,\ q_m)\in M^m,\quad p=0,\quad |\mathscr{H}(p,\ q,\ \omega)|<\varepsilon.$$

Define the *bicharacteristics of the operator* $f\left(\overset{1}{A}_1, \dots, \overset{n}{A}_n, \overset{n+1}{B}\right)$ as a solution of the Hamiltonian system

$$\frac{dq_j}{d\tau}=\frac{\partial H}{\partial p_j}(p,\ q,\ \omega),\quad q_j|_{\tau=0}=q_j^0,$$
$$\frac{dp_j}{d\tau}=-\frac{\partial H}{\partial q_j}(p,\ q,\ \omega),\quad p_j|_{\tau=0}=p_j^0,\quad j=1,\ 2,\ \dots,\ s+m,$$
$$(p^0,\ q^0,\ \omega)\in\Omega_\varepsilon. \tag{9.49}$$

In the following examples we denote sometimes the arguments of a symbol by $y_1, \dots, y_n$, α to avoid ambiguity.

Examples. Turn to the construction of the Hamiltonian function and consider an example of a polynomial in the operators $-i\frac{\partial}{\partial x}$, $x=x_1, \dots, x_n$ with coefficients dependent on x. In this case the second commutator is zero and the representation of the operator $A_k=-i\frac{\partial}{\partial x_k}$ has the form $y_k-i\frac{\partial}{\partial\alpha_k}$ and $B_k=x_k=\alpha_k$. The polynomial

$$-\sum_i a_i(x)\frac{\partial^2}{\partial x_i^2} \tag{9.50}$$

has the form $\sum\limits_i a_i\left(\overset{2}{B}\right)\left(\overset{1}{A}_i\right)^2$. Its representation has the form

$$\sum_k a_k\left(\overset{2}{\alpha}\right)\left(y_k-i\frac{\partial}{\partial\alpha_k}\right)^2. \tag{9.51}$$

The Hamiltonian function (for $\rho_i = 1$, $\forall i$) has the form

$$H(p, q, \omega) = \sum_{k=1}^{n} a_k(q_1, \ldots, q_n)(\omega_k - p_k + p_{n+k})^2. \tag{9.52}$$

Consider also the Hamiltonian system defined by the Hamiltonian function

$$H_1(q, p) = \sum a_k(q)\, p_k^2, \tag{9.53}$$

where $q = q_1, \ldots, q_n$, $p = p_1, \ldots, p_n$.

A solution of the Hamiltonian system

$$\begin{aligned} &\dot{p}_{n+j} = 0 \\ &\dot{p}_j = H_{q_j} \\ &\dot{q}_j = -H_{p_j} \quad j = 1, \ldots, n \end{aligned} \tag{9.54}$$

satisfying the initial conditions $q(0) = q_0$, $p(0) = 0$, $\sum_{k=1}^{n} \omega_k^2 = 1$, $H(q_0, \omega_k) = 0$ is defined as bicharacteristics of operator (9.51).

Note, that these bicharacteristics can be defined by a solution of the system

$$\dot{q} = H_{1p}, \quad \dot{p} = -H_{1q}$$

satisfying the conditions

$$q_k(0) = q_{0k}, \quad p_k(0) = \omega_k.$$

Thus the definition given above is reduced to the definition of bicharacteristics of the wave equation, given in Sec. 8.

Now consider an operator

$$-\sum_{i=1}^{n-1} a_i(x) \frac{\partial^2}{\partial x_i^2} + i b_n(x) \frac{\partial}{\partial x_n}. \tag{9.55}$$

In this case the Hamiltonian function equals

$$H(p, q, \omega) = \sum_{k=1}^{n-1} a_k(q_1, \ldots, q_n)(\omega_k - p_k + p_{k+n}) - \\ - b_n(q_1, \ldots, q_n)\,\omega_n.$$

Denote

$$H_1(p, \eta, \alpha) = \sum_{k=1}^{n-1} a_k(q)\, p_k^2 + b_n(q)\,\omega_n.$$

The solution of the problem

$$\dot{p}=-\frac{\partial H_1}{\partial q},\quad p(0)=p_0,$$
$$\dot{q}=\frac{\partial H_1}{\partial p_2},\quad q(0)=q_0. \tag{9.56}$$
$$\left|\sum_{k=1}^{n-1} a_n(q_0)\,\omega_k^2-b_n(q_0)\,\omega_n\right|<\varepsilon,\quad \sum_{k=1}^{n-1}\omega_k^2+|\omega_n|=1$$

is defined as bicharacteristics of operator (9.55).

Now we shall consider the Schrödinger operator

$$-ih^{-1}\frac{\partial}{\partial t}-\frac{1}{2m}\frac{\partial^2}{\partial x^2}+h^{-2}v(x)=\mathscr{L},\quad m=\text{const},\quad x\in\mathbf{R}. \tag{9.57}$$

Take

$$A_1=-i\frac{\partial}{\partial t},\quad A_2=-i\frac{\partial}{\partial x},\quad A_3=\frac{1}{h},\quad B_1=t,\quad B_2=x.$$

Then

$$\mathscr{L}=+A_1A_3+\frac{1}{2m}A_2^2+v(B)\,A_3^2.$$

The representation of the operators has the form

$$L_1=-i\frac{\partial}{\partial\alpha_1}+y_1,\quad L_2=-i\frac{\partial}{\partial\alpha_2}+y_2,\quad L_3=y_3.$$

Thus, the representation of $\mathscr{L}$ has the form

$$\hat{\mathscr{L}}=+y_3\left(-i\frac{\partial}{\partial\alpha_1}+y_1\right)+\frac{1}{2m}\left(-i\frac{\partial}{\partial\alpha_2}+y_2\right)^2+y_3^2v(\alpha_2).$$

The Hamiltonian function (for $\rho_i=1$, $\forall i$) has the form

$$H=(\omega_3+p_6)(\omega_1+p_1+p_n)+\frac{1}{2m}(\omega_2+p_2+p_5)^2+$$
$$+(\omega_3+p_6)^2\,v(q_2).$$

Reduce the equations of bicharacteristics to the system

$$\dot{E}=0,\quad \dot{t}=1,\quad \dot{p}=-v'(q),\quad \dot{q}=+\frac{p}{m},$$

with the inital conditions and the Hamiltonian function

$$q(0)=\frac{q_0}{\omega_3},\quad p(0)=\frac{\omega_2}{\omega_3},\quad H_1(q,p)=\frac{p^2}{2m}+v(q)$$

with the help of a substituion of the form

$$\frac{p_2+\omega_2}{\omega_3}=p,\quad \frac{q_1}{\omega_3}=t,\quad \frac{q_2}{\omega_3}=q.$$

We have obtained the equations of the classical mechanical problem, corresponding to the quantum problem, which is defined by the Schrödinger operator (9.57).

Problem. Consider a differential-difference operator, defining the equation (8.4). Taking

$$A_1 = i\frac{\partial}{\partial t}, \quad A_2 = i\frac{\partial}{\partial x}, \quad A_3 = h, \quad \alpha = x$$

obtain the solutions of system (8.36) as bicharacteristics.

Definition. We shall say that *the absorption conditions are fulfilled if there exist the constants* $T > 0$, $\varepsilon > 0$ *and the number* $\tau' = \tau'(q^0, \omega, p^0)$, $0 < \tau' < T$ *such that*

(1) *when* $0 \leqslant \tau \leqslant \tau'$, *there exists a solution*

$$q(q^0, \omega, p^0, \tau); \quad p(q^0, \omega, p^0, \tau)$$

of problem (9.49) *of the class* C^∞,

(2) *when* $0 \leqslant \tau \leqslant \tau'$ *and* $(q^0, \omega, p^0) \in \Omega_\varepsilon$ *the function*

$$-\widetilde{H}(p(q^0, \omega, p^0, \tau); \quad q(q^0, \omega, p^0, \tau); \omega)$$

is non-negative, and when $\tau = \tau'$ *the function is strictly positive.*

A few words regarding this condition. A condition similar to this one was first strictly formulated by Zommerfeld for the needs of electrodynamics and is now called Zommerfeld's radiation conditions at infinity. It enabled Zommerfeld to solve difficult problems in the mathematical theory of diffraction. Physicists have come to the conclusion that this problem (the problem of an absolutely black body) cannot be solved by putting only boundary conditions without additional conditions on the coefficients of the equation itself and its form. This problem is especially often dealt with in modern physical problems (for example, in the theory of laser resonators). However, analysis of physical problems shows that the non-smooth part of a solution is more often absorbed, since the non-smooth part is defined by higher Fourier harmonics and precisely these higher Fourier harmonics are absorbed in many problems. For example, the walls of a house absorb light and let radio waves through. What are the mathematical devices modelling the phenomenon? It seems likely that the only way is to introduce a strong absorption potential, which is equal to zero inside a room and small for low frequences.

The absorption condition introduced above is in some sense a substitution for the missing condition of the absolutely black body and in many problems it is quite adequate. From this point of view the ideal degenerate case is the one when short waves do not fade. This is the case of constant coefficients.

The main theorem (the simplest form). *Let* $A_1, \ldots, A_n$, B *be generators of a nilpotent Lie algebra. An operator* $f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)$ *is quasi-inverse, if its Hamiltonian is such that the absorption condition is satisfied.*

This theorem is proved in the last chapter and at the same time (and this is the main thing) the construction of quasi-inverse sequence is given there.

Note 1. Let $M^n = M^k \times \mathbf{R}^{n-k}$, $k \leqslant n$, where M^k is a torus, then in the simplest form the main theorem remains true, if there is a function $P(B)$ of a vector operator B with a compact support equal to 1 in a domain Ω in the right-hand side of (9.1) instead of the unity. Note that when M^n is compact, the unity is a function of B with a compact support. In this case it is necessary that $q_i^0 \in \operatorname{supp} P(\alpha)$ for $i > n$ in conditions of absorption.

Example 1. Consider the equation

$$-i\frac{\partial u}{\partial t} + i\frac{\partial u}{\partial y} = 1.$$

Let $-i\,\frac{\partial}{\partial t} = A_1$, $-i = \frac{\partial}{\partial x} = A_2$, $t = B_1$, $y = B_2$, $B = (B_1, B_2)$, then we obtain the equation

$$(A_1 - A_2)\,u = 1.$$

We shall construct a quasi-inverse sequence for the operator $A_1 - A_2$ within the framework of the introduced definitions.

The representations of the operators A_1, A_2 are

$$L_1 = x_1 - i\frac{\partial}{\partial\alpha_1}, \qquad L_2 = x_2 - i\frac{\partial}{\partial\alpha_2}.$$

Hence the Hamiltonian of the operator $A_1 - A_2$ has the form

$$f(L_1, L_2) = \left(x_1 - i\frac{\partial}{\partial\alpha_1}\right) - \left(x_2 - i\frac{\partial}{\partial\alpha_2}\right).$$

Thus

$$f(L_1(y), L_2(y)) = (y_1 + y_3) - (y_2 + y_4).$$

Let $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 1$, then

$$\pi(y, \eta, \alpha) = \pi(y) = f(L(y)).$$

The absorption condition is not satisfied in this case, since $\operatorname{Im}\pi(y) = 0$. It looks like the operator $A_1 - A_2$ has no quasi-inverse. Indeed, it has not. The operator $A_1 - A_2$, however, can be modified in a way that the obtained operator will have a quasi-inverse and its symbol will coincide with the symbol $A_1 - A_2$ in a closed domain $\Delta \subset \mathbf{R}_t^1 \times \mathbf{R}_y^1$.

Consider the operator

$$f\left(\overset{1}{A}_1, \overset{1}{A}_2, \overset{2}{B}\right) = A_1 - A_2 - i\varphi\left(\overset{2}{B}\right)\overset{1}{[\![}\sqrt{1 - A_1^2 - A_2^2}]\!],$$

where $\varphi(\alpha_1, \alpha_2) \geqslant 0$ is a smooth bounded function equal to zero in a closed domain Δ and strictly positive outside Δ. We shall not discuss the form of Δ.

We shall prove somewhat later that an addition of the operator $-i\varphi\left(\overset{2}{B}\right)\overset{1}{[\![}\sqrt{1-A_1^2-A_2^2}]\!]$ to the operator $A_1 - A_2$ provides the fulfilment of the absorption conditions.

The Hamiltonian of the operator $f\left(\overset{1}{A}_1, \overset{1}{A}_2, \overset{2}{B}\right)$ has the form

$$f\left(\overset{1}{L}_1, \overset{1}{L}_2, \overset{2}{\alpha}\right) = \left(x_1 - i\frac{\partial}{\partial\alpha_1}\right) - \left(x_2 - i\frac{\partial}{\partial\alpha_2}\right) -$$

$$-i\varphi\left(\overset{2}{\alpha}\right)[\![\sqrt{\left(x_1 - i\frac{\partial}{\partial\alpha_1}\right)^2 + \left(x_2 - i\frac{\partial}{\partial\alpha_2}\right)^2 + 1}]\!],$$

thus

$$f(L_1(y), L_2(y), \alpha) = (y_1 + y_3) - (y_2 + y_4) -$$

$$-i\varphi(\alpha_1, \alpha_2)\sqrt{(y_1+y_3)^2 + (y_2+y_4)^2 + 1}.$$

Let $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 1$. Then

$$\pi(y, \alpha) = (y_1 + y_3) - (y_2 + y_4) -$$

$$-i\varphi(\alpha_1, \alpha_2)\sqrt{(y_1+y_3)^2 + (y_2+y_4)^2 + 1}.$$

Therefore the Hamiltonian function of the operator $f\left(\overset{1}{A}_1, \overset{1}{A}_2, \overset{2}{B}\right)$ has the form

$$\mathscr{H}(p, q, \omega) = (\omega_1 - p_1 + p_3) - (\omega_2 - p_2 + p_4) -$$

$$-i\varphi(q_3, q_4)\sqrt{(\omega_1 - p_1 + p_3)^2 + (\omega_2 - p_2 + p_4)^2}.$$

The manifold Ω_ε (cf. p. 129) is determined by the equations

$$\begin{gathered}\omega_1^2 + \omega_2^2 = 1, \quad p_1 = p_2 = p_3 = p_4 = 0, \quad (q_3, q_4) \in \mathbf{R}^2 \\ q_1^2 + q_2^2 < \varepsilon, \quad |\mathscr{H}(p, q)| < \varepsilon.\end{gathered} \tag{9.58}$$

Consider the bicharacteristics of the operator $f\left(\overset{1}{A}_1, \overset{1}{A}_2, \overset{2}{B}\right)$

$$\begin{aligned}\frac{dq_j}{d\tau} &= \frac{\partial \operatorname{Re}\mathscr{H}}{\partial p_j}, \quad & q_j|_{\tau=0} = q_j^0 \\ \frac{dp_j}{d\tau} &= -\frac{\partial \operatorname{Re}\mathscr{H}}{\partial q_j}, \quad & p_j|_{\tau=0} = p_j^0.\end{aligned} \tag{9.59}$$

The solutions of (9.59) are the functions

$$p_j = 0, \qquad j = 1, 2, 3, 4;$$

$$q_i = q_i^0, \qquad i = 1, 2;$$

$$q_3 = q_3^0 + \tau, \quad q_4 = q_4^0 - \tau.$$

Consider the function

$$\operatorname{Im} \mathscr{H}(p(q^0, \tau), q(q^0, \tau)) = -\varphi(q_3^0+\tau, q_4^0-\tau)(\omega_1^2+\omega_2^2)^{1/2}. \tag{9.60}$$

From (9.60) we infer that the absorption conditions will be verified if the trajectory $q_3 = q_3^0 + \tau$, $q_4 = q_4^0 - \tau$ of Eqs. (9.59) intersects the boundary of the domain Δ for any $(q_3^0, q_4^0) \in \Delta$ and $\tau \geqslant T'$, where T' is a constant not depending on (q_3^0, q_4^0). Therefore, in the space $R^2_{q_3q_4}$ we may take any domain containing no direct lines parallel to the line $q_3 = -q_4$ as the domain. For example we may take the stripe

$$\Delta = \{(q_1, q_2) \mid 0 \leqslant q_1 \leqslant Q, \quad q_2 \in \mathbf{R}^1\} \tag{9.61}$$

as the domain Δ. Note that in this example a condition stronger than the absorption condition is verified. Indeed, the inequality

$$\operatorname{Im} \mathscr{H}(p(q^0, T'), q(q^0, T')) < 0$$

is valid when $(q^0, p^0) \in \Omega_{1\varepsilon}$, where the manifold $\Omega_{1\varepsilon}$ is defined by the equations (cf. (9.58))

$$\omega_1^2 + \omega_2^2 = 1, \quad p_1 = p_2 = p_3 = p_4 = 0, \quad (q_3, q_4) \in \mathbf{R}^2$$
$$q_1^2 + q_2^2 < \varepsilon,$$

i.e., all the bicharacteristics are absorbed.

We shall construct the symbol of a quasi-inverse operator for an operator $f\left(\overset{1}{A}_1, \overset{1}{A}_2, \overset{2}{B}\right)$ and, at the same time, explain the meaning of the absorption conditions. Take the stripe defined by conditions (9.61) as the set Δ. Then the function $\varphi(\alpha_1, \alpha_2)$ can be chosen not depending on α_2. Let $\varphi(\alpha_1, \alpha_2) = \nu(\alpha_1)$ and consider the operator

$$g\left(\overset{1}{A}_1, \overset{1}{A}_2, \overset{2}{B}\right) = A_1 - A_2 - i\varphi\left(\overset{1}{B_1}\right)[\![\sqrt{1-A_1^2-A_2^2}]\!]. \tag{9.62}$$

By the reduction rule for the construction of a quasi-inverse operator we must find a function satisfying the conditions

$$\begin{cases} [\![-i\dfrac{\partial}{\partial\tau} + g\left(\overset{1}{L}_1, \overset{1}{L}_2, \overset{2}{\alpha}\right)]\!]\, \psi_N\left(\overset{1}{x}, -i\overset{2}{\dfrac{\partial}{\partial x}}, \alpha, \tau\right) \doteq \\ \qquad \doteq B_N\left(\overset{1}{x}, -i\overset{2}{\dfrac{\partial}{\partial x}}, \alpha, \tau\right) \\ \psi_N\left(\overset{1}{x}, -i\overset{2}{\dfrac{\partial}{\partial x}}, \alpha, 0\right) = \rho\left(-i\dfrac{\partial}{\partial x}\right) \\ B_N(x, \eta, \alpha, \tau) = O_{\mathscr{L}}(|x|^{-N}). \end{cases} \tag{9.63}$$

There is $T > 0$ such that the estimate

$$\psi_N(x, \eta, \alpha, T) = O_{\mathscr{L}}(|x|^{-M})$$

is true when $\tau = T$ and $M > 0$ is any.

Suppose that the function ψ_N can be put in the form

$$\psi_N\left(\overset{1}{x}, -i\overset{2}{\frac{\partial}{\partial x}}, \alpha, \tau\right) = e^{-i\overset{1}{\Lambda}S\left(\overset{1}{x}, -i\overset{2}{\frac{\partial}{\partial x}}, \alpha, \tau\right)}\theta_N\left(\overset{1}{x}, -i\overset{2}{\frac{\partial}{\partial x}}, \alpha, \tau\right), \tag{9.64}$$

where $\Lambda = (x_1^2 + x_2^2)^{1/2}$.

The function ψ_N satisfies the initial conditions of problem (9.63) if

$$S(x, \eta, \alpha, 0) = 0 \tag{9.65}$$

$$\theta_N(x, \eta, \alpha, 0) = \rho(\eta). \tag{9.66}$$

On substituting ψ_N in (9.63) by the commutation rules we obtain

$$e^{i\overset{1}{\Lambda}S\left(\overset{1}{x}, -i\overset{4}{\frac{\partial}{\partial x}}, \overset{4}{\alpha}, t\right)} \overset{31'}{[\![}\Lambda\frac{\partial S}{\partial\tau} - i\frac{\partial}{\partial\tau} + x_1 + \overset{1'}{\Lambda}\frac{\partial S}{\partial\alpha_1} - \overset{1'}{\Lambda}\frac{\partial S}{\partial\eta_1} - i\frac{\partial}{\partial\alpha_1} - x_2 + \overset{1'}{\Lambda}\frac{\partial S}{\partial\eta_2} - \overset{1'}{\Lambda}\frac{\partial S}{\partial\alpha_2} - i\frac{\partial}{\partial\alpha_2} - i\nu\left(\overset{2}{\alpha_1}\right)\left[\left(x_1 + \overset{1'}{\Lambda}\frac{\partial S}{\partial\alpha_1} - \overset{1'}{\Lambda}\frac{\partial S}{\partial\eta_1} - i\frac{\partial S}{\partial\alpha_1}\right)^2 + \left(x_2 - \overset{1'}{\Lambda}\frac{\partial S}{\partial\eta_2} + \overset{1'}{\Lambda}\frac{\partial S}{\partial\alpha_2} - i\frac{\partial}{\partial\alpha_2}\right)^2 + 1\right]^{1/2}]\!]\,\theta_N = 0.$$

On ordering the operators by the K-formula we obtain

$$\begin{aligned}
&[\![\overset{1'}{\Lambda}\frac{\partial S}{\partial\tau} - i\frac{\partial}{\partial\tau} + x_1 + \overset{1'}{\Lambda}\frac{\partial S}{\partial\alpha_1} - \overset{1'}{\Lambda}\frac{\partial S}{\partial\eta_1} - i\frac{\partial}{\partial\alpha_1} - \\
&\quad - \left(x_2 - \overset{1'}{\Lambda}\frac{\partial S}{\partial\eta_2} + \overset{1'}{\Lambda}\frac{\partial S}{\partial\alpha_2} - i\frac{\partial}{\partial\alpha_2}\right) - i\nu\left(\overset{2}{\alpha_1}\right) \times \\
&\quad \times\left[\left(x_1 + \overset{1'}{\Lambda}\frac{\partial S}{\partial\alpha_1} - \overset{1'}{\Lambda}\frac{\partial S}{\partial\eta_1} - i\frac{\partial}{\partial\alpha_1}\right)^2 + \left(x_2 - \overset{1'}{\Lambda}\frac{\partial S}{\partial\eta_2} + \right.\right. \\
&\quad \left.\left. + \overset{1'}{\Lambda}\frac{\partial S}{\partial\alpha_2} - i\frac{\partial}{\partial\alpha_2}\right)^2 + 1\right]^{1/2}]\!]\,\theta_N = \overset{1}{\Lambda}\left\{\frac{\partial S}{\partial\tau} + \omega_1 + \frac{\partial S}{\partial\alpha_1} - \right. \\
&\quad - \frac{\partial S}{\partial\eta_1} - \omega_2 - \frac{\partial S}{\partial\eta_2} - \frac{\partial S}{\partial\alpha_2} - i\nu(\alpha_1)\left[\left(\omega_1 + \frac{\partial S}{\partial\alpha_1} - \frac{\partial S}{\partial\eta_1}\right)^2 + \right. \\
&\quad \left.\left. + \left(\omega_2 + \frac{\partial S}{\partial\alpha_2} - \frac{\partial S}{\partial\eta_2}\right)^2\right]^{1/2} + \sum_{k=0}^{N-1}\left(-i\overset{1}{\Lambda}{}^{-1}\right)^{k+1}R_k\right\}\theta_N \times \\
&\quad \times\left(\overset{1}{x}, -i\overset{2}{\frac{\partial}{\partial x}}, \alpha, t\right) + R_N\left(\overset{3}{x}, -i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}, \alpha, t\right),
\end{aligned} \tag{9.67}$$

$$R_N(y, \eta, x, \alpha, t) = O_{\mathscr{L}}((|x| + |y|)^{-N}).$$

where R_k, $k = 0, \ldots, N-1$ are differential operators of order not higher than k with coefficients being smooth functions of x, α, τ and the derivatives of the functions S, $\omega_1 = x_1\Lambda^{-1}$, $\omega_2 = x_2\Lambda^{-1}$. Let

$$S_1 = \operatorname{Re} S, \quad S_2 = \operatorname{Im} S,$$

$$\frac{\partial S}{\partial \tau} + \omega_1 + \frac{\partial S}{\partial \alpha_1} - \frac{\partial S}{\partial \eta_1} - \omega_2 - \frac{\partial S}{\partial \alpha_2} - \frac{\partial S}{\partial \eta_2} - $$
$$- i\nu(\alpha_1)\left[\left(\omega_1 + \frac{\partial S}{\partial \alpha_1} - \frac{\partial S}{\partial \eta_1}\right)^2 + \left(\omega_2 + \frac{\partial S}{\partial \alpha_2} - \frac{\partial S}{\partial \eta_2}\right)^2\right]^{1/2}. \tag{9.68}$$

Equation (9.68) is the Hamilton-Jacobi equation with dissipation with the Hamiltonian function $\mathscr{H}$ of the operator $f\left(\overset{1}{A}_{,1}\ \overset{1}{A}_2,\ \overset{2}{B}\right)$:

$$\mathscr{H}(p, q) = (\omega_1 - p_1 + p_3) - (\omega_2 - p_2 + p_4) - i\varphi(q_3, q_4) \times$$
$$\times [(\omega_1 - p_1 + p_3)^2 + (\omega_2 - p_2 + p_4)^2]^{1/2}.$$

Applying the formulas of Ch. IV, Eq. (9.68) and equating to zero the remaining terms in the parenthesis in the right-hand side of (9.67) we obtain the transport equation with dissipation for the function θ_N. This equation can be solved by the method of Sec. 6 of Ch. IV. Let

$$D = -\int_0^\tau \operatorname{Im} \widetilde{H}(p(q^0, \tau'), q(q^0, \tau'))\, d\tau' \big|_{q^0 = q^0(\alpha, \tau)}. \tag{9.69}$$

We shall prove in Ch. IV that the solution of (9.20) satisfies the dissipation inequality

$$\operatorname{Im} S(x, \alpha, \tau) \geqslant \gamma D,$$

where γ is a constant.

When $\tau = 2Q + 1 \overset{\text{def}}{=} T$ (cf. (9.61)) the inequality $D \geqslant \delta > 0$ follows from (9.69) and the absorption conditions. Therefore, the function

$$\psi_N(x, \eta, \alpha, \tau) = e^{i\Lambda S(x, \eta, \alpha, \tau)}\theta_N(x, \eta, \alpha, \tau)$$

satisfies the estimate

$$|\psi_N(x, \eta, \alpha, T)| \leqslant \text{const}\, e^{-\delta_1 |x|}.$$

Thus the functions ψ_N satisfy (i), (ii) and the operator with the symbol

$$\varkappa_N(x, \alpha) = i\int_0^T \left(\left[\!\left[e^{i\Lambda S\left(\overset{1}{x},\ -i\overset{2}{\frac{\partial}{\partial x}},\ \alpha,\ \tau\right)} \times\right.\right.\right.$$
$$\left.\left.\left.\times\, \theta_N\left(\overset{1}{x},\ -i\overset{2}{\frac{\partial}{\partial x}},\ \alpha,\ \tau\right)\right]\!\right] \cdot 1(x)\right) d\tau$$

is a quasi-inverse of the operator $f\left(\overset{1}{A}_1, \overset{1}{A}_2, \overset{2}{B}\right)$ by the reduction rule (cf. p. 108).

Note 2. Introduce the construction of the first term of the quasi-inverse sequence in the simplest case of the main theorem in a particular case. The general case is not within our scope for the present time, since it demands a number of topological concepts which will be introduced only in Chapter IV. In order to orient the reader, however, we shall provide the construction in a particular case and consider one example. In particular, we shall assume:

(1) The Hamiltonian $\hat{H}$ is a homogeneous function in p of degree k and does not depend on ζ and η.

(2) The operator $f\left(\overset{1}{L}_1, \ldots, \overset{n}{L}_n, \overset{n+1}{\alpha}\right)$ does not change if all indices over the operators (and inside every L_i as well) are written in the inverse order (i.e., $n+1 \to 1$, $n \to 2$, $\ldots$, $1 \to n+1$, i.e., the operator $f\left(\overset{1}{L}_1, \ldots, \overset{n}{L}_n, \overset{n+1}{\alpha}\right)$ is "formally" self-adjoint.

These conditions are not vital and introduced merely for the sake of simplicity. Now let us introduce the principal conditions without which we shall get rather complicated formulas. These two conditions will invoke the non-existence of the focal points, and we saw on the example of Sec. 8 their principal importance for the asymptotics of solution.

(3) The Jacobian $\frac{dq}{dq_0} \neq 0$ for $0 \leqslant t \leqslant T$ and the Jacobian $J(q_0, t) = \frac{d(q+z)}{dq_0} \neq 0$ in the same segment. Here the function $z(\alpha, t)$ is defined by a linear system of equations (the "germ" equations)

$$\begin{aligned} \dot{z} &= i\widetilde{H}_p(q, p) + \mathcal{H}_{pq}(q, p)\, z + \mathcal{H}_{pp}(q, p)\, w, \\ \dot{w} &= -i\widetilde{H}_q(q, p) - \mathcal{H}_{qq}(q, p)\, z - \mathcal{H}_{qp}(q, p)\, w \end{aligned} \tag{9.70}$$

and the conditions

$$z|_{t=0} = w|_{t=0} = 0, \quad z = z_1, \ldots, z_n, \quad w = w_1, \ldots, w_n,$$

where $q = q(q^0, t)$, $p = p(q^0, t)$, $q = q_1, \ldots, q_n$, $p = p_1, \ldots, p_n$ are the solutions of the Hamiltonian system

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \quad q|_{t=0} = q^0, \quad p|_{t=0} = 0. \tag{9.71}$$

Introduce the construction of the symbol $g_1(y, \alpha)$, i.e., of the first term of the quasi-inverse sequence. Denote by $D = -\int_0^t \widetilde{H}\, dt$

(all integrals in t are defined as integrals along trajectories), for example,

$$\int_0^t \widetilde{H}\,dt \overset{\text{def}}{\equiv} \int_0^t \widetilde{H}(p(q^0, t), q(q^0, t))\,dt;$$

$$\mu = \langle z, C^{-1}Bz\rangle, \quad \text{where } C_{ij} = \frac{\partial(q_j+z_i)}{\partial q_i^0},$$

$$B_{ij} = \frac{\partial(p_j+w_j)}{\partial q_i^0}; \quad \mathcal{L} = \frac{1}{2}\Big[\langle p, H_p\rangle + \langle q, H_q\rangle - i\langle \widetilde{H}_p, w\rangle -$$

$$- i\langle \widetilde{H}_q, z\rangle + \frac{d}{dt}(\langle p, q\rangle - \langle w, z\rangle)\Big] - \mathcal{H}, \tag{9.72}$$

where the symbol $\langle a, b\rangle$ denotes the sum $\sum a_i b_i$. Denote by

$$\Phi(q^0, t) = \mu(q^0, t) + \int_0^t \mathcal{L}\,dt. \tag{9.73}$$

The absorption condition involves an important inequality (the so-called inequality of dissipation)

$$\operatorname{Im}\Phi \geqslant c_1 D - c_2 D^{3/2-\varepsilon} \tag{9.74}$$

which is true for some $c_1 > 0$, $c_2 > 0$, and $\varepsilon > 0$. Let M be the lower boundary of such constants c_2 for which the dissipation inequality is true. Then

$$g_1(y_1, \ldots, y_n, \alpha) =$$

$$= i\int_0^T \frac{P(q^0)}{\sqrt{|J(q^0, t)|}} e^{i\{\Phi(q^0, t)+tMD^{3,2-\varepsilon}(q_0, t)\}}\Big|_{q^0=q^0(\alpha, yt)}\,dt, \tag{9.75}$$

where $q^0(\alpha, y, t)$ is a solution of the equation $q_i(q^0, t) = \alpha_i$, $m+n \geqslant i > n$; $q_i(q^0, t) = y_i$, $i \leqslant n$.

Example. Consider an operator

$$\overset{1}{A}_1 - i\varphi\left(\overset{2}{B}\right)\overset{1}{A}_2 \tag{9.76}$$

with the symbol

$$y_1 - i\varphi(\alpha)\,y_2,$$

where $\varphi(\alpha) \geqslant 0$ equalszero for $|\alpha| < \frac{T}{2}$ and is strictly positive for $\alpha > T/2$. If the operators A_1, A_2, B commute then the quasi-inverse element obviously does not exist (cf. p. 102).

Take

$$[A_1, B] = i, \quad [A_2, A_1] = 0, \quad [A_2, B] = 0.$$

Let the spectrum of A_2 lie on the half-axis $y_2 > \delta > 0$. In this case it is sufficient that the symbol should belong to $C^\infty_{\mathscr{L}}$ only for $y_2 > \delta$.

For example, let $A_1 = -i \frac{\partial}{\partial x}$, $B = x$, $A_2 = k$, where k is a parameter $\delta < k < \infty$ and let $F(x, k) \overset{\text{def}}{=} F(x)$ be a function with a support in Ω, such that $F(x, k) P(x) \equiv F(x, k)$. The function $P(\alpha)$ was defined earlier in the remark to the main theorem. Consider the equation for $u(x, k)$:

$$\left(\overset{1}{A_1} - i\varphi\left(\overset{2}{B}\right) \overset{1}{A_2}\right) u(x, k) = P(B) F(x, k),$$
$$|F(x, k)| < \text{const}. \tag{9.77}$$

We can tell by formula (9.77), that (see Sec. 6 of Ch. V) an asymptotic solution of the problem is obtained by a "quasi-inverse" sequence applied to $F(x)$, which satisfies the equation within the accuracy to any function, tending to zero as $k \to \infty$ faster than k^{-N} for any N.

We shall verify this by the given example, using formula (9.75). The Hamiltonian of the operator (9.76) is of the form

$$y_1 - i\frac{\partial}{\partial \alpha} - i\varphi(\alpha)\,|y_2|. \tag{9.78}$$

The Hamiltonian function, corresponding to the Hamiltonian (9.78), is of the form $\mathscr{H} = \omega_1 - p_1 + p_3 - i\varphi(q_1)\,|\,\omega_2 - p_2 + p_4\,|$, $\omega_2 > 0$. The systems of Hamilton and of the germ are the following in this case

$$\begin{aligned}
&\dot q_1 = 1, \quad \dot q_2 = \dot q_3 = \dot q_4 = 0; \quad q(0) = q_0,\\
&\dot p = 0; \quad p(0) = 0.\\
&\dot z_1 = \dot z_3 = 0, \quad \dot z_2 = \dot z_4 = -i\varphi(q_1); \quad z(0) = 0\\
&\dot w_1 = i\varphi'(q_1)\,\omega_2, \quad \dot w_2 = \dot w_3 = \dot w_4 = 0; \quad w(0) = 0.
\end{aligned} \tag{9.79}$$

Calculate the values entering formula (9.75). We have

$$\Phi(q_0, \omega, t) = -\omega_1 t + i\omega_2 \int_0^t \varphi(q_{10} + \tau)\,d\tau. \tag{9.80}$$

We see that $z(q_0, t) \equiv 0$ by the germ system. Thus $\mu(q_0, t) = 0$. We shall calculate the dissipation D.

By the definition we have

$$D(q_0, t) = -\int_0^t \widetilde{H}\,dt = \omega_2 \int_0^t \varphi(q_0 + t)\,dt. \tag{9.81}$$

By (9.80) and (9.81), the inequality of dissipation is satisfied when $c_1 = 1$, $c_2 = 0$ in our case. We shall find the solution of the equation $q_3(q_0, t) = \alpha$. We have $q_3 = q_{03} - t = \alpha$ by the Hamiltonian system. Therefore $q_{03} = \alpha - t$. It is easy to see that $J(q_0, t) = 1$. On substituting the necessary functions in formula (9.75), we obtain

$$g(y, \alpha) = i \int_0^T e^{-iy_1 t - |y_2| \int_{\alpha - t}^{\alpha} \varphi(\xi)\, d\xi} P(\alpha - t)\, dt =$$

$$= i \int_0^\infty e^{-iy_1 t - |y_2| \int_{\alpha - t}^{\alpha} \varphi(\xi)\, d\xi} P(\alpha - t)\, dt + O(|y_2|^{-\infty}). \qquad (9.82)$$

Transform the integral in the right-hand side into the form

$$-i \int_\alpha^{-\infty} e^{-iy_1(\alpha - \beta) - y_2 \int_\beta^\alpha \varphi(\xi)\, d\xi} P(\beta)\, d\beta =$$

$$= +ie^{-iy_1\alpha - |y_2| \int_0^\alpha \varphi(\xi)\, d\xi} \int_{-\infty}^{\alpha} e^{iy_1\beta - |y_2| \int_\beta^0 \varphi(\xi)\, d\xi} P(\beta)\, d\beta \qquad (9.83)$$

with the help of the substitution $\alpha + t = \beta$. Considering that $e^{ix\overset{2}{A}} e^{-i\beta \overset{1}{A}} F(x) = F(\beta)$, we obtain

$$g\left(\overset{1}{A}_1, \overset{1}{A}_2, \overset{2}{B}\right) F(x) = ie^{-k \int_0^x \varphi(\xi)\, d\xi} \int_{-\infty}^{x} e^{k \int_0^\beta \varphi(\xi)\, d\xi} F(\beta)\, d\beta + O(k^{-\infty})$$

applying the operator with symbol (9.83) to the function $F(x) \overset{\text{def}}{=} F(x, k)$.

It is easy to verify that the first term in the right-hand side produces the exact solution of equation (9.77). Thus in this example the first term of the asymptotic (i.e., of the "quasi-inverse" sequence) happens to coincide, correct to $O(1/k^n)$, for any n, with the exact solution of the problem.

Note the following circumstance. If $F(x, k) = 0$ when $x < 0$ and $x < T$, then the obtained integral equals $\int_0^x F(x, k)\, dx$.

Thus we may take the integral of the function $F(x, k)$ from 0 to x, i.e., we may solve the problem (Sec. 6 of Ch. V)

$$i\frac{\partial u}{\partial x}=F(x, k),\ u|_{x=0}=0 \tag{9.84}$$

with the help of the main theorem by the following procedure:

1. Add such a term to the operator $\frac{\overset{\bullet}{\partial}}{\partial x}$, that the absorption conditions might be fulfilled;
2. Take an ordered representation of the obtained operator;
3. Apply the formula of the main theorem (considered above in the particular case).

Then the above-mentioned formula of the main theorem has merely stated that the solution of problem (9.84) is given by the integral $\int_0^x F(x, k)\,dx$ correct to $O(k^{-1/2+\varepsilon})$; indeed, the solution of problem (9.84) is exactly equal to this integral by a formula indicated in Chapter V. The same procedure provides a solution of a broad class of differential equations.

Note. Consider a square of an operator of the form (9.77):

$$-\frac{\partial^2}{\partial x^2}+2k\varphi(x)\frac{\partial}{\partial x}-k^2\varphi^2(x).$$

Here the absorption condition in the form indicated above is obviously not fulfilled; nonetheless, it is clear that the operator is quasi-inverse. There is a generalization of the theorem for this case in the following item. This generalization also contains the example in Sec. 8, which does not satisfy the condition of this item.

Example. Consider the wave operator (cf. (8.6))

$$\hat{H}=\frac{\partial^2}{\partial t^2}-c^2(x)\Delta.$$

The Hamiltonian system corresponding to it has the form

$$\begin{cases} \dot{p}_4=0, \\ \dot{p}=2c\nabla c\,|p|^2, \quad p_4^0=\pm c(x^0)\,|p^0|; \\ \dot{t}=2p_4, \quad (p_4^0)^2+|p^0|^2=1, \\ \dot{x}=2pc^2(x), \end{cases} \tag{9.85}$$

where $x=x_1, x_2, x_3$; $p=p_1, p_2, p_3$. It follows from equations (9.85) that

$$(c^2(x^0)+1)\,|p^0|^2=1$$

and

$$p_4^0 = \sqrt{\frac{1}{1+\frac{1}{c^2(x^0)}}}.$$

Hence

$$t = \pm 2\,\frac{|c(x^0)|}{1+|c(x^0)|}\,\tau + t_0.$$

Add to the wave operator a complex term of the form

$$i\widetilde{H} = -i\varphi_1\left(\overset{2}{t^2}\right)\overset{1}{p^2}\varphi_2\left(\overset{1}{p^2}\right), \quad p = i\nabla,$$

where $\varphi_{1,2}(t^2) = 0$, when $|t| \leqslant T$, $\varphi_{1,2}(t^2) = \alpha > 0$ when $t = T + 1$. Then the absorption condition of each trajectory is fulfilled, since $p^2 \neq 0$ is on a trajectory as it follows from equation (9.85).

Therefore, the operator $\hat{H} + i\widetilde{H}$ is quasi-inverse. It is not difficult to see that the solution of the problem

$$\left[\hat{H} + i\widetilde{H}\, u = F(x, t),\right.$$

where $F(x, t)$ is finite, does not depend on the form of $\varphi(t^2)$, when $t^2 < T^2$ and $\hat{H}$, $\widetilde{H}$ are defined above.

Thus the solution of the problem for sufficiently small t ($t < T$) does not change, when we introduce the absorption term.

Note. An addition of the summand $-i\varphi(x)\,p^2\varphi(p)$, where $\varphi(x) = 0$ when $|x| < M$ and $\varphi(x) = \alpha > 0$ when $|x| = M + 1$ provides the absorption condition for every trajectory as well, since for sufficiently large τ $|x(x^0, \tau)| > M + 1$ by equations (9.84). But in this case only the asymptotic of the non-smooth part of the solution does not depend on the form of the absorption term. Nonetheless the addition of the term makes sense (from a physical viewpoint) for some problems.

(4) **The main theorem.** Consider a one-parameter family of symbols $f(x_1, \ldots, x_n, \alpha, \xi)$ and symbols of order n, $r(x_1, \ldots, x_n, \alpha, \xi)$, where ξ is a parameter, $0 < \xi \leqslant \infty$. Let

$$\lim_{\xi\to\infty} f(x_1, \ldots, x_n, \alpha, \xi) = f_0(x_1, \ldots, x_n, \alpha,) \in \mathscr{S}^\infty,$$

$$\lim_{\xi\to\infty} r(x_1, \ldots, x_n, \alpha, \xi) = r_0(x_1, \ldots, x_n, \alpha) \in \mathscr{S}^\infty,$$

$$f_0 \neq 0, \quad r_0 \neq 0.$$

Let $P_0(x)$ be a function of the class $C_0^\infty(\mathbf{R}^n)$ equal to unity in the domain $|x| \leqslant d$, where d is a constant.

Let $A_1, \ldots, A_n, B \in X$, $\hat{r} \equiv r\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}, \xi\right) \in X$ and the Hamiltonian of the operator $f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}, \xi\right)$ with respect to r is asymptotically ρ-quasi-homogeneous in the variables $y_i = x_i$, $0 \leqslant i \leqslant n$; $y_{n+j} = -i\dfrac{\partial}{\partial \alpha_j}$, $j \leqslant m$; ξ, and the numbers ρ_i, corresponding to the last $(m+1)$ arguments, are equal to unity. This Hamiltonian depends also on the parameters α, $\varphi = -i\dfrac{\partial}{\partial x}$. The leading term of this symbol is denoted by $\pi(y, \varphi, \alpha, \xi)$ and the essential part is denoted by $\pi_0(y, \varphi, \alpha, \xi)$, where $y = (y_0, y_1, \ldots, y_{n+m})$, $\varphi = (\varphi_1, \ldots, \varphi_n)$, $\alpha = (\alpha_1, \ldots, \alpha_m)$.

Let $\rho_i \neq 1$ when $s < i \leqslant n$, $\rho_i = 1$ when $i \leqslant s$ (in the opposite case, we shall change the indices of the arguments y). Introduce the Hamiltonian coordinates. Denote

$$y_1 = \omega_1 - p_{m+1}, \ldots, y_s = \omega_s - p_{m+s}; \quad y_{s+1} = \omega_{s+1}, \ldots,$$
$$y_n = \omega_n; \quad y_{n+1} = p_1, \ldots, y_{n+m} = p_m; \quad \varphi_1 = q_{m+1}, \ldots$$
$$\ldots, \varphi_n = q_{m+n}.$$
$$\alpha_1 = q_1, \ldots, \alpha_m = q_m.$$

Besides this, take the parameter $\nu = \xi$ instead of the parameter ξ. Denote by $\mathscr{H}(p, q, \omega, \nu)$ the function π considered in the new arguments and by $\mathscr{H}_0(p, q; \omega, \nu)$ the function π_0.

Define $\mathscr{H}(p, q, \omega, \nu)$ as *a Hamiltonian function*. Denote by $H = \operatorname{Re}\mathscr{H}$, $\tilde{\tilde{H}} = \operatorname{Im}\mathscr{H}$ as previously and let Ω_ε be a set in the space of the variables q, ω, p, ν defined by the following system of equations

$$\sum_{i=1}^{n} (\omega_i)^{2/\rho_i} = 1, \quad \sum_{i=1}^{n} (q_{m+i})^2 < \varepsilon$$
$$(q_1, \ldots, q_m) \in M^m, \quad p = 0, \quad |\mathscr{H}(p, q, \omega, \nu)| < \varepsilon$$
$$|\nu| \leqslant d.$$

Now we shall formulate the absorption conditions, generalizing the conditions of item 3.

Definition. *The absorption conditions are fulfilled, if there exist constants* $\varepsilon > 0$, $T > 0$, *a number* $\tau' = \tau'(q^0, \omega, p^0, \nu)$, $0 < \tau' < T$ *and a function* $\mathscr{A}(q, \omega) \in C^\infty$ *which is not equal to zero when* $(q, \omega) \in \{(q, \omega)\colon (q, \omega, 0, 0, \nu) \in \Omega_0\}$ *such that*: (1) *when* $0 \leqslant \tau \leqslant \tau'$ *and* $(q^0, \omega, p^0, \nu) \in \Omega_\varepsilon$ *there exists a solution*

$$q(q^0, \omega, p^0, \nu, \tau); \quad p(q^0, \omega, p^0, \nu, \tau)$$

of the Hamiltonian system

$$\dot{q} = H_p, \qquad q|_{\tau=0} = q^0,$$
$$\dot{p} = -H_q, \qquad p|_{\tau=0} = p^0,$$

belonging to the class C^∞;

(2) *when* $0 \leqslant \tau \leqslant \tau'$ *and* $(q^0, \omega, p^0, \nu) \in \Omega_\varepsilon$ *the function*

$$\mathcal{A}(q^0, \omega)\cdot\tilde{\tilde{H}}(p(q^0, \omega, p^0, \nu, \tau); \quad q(q^0, \omega, p^0, \nu, \tau); \omega; \nu)$$

is non-negative and when $\tau = \tau'$ *the function is strictly positive.*

Theorem 9.1. (The main theorem.) *Let the absorption conditions be satisfied for a Hamiltonian function* $\mathcal{H}$ *corresponding to a symbol* $f(x, \alpha, \xi)$. *Then there exists a sequence of symbols* $g_N(x, \alpha, x_0, \xi) \in \in C^\infty_{\mathcal{L}}$ *depending on the parameter* ξ, *such that*

$$[\![f\left(\overset{1}{A_1}, \overset{2}{A_2}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}, \xi\right)]\!] \times$$
$$\times \left[g_N\left(\overset{2}{A_1}, \overset{3}{A_2}, \ldots, \overset{n+1}{A_n}, \overset{n+2}{B}, \overset{1}{r}, \xi\right)\right] =$$
$$= P_0\left(\xi^{-\rho_1}\overset{1}{A_1}, \xi^{-\rho_2}\overset{2}{A_2}, \ldots, \overset{n}{A_n}\xi^{-\rho_n}\right) +$$
$$+ R_N\left(\overset{2}{A_1}, \ldots, \overset{n+1}{A_n}, \overset{n+2}{B}, \overset{1}{r}, \xi\right), \tag{9.86}$$

where the function $R_N(x_1, \ldots, x_n, \alpha, x_0, \xi) = O_{\mathcal{L}}(|x|^{-N})$ *for any* N *uniformly in* ξ *and* α.*

The construction of $g_N\left(\overset{2}{A_1}, \ldots, \overset{n+1}{A_n}, \overset{n+2}{B}, \xi\right)$ is contained in Chapter V. In the case $\xi = \infty$ we obtain the simplest version of the main theorem formulated earlier. Though in the right-hand side of (9.85) there is the function P_0 instead of 1, which was in the main problem, this theorem gives a direct answer to the question stated by the main problem. Indeed, it is possible to take a unity instead of the function P_0 in the right-hand side of (9.86), if the common spectrum of the sequence $\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}$ belongs to the domain $\Omega_\xi \times \Omega$. This condition is not fulfilled as we saw in the problem on the crystal (there is the spectrum of $ih\frac{\partial}{\partial x}$ on M_N in the segment

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* Under the same conditions there is a left quasi-inverse sequence $g'_N = g'_N\left(\overset{n+1}{A_1}, \ldots, \overset{2}{A_n}, \overset{1}{B}, \overset{n+2}{r}, \xi\right)$:

$$g'_N f - P_0 = O_{\mathcal{L}}(|x|^{-N}).$$

See the Main Theorem in the general form in Ch. V.

$[-\pi, \pi]$ and $\xi = \frac{1}{h}\Big)$. But it is true for general difference schemes if a convenient space of smooth functions is taken instead of M_N. There is a similar situation in the case, when B is the Hamiltonian of the oscillator considered in Sec. 8 and therefore has a discrete spectrum located on a lattice of a step h, where h is Planck's constant. The construction of $g_N\left(\overset{1}{A_1}, \ldots, \overset{n+1}{A_n}, \overset{n+2}{B}\right)$ in these problems permits to obtain the quasi-classical asymptotics of the secondary quantized equations.

Indeed, all the results obtained for the equations of vibrations of a crystal lattice can be extended to this general case, moreover, if the absorption term in the given equation is missing, it must be introduced in the same way as it was done in the example with the wave operator.

The expansion in a power series in $\frac{1}{\xi}$ can be obtained in just the same way in the general case. Here the parameter ν has the function of ω in the crystal equation.

Problem. Find the effect of Cherenkov's type for a difference scheme approximating, as $h \to 0$, the wave equation:

$$\frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{h^2} - C_{(m)}^2 \frac{u_n^{m+1} - 2u_n^m + u_n^{m-1}}{\tau_2} = F(n, m)$$

$$\frac{h}{\tau} = d = \text{const}, \quad c(x) \in C^\infty, \quad F(n, m) = 0,$$

when $n = -1, -2, \ldots$.

Find an asymptotic of the solution correct to h^N.

Hint. The cited Hamiltonian function has the form

$$\sin^2 \nu p_1 = c^2(x_2)\, d^2 \cdot \sin^2 \nu p_2.$$

Use a space of integer functions of the first order of the type $\frac{\pi}{h}$ as an analogue of M_N (Kotelnikov's theorem).

Note. There is no operation "prime" in the statement of the main theorem. Indeed, we have used the operation proving the theorem only for $M = \left\{ i\, \frac{\partial}{\partial x}, x \right\}$; the rule of the reduction of the main problem remains true without the operation "prime". Therefore, the main theorem is true, if the structures satisfy axioms (μ_1)-(μ_8). If the axioms (μ_1)-(μ_6) are true, then the main theorem can be proved only in the case, when the symbol $f(x, \xi, \alpha)$ is a polynomial in x.

In conclusion, the following statement can be taken as an axiom instead of $\mu(8)$: (9.5) holds for any $P \in \mathscr{S}^\infty$ (so to say, the μ-structure agrees with the ordered representation of the nilpotent Lie algebra).

I. FUNCTIONS OF A REGULAR OPERATOR

In this chapter we shall develop an operational calculus corresponding to a μ-structure (see Introduction) based on the commutative unbounded operator algebra in a Banach space B. It will be our purpose to determine how wide an operator class constitutes a set M. In addition we shall clarify which symbols correspond to bounded operators on B. This will facilitate the forming of a closed extension of the C^∞-symbol class.

For simplicity our presentation will not depend on the μ-structure notion. The symbol concept introduced in this chapter, however, proves to be coincident with a similar notion within the μ-structure framework.

To illustrate the function of an operator concept we shall consider a function of multiplication operator A by an independent variable x defined on a Sobolev space $W_2^k(\mathbf{R}^n)$. It is easy to see that if $\varphi \in \in C^k(\mathbf{R}^n)$ then one can readily define the function of multiplication operator A by x, this operator being in turn a multiplication operator by $\varphi(x)$. It will be seen that $\varphi(A)$ is a bounded operator on W_2^k. If the kth derivative of $\varphi(x)$ is unbounded then the multiplication operator by $\varphi(x)$ is also unbounded in W_2^k.

On the other hand a multiplication operator by x defined on W_2^k has the following properties:

(1) it generates a one-parameter group of multiplication operators by the exp $\{ixt\}$ family;

(2) the multiplication operator satisfies the condition: for any $g \in W_2^s(\mathbf{R}^n)$, where $s \geqslant k$, the following inequality is valid:

$$\| e^{ixt} g(x) \|_{W_2^k} \leqslant c(1 + |t|^k) \| g(x) \|_{W_2^s},$$

that is,

$$\| e^{ixt} \|_{W_2^s}^{W_2^k} \leqslant c(1 + |t|^k),$$

$c = \text{const}.$

In terms of the corresponding evolution equation we can formulate the conditions (1), (2) as follows: there exists a unique solution u of the evolution equation

$$i\,\frac{du(t)}{dt}=Au(t), \tag{$*$}$$

$$u(0)=g, \quad g\in B_2, \quad B_2=W_2^l,$$

u being a continuous function of t with values in $B_1=W_2^k$, $k\leqslant l$ satisfying the condition

$$\|u(t)\|_{B_1}\leqslant c(1+|t|^s)\cdot\|u(0)\|_{B_2} \tag{$**$}$$

for all $u(0)\in B_2=W_2^l$.

As we shall show, a wide operator class is covered by the relation between the number s, featured in the estimate on the solution of evolution equation (*) (where, as a given operator, we have a multiplication operator by x), and the boundedness of a multiplication operator by $\varphi(x)$.

The condition (**) for the growth rate of the solution of the evolution equation generated by unbounded operator A becomes necessary, if it is required that the finite k times differentiable function of A should be a bounded operator. If we skip this requirement and restrict ourselves only to the infinitely differentiable functions of an operator, then the growth rate condition (**) may be essentially relaxed.

Consider a differentiation operator in a space $C(\mathbf{R})$ of continuous functions bounded in infinity (or in a space C_N of continuous functions increasing less than $|x|^N$). As it is easily seen, the totality of functions of such an operator form an algebra $\mathscr{B}_0$ (or an algebra $\mathscr{B}_N$, respectively). This example shows the necessity of using the algebra $\mathscr{B}_0$ (the algebra $\mathscr{B}_N$ is the case of unbounded continuous functions) if we wish to consider an algebra of functions of operators drawn from a class which includes a differentiation operator in a continuous function space. With these two examples we wish to emphasize "the necessity" of using a growth rate condition of type (**) as well as the algebra $\mathscr{B}_N$.

Finally, let us consider a multiplication operator A by $x+iy$ defined on $W_2^k(\mathbf{R}^2)$. It is natural to define a k times differentiable function of $x+iy$ as a function $\varphi(x, y)$ of two arguments x and y. It is evident that each operator, $x=\operatorname{Re}A$ and $y=\operatorname{Im}A$, will satisfy the condition (**), $\operatorname{Re}A$ being commutative with $\operatorname{Im}A$.

We shall introduce operators satisfying a condition of type (**) and call them generators of degree s. Then we shall consider operators of type A_1+iA_2, where A_1 and A_2 are generators of degree s, A_1 being commutative with A_2. Such operators will be called regular operators and functions of them will be considered as functions of A_1 and A_2.

In Sec. 7 we shall prove the theorem which will show the sufficiency of a condition of type (**). Specifically, we shall prove that in the case of a purely discrete operator spectrum the existence of a complete system of eigenelements and associated elements (the number of the latter not exceeding N for a given eigenvalue) is equivalent to the regularity of the operator (of degree N). Thus, for a discrete spectrum at least, we shall show how wide the regular operator class is.

When the spectrum is not discrete the concept of a system of eigenvectors and associated vectors is introduced in the case of the regular operator in Sec. 8. The theorem of completeness of such a system will be proved there. A similar concept in the case of an arbitrary operator, however, has not yet been involved. Therefore, there is no inverse theorem as yet.

Sec. 1. Certain Spaces of Continuous Functions and Related Spaces

We shall now consider certain Banach algebras of continuous functions. First of all, consider the set of continuous bounded complex-valued functions with the field of definition $\Omega \subset \mathbf{R}^n$. As usual, we introduce the structure of a vector space on this set. Define the norm of function f by the formula

$$\| f \| = \sup_{x \in \Omega} | f(x) |. \tag{1.1}$$

Denote such normed space of functions by $C(\Omega)$. The space $C(\Omega)$ is a Banach space.

In analogy with $C(\Omega)$ we define the Banach space $C^{(k)}(\Omega)$, $k > 0$ being an integer. The norm in $C^{(k)}(\Omega)$ is defined by the formula

$$\| f \|_{C^k(\Omega)} = \max_{0 \leq |j| \leq k} \sup_{x \in \Omega} | D^j f(x) |, \tag{1.2}$$

where $j = (j_1, j_2, \ldots, j_n)$, $|j| = \sum_{j=1}^{n} j_i$, $D^j = \dfrac{\partial^{|j|}}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}}$.

Let $C^*(\mathbf{R}^n)$ be the space conjugated to $C(\mathbf{R}^n)$. *The Dirac δ_ξ-function* is called the functional on $C(\mathbf{R}^n)$, which gives the value $g(\xi)$ at point $\xi \in \mathbf{R}^n$ for every function $g \in C(\mathbf{R}^n)$. The functional δ_ξ belongs to the space $C^*(\mathbf{R}^n)$.

Let $N \geqslant 0$. By $C_N(\mathbf{R}^n)$ we denote the space of continuous functions in $\mathbf{R}^n$ with finite norms

$$\| g \|_{C_N(\mathbf{R}^n)} = \sup_{x \in \mathbf{R}^n} \left| \frac{g(x)}{(1 + |x|)^N} \right|.$$

Thus, by definition, the space $C_N(\mathbf{R}^n)$ will include all continuous functions increasing not faster than $|x|^N$.

In this respect it resembles "a superstructure" on $C(\mathbf{R}^n)$ which is similar to $\widetilde{W}_2^{-k}(\mathbf{R}^n)$ being "a superstructure" on $L_2(\mathbf{R}^n)$.

Obviously, the Dirac function δ_ξ can be considered as an element of the space $C_N^*(\mathbf{R}^n)$.

The functional $\mathscr{F}_\varphi$ provides another example of an element of $C_N^*(\mathbf{R}^n)$:

$$\mathscr{F}_\varphi(g) = \int_{\mathbf{R}^n} \varphi(x)\, g(x)\, dx, \tag{1.3}$$

where φ is an element of the vector space $C_0^\infty(\mathbf{R}^n)$ of infinitely differentiable functions with compact carriers.

It will be convenient to identify the functional $\mathscr{F}_\varphi$ with the function φ (it is easy to see that $(\varphi \neq 0) \Rightarrow (\mathscr{F}_\varphi \neq 0)$, i.e., it is easier to write down $\varphi(x) \in C_N^*(\mathbf{R}^n)$ instead of $\mathscr{F}_\varphi \in C_N^*(\mathbf{R}^n)$). We shall denote any functional $\mathscr{F} \in C_N^*(\mathbf{R}^n)$ by $\varphi(x)$ (though $\varphi(x)$ may not be the function in the ordinary sense of the word); the functional $\mathscr{F}$ will be denoted as $\int \varphi(x)\, g(x)\, dx$, though the integral does not exist in the Riemann sense. In particular, the functional δ_ξ will be written $\int \delta(x-\xi)\, f(x)\, dx$, as accepted in physics literature.

It will be sufficient for us to consider a space, simpler and easier to understand, namely, the subspace of $C_N^*(\mathbf{R}^n)$ which is the linear core of the association of the Dirac δ-function and $C_0^\infty(\mathbf{R}^n)$.

Let us denote by $C_N^+(\mathbf{R}^n)$ the closure in $C_N^*(\mathbf{R}^n)$ of the linear core of the set $C_0^\infty(\mathbf{R}^n) \cup \{\delta_\xi\}_{\xi\in\mathbf{R}^n}$.

Let us now define the *Fourier transformation* in $C_N^+(\mathbf{R}^n)$. As usual, the *Fourier transform* of function $\varphi \in C_0^\infty(\mathbf{R}^n) \subset C_N^+(\mathbf{R}^n)$ will be denoted by function $\tilde{\varphi}$ defined by the formula

$$\tilde{\varphi}(p) \overset{\text{def}}{=} (F\varphi)(p) \overset{\text{def}}{=} \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} \varphi(x)\, e^{-ip\cdot x}\, dx. \tag{1.4}$$

The function defined by the formula

$$\tilde{\delta}_\xi(p) \overset{\text{def}}{=} (F\delta_\xi)(p) \overset{\text{def}}{=} \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} \delta_\xi(x)\, e^{ip\cdot x}\, dx \overset{\text{def}}{=} \frac{1}{(2\pi)^{n/2}} e^{-ip\cdot\xi} \tag{1.5}$$

will be called the *Fourier transform of the* δ_ξ*-function*.

Denote by $\mathscr{B}_N^0(\mathbf{R}^n)$ a vector space which consists of linear combinations of the functions of type (1.4), (1.5).

The formulas (1.4) and (1.5) define an operator which maps a linear core of the set $C_0^\infty(\mathbf{R}^n) \cup \{\delta_\xi\}_{\xi\in\mathbf{R}^n}$ into $\mathscr{B}_N^0(\mathbf{R}^n)$. The operator F will be shown to have an inverse operator.

Let us now introduce the norm in space $\mathscr{B}_N^0(\mathbf{R}^n)$:

$$\|\tilde{\varphi}\|_{\mathscr{B}_N^0(\mathbf{R}^n)} \stackrel{\text{def}}{=} \|F^{-1}\tilde{\varphi}\|_{C_N^+(\mathbf{R}^n)}. \tag{1.6}$$

Denote by $\mathscr{B}_N(\mathbf{R}^n)$ the completion of the space $\mathscr{B}_N^0(\mathbf{R}^n)$ in the norm (1.6). Then the operator F can be uniquely extended to the isometric isomorphism of space $C_N^+(\mathbf{R}^n)$ on $\mathscr{B}_N(\mathbf{R}^n)$. We shall denote this isometric isomorphism by F and call it the *Fourier transformation of the elements of space* $C_N^+(\mathbf{R}^n)$. We shall later see that the space $\mathscr{B}_N(\mathbf{R}^n)$ is nothing but the space of continuous (smooth) functions.

In the linear core of the set

$$[C_0^\infty(\mathbf{R}^n) \cup \{\delta_\xi\}_{\xi\in\mathbf{R}^n}] \subset C_N^+(\mathbf{R}^n)$$

there exists a commutative composition which transforms this variety into the algebra, namely

$$(\varphi * \psi)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} \varphi(y)\,\psi(x-y)\,dy \quad \text{for } \varphi,\ \psi \in C_0^\infty(\mathbf{R}^n);$$

$$\delta_\xi * \delta_\eta = \frac{1}{(2\pi)^{n/2}}\,\delta_{\xi+\eta};$$

$$(\delta_\xi * \psi)(x) = \frac{1}{(2\pi)^{n/2}}\,\psi(x-\xi) \quad \text{for } \psi \in C_0^\infty(\mathbf{R}^n).$$

Show that there is a constant c, such that

$$\|\varphi * \psi\| \leqslant c\,\|\varphi\|\,\|\psi\|.$$

Let $f \in C_N(\mathbf{R}^n)$. Consider the integral

$$I = \int_{\mathbf{R}^n} [\varphi * \psi](x)\,f(x)\,dx. \tag{1.7}$$

The integral (1.7) can be transformed to the form

$$(2\pi)^{n/2} I = \int_{\mathbf{R}^n} \psi(t)\,dt \int_{\mathbf{R}^n} f(y+t)\,\varphi(y)\,dy. \tag{1.8}$$

For a proof thereof, it is sufficient to consider three particular cases since we consider the linear core of the set $C_0^\infty(\mathbf{R}^n) \cup \{\delta_\xi\}$:

(a) $\varphi,\ \psi \in C_0^\infty(\mathbf{R}^n)$;

(b) $\varphi \in C_0^\infty(\mathbf{R}^n),\ \psi = \delta_\xi$;

(c) $\varphi = \delta_\xi,\ \psi = \delta_\eta$.

In the (a) case

$$(2\pi)^{n/2}\int\limits_{\mathbf{R}^n}[\varphi*\psi](x)f(x)\,dx=\int\limits_{\mathbf{R}^n}f(x)\,dx\int\limits_{\mathbf{R}^n}\varphi(y)\psi(x-y)\,dy=$$

$$=\int\limits_{\mathbf{R}^n}\varphi(y)\,dy\int\limits_{\mathbf{R}^n}f(x)\psi(x-y)\,dx=\int\limits_{\mathbf{R}^n}\varphi(y)\,dy\int\limits_{\mathbf{R}^n}\psi(t)f(y+t)\,dt=$$

$$=\int\limits_{\mathbf{R}^n}\psi(t)\,dt\int\limits_{\mathbf{R}^n}\varphi(y)f(y+t)\,dy.$$

In the (b) case

$$(2\pi)^{n/2}I=\int\limits_{\mathbf{R}^n}\varphi(x-\xi)f(x)\,dx;$$

on the other hand,

$$\int\limits_{\mathbf{R}^n}\delta(t-\xi)\,dt\int\limits_{\mathbf{R}^n}f(y+t)\varphi(y)\,dy=$$

$$=\int\limits_{\mathbf{R}^n}f(y+\xi)\varphi(y)\,dy=\int\limits_{\mathbf{R}^n}f(x)\varphi(x-\xi)\,dx.$$

Finally, in the (c) case

$$(2\pi)^{n/2}I=\int\limits_{\mathbf{R}^n}\delta(x-\xi-\eta)f(x)\,dx=f(\xi+\eta),$$

$$\int\limits_{\mathbf{R}^n}\delta(t-\eta)\,dt\int\limits_{\mathbf{R}^n}f(y+t)\delta(y-\xi)\,dy=$$

$$=\int\limits_{\mathbf{R}^n}\delta(t-\eta)f(\xi+t)\,dt=f(\xi+\eta).$$

This completes the proof (1.8). Using (1.8) we have

$$\left|\int\limits_{\mathbf{R}^n}(\varphi*\psi)(x)f(x)\,dx\right|\leqslant(2\pi)^{-n/2}\|\psi\|_{C_N^*(\mathbf{R}^n)}\times$$

$$\times\left(\sup_{t\in\mathbf{R}^n}\left|(1+|t|)^{-N}\int\limits_{\mathbf{R}^n}f(y+t)\varphi(y)\,dy\right|\right)\leqslant$$

$$\leqslant(2\pi)^{-n/2}\|\psi\|_{C_N^*(\mathbf{R}^n)}\|\varphi\|_{C_N^*(\mathbf{R}^n)}\times$$

$$\times\sup_{t\in\mathbf{R}^n}\left[(1+|t|)^{-N}\sup_{y\in\mathbf{R}^n}\frac{f(y+t)}{(1+|y|)^N}\right]\leqslant$$

$$\leqslant(2\pi)^{-n/2}\|\psi\|_{C_N^*(\mathbf{R}^n)}\|\varphi\|_{C_N^*(\mathbf{R}^n)}\times$$

$$\times\sup_{t,\,y\in\mathbf{R}^n}\left|\frac{f(y+t)}{(1+|t|)^N(1+|y|)^N}\right|.$$

Since

$$\sup_{t,\, y\in \mathbf{R}^n}\left|\frac{f(y+t)}{(1+|t|)^N(1+|y|)^N}\right|\leqslant$$

$$\leqslant \|f\|_{C_N(\mathbf{R}^n)}\sup_{t,\, y\in\mathbf{R}^n}\left(\frac{1+|t+y|}{(1+|t|)(1+|y|)}\right)^N\leqslant \|f\|_{C_N(\mathbf{R}^n)},$$

we have

$$\left|\int_{\mathbf{R}^n}(\varphi*\psi)(x)f(x)\,dx\right|\leqslant(2\pi)^{-n/2}\|\psi\|_{C_N^*(\mathbf{R}^n)}\|\varphi\|_{C_N^*(\mathbf{R}^n)}\|f\|_{C_N(\mathbf{R}^n)}.$$

Thus,

$$\|\varphi*\psi\|_{C_N^*(\mathbf{R}^n)}\leqslant(2\pi)^{-n/2}\|\varphi\|_{C_N^*(\mathbf{R}^n)}\|\psi\|_{C_N^*(\mathbf{R}^n)}, \tag{1.9}$$

Q.E.D.

By virtue of the estimate (1.9) we can extend the operation $*$ to the whole space $C_N^+(\mathbf{R}^n)$ in a unique fashion, the inequality (1.9) remaining valid. $C_N^+(\mathbf{R}^n)$ will then become converted into the Banach algebra (to be more precise, into a fully normalized one). Since the Fourier transformation is an isometric isomorphism of the Banach space $C_N^+(\mathbf{R}^n)$ on $\mathscr{B}_N(\mathbf{R}^n)$, the operation $*$ will induce a binary operation which transforms $\mathscr{B}_N(\mathbf{R}^n)$ into Banach algebra and is defined by the formula

$$\tilde{\varphi}\cdot\tilde{\psi}=F[(F^{-1}\varphi)*(F^{-1}\psi)].$$

Let us see how the operation $\cdot$ acts on $\mathscr{B}_N^0(\mathbf{R}^n)$. Let $\varphi_0,\ \psi_0\in C_0^\infty(\mathbf{R}^n)$ and let

$$\varphi=\varphi_0+\sum_j\alpha_j\delta_{\xi_j},\quad \psi=\psi_0+\sum_k\beta_k\delta_{\eta_k}. \tag{1.10}$$

Then

$$\varphi*\psi=\varphi_0*\psi_0+\sum_j\alpha_j\delta_{\xi_j}*\psi_0+\varphi_0*\sum_k\beta_k\delta_{\eta_k}+$$
$$+\sum_{j,\,k}\alpha_j\beta_k(2\pi)^{-n/2}\delta_{\xi_j}+\eta_k,$$

and we thus have

$$(F\varphi\cdot F\psi)(p)=(F\varphi_0\cdot F\psi_0)(p)+(2\pi)^{-n/2}\sum_j\alpha_je^{ip\cdot\xi_j}(F\psi_0)(p)+$$
$$+(2\pi)^{-n/2}\sum_k\beta_ke^{ip\cdot\eta_k}(F\varphi_0)(p)+\sum_{j,\,k}(2\pi)^{-n}\alpha_j\beta_ke^{ip\cdot(\xi_j+\eta_k)}.$$

It is easy to verify that $(F\varphi_0\cdot F\psi_0)(p)=\varphi_0(p)\psi_0(p)$. So $(F\varphi\cdot F\psi)(p)=\varphi(p)\psi(p)$ for any φ,ψ of the form (1.10), i. e., in $\mathscr{B}_N^0(\mathbf{R}^n)$ the operation $\cdot$ is in fact a point-by-point multiplication of functions. For this reason we shall drop the point in the notation of this operation:

$$\tilde{\varphi}\tilde{\psi}=\tilde{\varphi}\cdot\tilde{\psi},\quad \tilde{\varphi},\ \tilde{\psi}\in\mathscr{B}_N(\mathbf{R}^n).$$

Sec. 2. Embedding Theorems

Quite often it is difficult to determine whether functions belong to the Banach algebra C or, in particular, the Banach algebra $\mathscr{B}_N$. For this reason it is important for us to establish the salient features which show that the belonging to a Sobolev space includes the belonging to $\mathscr{B}_N$ or C. To be more precise, we have in mind the existence of the natural embedding of Sobolev spaces in C or $\mathscr{B}_N$.

Theorem 2.1. *For $k > N + \frac{n}{2}$ there exists an embedding $W_2^k(\mathbf{R}^n) \subset \subset \mathscr{B}_N(\mathbf{R}^n)$.*

Proof. Let $f \in S$. Then

$$\|f\|_{C_N^*(\mathbf{R}^n)} = \sup_{\|\varphi\|_{C(\mathbf{R}^n)}=1} \left| \int_{\mathbf{R}^n} \varphi(x)(1+|x|)^N f(x)\,dx \right| =$$

$$= \sup_{\|\varphi\|_{C(\mathbf{R}^n)}=1} \left| \int_{\mathbf{R}^n} f(x)(1+|x|)^k \frac{\varphi(x)}{(1+|x|)^{k-N}}\,dx \right| \leqslant$$

$$\leqslant \sup_{\|\varphi\|_{C(\mathbf{R}^n)}=1} \sqrt{\int_{\mathbf{R}^n} \frac{|\varphi(x)|^2\,dx}{(1+|x|)^{2(k-N)}}} \times$$

$$\times \sqrt{\int_{\mathbf{R}^n} |f(x)|^2 (1+|x|)^{2k}\,dx}. \tag{2.1}$$

We have

$$\sup_{\|\varphi\|_{C(\mathbf{R}^n)}=1} \sqrt{\int_{\mathbf{R}^n} \frac{|\varphi(x)|^2\,dx}{(1+|x|)^{2(k-N)}}} = \sqrt{\int_{\mathbf{R}^n} \frac{dx}{(1+|x|)^{2(k-N)}}} < \infty, \tag{2.2}$$

since $2(k-N) > n$. Next, $(1+|x|)^{2k} \leqslant (1+x^2)^k 2^k$, therefore

$$\int_{\mathbf{R}^n} |f(x)|^2 (1+|x|)^{2k}\,dx \leqslant 2^k \|f\|_{\widetilde{W}_2^k(\mathbf{R}^n)}. \tag{2.3}$$

By virtue of (2.1), (2.2), (2.3) we obtain $\|f\|_{C_N^*} \leqslant c\|f\|_{\widetilde{W}_2^k(\mathbf{R}^n)}$, where the constant c depends on N, n and k, but is independent of $f \in S$. This means that for any $\varphi \in \widetilde{C}_0^\infty(\mathbf{R}^n)$* the following estimate is valid:

$$\|\varphi\|_{\mathscr{B}_N(\mathbf{R}^n)} \leqslant c\|\varphi\|_{W_2^k(\mathbf{R}^n)}, \quad c = \text{const}.$$

It remains to be shown that if the sequence $\{\varphi_j\}$ of functions $\varphi \in \widetilde{C}_0^\infty(\mathbf{R}^n)$ converges to zero in $\mathscr{B}_N(\mathbf{R}^n)$ and is fundamental in

* By $\widetilde{C}_0^\infty(\mathbf{R}^n)$ we denote the space of Fourier transforms of functions $\varphi \in C_0^\infty(\mathbf{R}^n)$.

$W_2^k(\mathbf{R}^n)$, then it converges to zero in $W_2^k(\mathbf{R}^n)$ as well. Let $\{\varphi_j\}$ be such a sequence and let $\widetilde{\varphi}_i = F^{-1}\varphi_j$. Then $\widetilde{\varphi}_i \in C_0^\infty(\mathbf{R}^n)$, $\widetilde{\varphi}_j \to 0$ in $C_N^*(\mathbf{R}^n)$ and $\{\widetilde{\varphi}_j\}$ is fundamental in $\widetilde{W}_2^k(\mathbf{R}^n)$. It is to be shown that $\widetilde{\varphi}_j \to 0$ in $\widetilde{W}_2^k(\mathbf{R}^n)$.

Since $\{\widetilde{\varphi}_j\}$ is fundamental in $\widetilde{W}_2^k(\mathbf{R}^n)$ there follows the convergence of this sequence in $L_2(\mathbf{R}^n) \supset \widetilde{W}_2^k(\mathbf{R}^n)$. It is sufficient to show that $\|\widetilde{\varphi}_j\|_{L_2(\mathbf{R}^n)} \to 0$. Let $\psi \in C_0^\infty(\mathbf{R}^n)$. Then

$$\left| \int_{\mathbf{R}^n} \varphi_j(x)\psi(x)\,dx \right| \leqslant \|\varphi_j\|_{C_N^*(\mathbf{R}^n)} \|\psi\|_{C_N(\mathbf{R}^n)} \to 0$$

for $j \to 0$. On the other hand, if $\widetilde{\varphi}_j \to 0$ in $L_2(\mathbf{R}^n)$ then

$$\lim_{j\to\infty} \int_{\mathbf{R}^n} \varphi_j(x)\psi(x)\,dx = (\varphi, \psi)_{L_2(\mathbf{R}^n)}.$$

Consequently, $(\varphi, \psi)_{L_2(\mathbf{R}^n)} = 0$ for any $\psi \in C_0^\infty(\mathbf{R}^n)$, whence it follows that $\varphi = 0$, Q.E.D.

Theorem 2.2. *There exists an embedding* $\mathscr{B}_N(\mathbf{R}^n) \subset C^{(N)}(\mathbf{R}^n)$. For the proof of Theorem 2.2 we shall need the following simple lemma.

Lemma 2.1. *Let* $f \in \overset{0}{\mathscr{B}}_N(\mathbf{R}^n)$, $\widetilde{f} = F^{-1}f$. *Then*

$$D^\alpha f(p) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} (-ix)^\alpha e^{ip\cdot x} \widetilde{f}(x)\,dx.$$

Here

$$\alpha = (\alpha_1, \ldots, \alpha_n),\quad D^\alpha = D_1^{\alpha_1}, \ldots, D_n^{\alpha_n},$$

$$(-ix)^\alpha = (-i)^{|\alpha|} x_1^{\alpha_1} \ldots x_n^{\alpha_n},\quad |\alpha| = \alpha_1 + \ldots + \alpha_n.$$

Proof. It is obviously sufficient to verify the statement of the lemma in two cases: $\widetilde{f} \in C_0^\infty(\mathbf{R}^n)$ and $\widetilde{f} = \delta_\xi$. In the first case the statement is obvious. Let $\widetilde{f} = \delta_\xi$. Then

$$f(p) = (2\pi)^{-n/2} e^{-ip\cdot\xi}, \text{ so}$$

$$D^\alpha f(p) = (2\pi)^{-n/2} (-i\xi)^\alpha e^{ip\cdot\xi} =$$

$$= (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-ip\cdot x} \delta(x-\xi)(-ix)^\alpha\,dx,$$

this proves the lemma.

Proof of the theorem. Let $f \in \mathscr{B}_N^0(\mathbf{R}^n)$, $\tilde{f} = F^{-1}f$. Then for $|\alpha| \leqslant N$

$$|D^\alpha f(p)| = (2\pi)^{-n/2}\left|\int_{\mathbf{R}^n} \tilde{f}(x)(-ix)^\alpha e^{-ip\cdot x}\,dx\right| \leqslant$$

$$\leqslant (2\pi)^{-n/2}\,\|\tilde{f}\|_{C_N^*(\mathbf{R}^n)} \sup_{x\in\mathbf{R}^n}\left|\frac{x_1^{\alpha_1}\dots x_n^{\alpha_n}e^{-ip\cdot x}}{(1+|x|)^N}\right| \leqslant$$

$$\leqslant (2\pi)^{-n/2}\,\|f\|_{\mathscr{B}_N(\mathbf{R}^n)}.$$

For this reason operator I_0 of the identical embedding $\mathscr{B}_N^0(\mathbf{R}^n) \subset \subset \mathscr{B}_N(\mathbf{R}^n)$ is bounded in $C^{(N)}(\mathbf{R}^n)$ and can be extended to cover the homomorphism I: $\mathscr{B}_N(\mathbf{R}^n) \to C^{(N)}(\mathbf{R}^n)$. It remains to be proved that $(I\varphi = 0) \Rightarrow (\varphi = 0)$.

Let $I\varphi = 0$ and let $\{\varphi_j\}$ be a sequence of functions belonging to $\mathscr{B}_N^0(\mathbf{R}^n)$ and such that $\varphi_j \to \varphi$ in $\mathscr{B}_N(\mathbf{R}^n)$ when $j \to \infty$. Then

$$\lim_{j\to\infty}\|\varphi_j\|_{C^{(N)}(\mathbf{R}^n)} = \lim_{j\to\infty}\|I\varphi_j\|_{C^{(N)}(\mathbf{R}^n)} = \|I\varphi\| = 0.$$

Let $\tilde{\varphi}_j = F^{-1}\varphi_j$, $\tilde{\varphi} = F^{-1}\varphi$. We have $\tilde{\varphi}_j \to \tilde{\varphi}$ in $C_N^+(\mathbf{R}^n)$, whence, in particular, for any $f \in C_N(\mathbf{R}^n)$,

$$\int_{\mathbf{R}^n} \tilde{\varphi}(x) f(x)\,dx = \lim_{j\to\infty}\int_{\mathbf{R}^n} \tilde{\varphi}_j(x) f(x)\,dx.$$

We shall show that for any infinitely differentiable function $f \in C_N(\mathbf{R}^n)$

$$\lim_{j\to\infty}\int_{\mathbf{R}^n} \tilde{\varphi}_j(x) f(x)\,dx = 0;$$

hence it follows that $\varphi = 0$ since $C^\infty(\mathbf{R}^n) \cap C_N(\mathbf{R}^n)$ is dense in $C_N(\mathbf{R}^n)$.

Thus, let f be a fixed infinitely differentiable function of $C_N(\mathbf{R}^n)$ and $\varepsilon > 0$ be a fixed number. Choose a number j_0 so that for $j > j_0$ the inequality $\|\tilde{\varphi}_j - \tilde{\varphi}_{j_0}\|_{C_N^*(\mathbf{R}^n)} < \varepsilon$ holds. Since $\tilde{\varphi}_{j_0}$ is the linear combination of the function of $C_0^\infty(\mathbf{R}^n)$ and δ_ξ-functions, a positive A will be found such that $\int_{\mathbf{R}^n} \tilde{\varphi}_{j_0}(x)\,g(x)\,dx = 0$ for any function $g \in C_N(\mathbf{R}^n)$, which becomes zero in a sphere $|x| < A$. Let $f_0 \in \in C_0^\infty(\mathbf{R}^n)$ be a function which coincides with f in this sphere and

which does not exceed modulo f. We have

$$\int_{\mathbf{R}^n} \tilde{\varphi}_j(x) f(x)\,dx = \int_{\mathbf{R}^n} \tilde{\varphi}_j(x) f_0(x)\,dx + \\ + \int_{\mathbf{R}^n} [\tilde{\varphi}_j(x) - \tilde{\varphi}_{j_0}(x)]\,[f(x) - f_0(x)]\,dx, \tag{2.4}$$

since

$$\int_{\mathbf{R}^n} \tilde{\varphi}_{j_0}(x)\,[f(x) - f_0(x)]\,dx = 0.$$

Consider the first term in the right-hand member of (2.4). We have

$$\left| \int_{\mathbf{R}^n} \tilde{\varphi}_j(x) f_0(x)\,dx \right| = \left| \int_{\mathbf{R}^n} \varphi_j(x)\,[Ff_0](x)\,dx \right| \leqslant \\ \leqslant \|\varphi_j\|_{C(\mathbf{R}^n)} \int_{\mathbf{R}^n} |[Ff_0](x)|\,dx \to 0$$

for $j \to \infty$. The second term in the right-hand member of (2.4) may be estimated in the following way:

$$\left| \int_{\mathbf{R}^n} [\tilde{\varphi}_j(x) - \tilde{\varphi}_{j_0}(x)]\,[f(x) - f_0(x)]\,dx \right| \leqslant \\ \leqslant \varepsilon \|f - f_0\|_{C_N(\mathbf{R}^n)} \leqslant 2\varepsilon \|f\|_{C_N(\mathbf{R}^n)}.$$

It means such a number $j_1 > j_0$ will be found that for $j > j_1$

$$\left| \int_{\mathbf{R}^n} \tilde{\varphi}_j(x) f(x)\,dx \right| \leqslant \varepsilon \left(1 + 2 \|f\|_{C_N(\mathbf{R}^n)}\right).$$

Since ε is arbitrary, it follows that

$$\lim_{j\to\infty} \int_{\mathbf{R}^n} \tilde{\varphi}_j(x) f(x)\,dx = 0,$$

Q.E.D.

Thus, from Theorems 2.1 and 2.2 it follows that for

$$k > \frac{n}{2} + N, \quad W_2^k(\mathbf{R}^n) \subset \mathscr{B}_N(\mathbf{R}^n) \subset C^{(N)}(\mathbf{R}^n).$$

We can see that the Banach algebra $\mathscr{B}_N(\mathbf{R}^n)$ is "in-between" the Sobolev space contained in $C^{(N)}(\mathbf{R}^n)$ and the Banach algebra $C^{(N)}(\mathbf{R}^n)$. So the $\mathscr{B}_N(\mathbf{R}^n)$ is a Banach algebra which is closer to the Sobolev space than $C^{(N)}(\mathbf{R}^n)$.

Problem. Show that for any $\varphi \in C_N^+(\mathbf{R}^n)$ the following formula is valid:

$$[F\varphi](p) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} \varphi(x)\, e^{-ip\cdot x}\, dx.$$

Consider a subalgebra $\mathscr{B}_N(\rho)$ of the Banach algebra $\mathscr{B}_N(\mathbf{R})$ obtained by closure of the set of functions of $\mathscr{B}_N(\mathbf{R})$, which are equal to zero in the neighborhood of set σ (one should note that for every function there is a corresponding neighborhood). Obviously $\mathscr{B}_N(\rho)$ is a closed ideal in $\mathscr{B}_N(\mathbf{R})$. The factor-algebra $\mathscr{B}_N(\mathbf{R})/\mathscr{B}_N(\rho)$ with the ordinary norm

$$\|\{\varphi\}\| = \inf_{\varphi \in \{\varphi\}} \|\varphi\|_{\mathscr{B}_N(\mathbf{R})}$$

will be denoted by $\mathscr{B}_N(\sigma)$.

Sec. 3. The Algebra of Functions of a Generator

Let E be a vector space given over field $\mathbf{C}$ with the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, where $c\,\|\cdot\|_1 \geqslant \|\cdot\|_2$, $c = \text{const}$. Let B_1 be a completion to space E in the norm $\|\cdot\|_1$ and B_2 be a completion to space E in the norm $\|\cdot\|_2$.

In this case let us write $B_1 \preceq B_2$.

Let $u(t)$ be a function with values in E defined on $\mathbf{R}$ and satisfying the equation

$$i\frac{du}{dt} - Au = 0, \tag{3.1}$$

and the initial condition

$$u(0) = g, \quad g \in E. \tag{3.2}$$

Here A is the linear operator $A: E \to E$ and the derivative $\frac{du}{dt} \in E$ is such that

$$\left\| \frac{du(t)}{dt} - \frac{u(t+h) - u(t)}{h} \right\|_1 \to 0 \text{ for } h \to 0.$$

Definition. *A homomorphism $A: E \to E$ is said to be a generator if, for any $g \in E$, the solution $u(t) \in E$ of equations* (3.1), (3.2) *exists and is unique for the class of functions satisfying the inequality*

$$\|u(t)\|_2 \leqslant c(1 + |t|)^k,$$

where $c > 0$, $k \geqslant 0$; k, c being constants.

We shall assume that A is a generator of degree $s \geqslant 0$ with the defining pair of spaces (B_1, B_2) if a constant c_1 exists such that

$$\|u(t)\|_2 \leqslant c_1(1 + |t|)^s \|g\|_1 \tag{3.3}$$

for any $t \in \mathbf{R}$ and any $g \in E$.

If $B_1 = B_2 = B$ then we shall assume that A is a generator on the Banach space B.

Note. By substituting the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ with equivalent norms whenever necessary we can put $c = 1$ in the above definition. We shall do likewise in the text that follows.

Lemma 3.1. *The mappings $\{U(t)\}_{t \in \mathbf{R}}$, $U(t): E \to E$ defined by the equality $U(t)\, g = u(t)$, g and U satisfying the conditions of the definition, are homomorphisms of the space E and form a one-parameter group, i.e.,*

$$U(t_2) \cdot U(t_1) = U(t_1 + t_2).$$

Proof. Let $u(t) = U(t)\, g$, $u_1(t) = U(t)$ and (t_1). By definition $u(t_1 + t_2) = U(t_1 + t_2)\, g$. This means that one needs only to show that $u(t_1 + t_2) = u_1(t)$. Denoting $u(t_1 + t)$ by $v(t)$ we obtain

$$v(0) = u(t_1), \quad i \frac{dv}{dt} = Av.$$

Since u_1 is a solution of the same Cauchy problem, $v(t) = u_1(t)$ due to the uniqueness of the solution, Q.E.D.

Definition. *The family $\{U(t)\}$ will be said to be a group generated by A.*

Lemma 3.2. *$AU(t) = U(t)\, A$ for any $t \in \mathbf{R}$.*

Proof. For any fixed $t \in \mathbf{R}$

$$\lim_{h \to 0} \left\| i \frac{U(t+h) - U(t)}{h} g - AU(t)\, g \right\|_1 = 0.$$

Presuming $t = 0$ and taking into account that $U(0)\, g = g$ we obtain

$$\lim_{h \to 0} \left\| i \frac{U(h) - 1}{h} g - Ag \right\|_1 = 0.$$

We shall apply $U(t)$ to the expression within $\|\cdot\|_1$. Then with $h \to 0$ (3.3) gives

$$\left\| U(t) \left(i \frac{U(h) - 1}{h} g - Ag \right) \right\|_2 \leqslant (1 + |t|)^s \left\| i \frac{U(h) - 1}{h} g - Ag \right\|_1 \to 0.$$

Hence

$$iU(t) \frac{U(\Delta) - 1}{\Delta} g = U(t)\, Ag + \sigma(\Delta),$$

where $\|\sigma(\Delta)\|_2 \to 0$ for $\Delta \to 0$. On the other hand,

$$iU(t) \frac{U(\Delta) - 1}{\Delta} g = i \frac{U(t+\Delta) - U(t)}{\Delta} g = AU(t)\, g + \delta(\Delta),$$

where $\| \delta(\Delta) \|_1 \to 0$ for $\Delta \to 0$. Therefore,

$$\| [U(t) A - AU(t)] g \|_2 = \| \delta(\Delta) - \sigma(\Delta) \|_2 \to 0$$

for $\Delta \to 0$. This completes the proof of the lemma.

Lemma 3.3. *Let A be a generator of degree N with the defining pair of spaces (B_1, B_2). Let $\{U(t)\}$ be a group generated by A and let $\varphi \in \in \mathscr{B}_N(\mathbf{R})$. Then the element $I[\varphi; g]$ of space B_2^{**} defined by the formula*

$$I[\varphi; g](h^*) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(t) h^*(U(t) g)\, dt, \quad h^* \in B_2^*, \quad \tilde{\varphi} = F^{-1}\varphi$$

belongs to B_2, where

$$\| I[\varphi; g] \|_{B_2} \leqslant \frac{1}{\sqrt{2\pi}} \| \varphi \|_{\mathscr{B}_N(\mathbf{R})} \| g \|_{B_1}.$$

Proof. Let $h^* \in B_2^*$. We have

$$| I[\varphi; g](h^*) | = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} \tilde{\varphi}(t) h^*(U(t) g)\, dt \right| \leqslant$$

$$\leqslant \frac{1}{\sqrt{2\pi}} \| \varphi \|_{\mathscr{B}_N(\mathbf{R})} \cdot \sup_{t \in \mathbf{R}} \frac{|h^*(U(t) g)|}{(1+|t|)^N}.$$

By the definition of the generator we have $\| U(t) g \|_{B_2} \leqslant \leqslant (1+|t|)^N \| g \|_{B_1}$; therefore

$$| I[\varphi; g](h^*) | \leqslant \frac{1}{\sqrt{2\pi}} \| \varphi \|_{\mathscr{B}_N(\mathbf{R})} \| h^* \|_{B_2^*} \| g \|_{B_1}.$$

Hence

$$\| I[\varphi; g] \|_{B_2^{**}} \leqslant \frac{1}{\sqrt{2\pi}} \| \varphi \|_{\mathscr{B}_N(\mathbf{R})} \| g \|_{B_1}. \tag{3.4}$$

It remains to be proved that $I[\varphi; g] \in B_2$.

If $\tilde{\varphi} \in C_0^\infty(\mathbf{R})$, then

$$I[\varphi; g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(t) U(t) g\, dt.$$

The above integral, being a Riemann integral of a B_2-valued function, is an element of B_2. If $\tilde{\varphi} = \delta_\xi$, then once again $I[\varphi; g] \in B_2$ since

$$I[\varphi; g] = \frac{1}{\sqrt{2\pi}} U(\xi) g.$$

Hence, if $\varphi \in \mathscr{B}_N^0(\mathbf{R})$ we have $I[\varphi; g] \in B_2$. From (3.4) it follows that the norm of the operator on $\mathscr{B}_N(\mathbf{R})$ to B_2 sending φ into $I[\varphi; g]$

is bounded by the number $\frac{1}{\sqrt{2\pi}} \| g \|_{B_1}$. Since B_2 is closed in B_2^{**} we obtain $I[\varphi; g] \in B_2$ for any $\varphi \in \mathscr{B}_N(\mathbf{R})$.

In this section A is presumed to be a generator of degree N with the defining pair of spaces (B_1, B_2) while $\{U(t)\}$ constitutes a group of homomorphisms generated by the operator A.

Set $\varphi \in \mathscr{B}_N(\mathbf{R})$. Denote by $\varphi_*(A)$ the operator $\varphi_*(A)\, g \overset{\text{def}}{=} \overset{\text{def}}{=} I[\varphi; g]$ on B_1 to B_2 which is defined on region E. Lemma 3.3 gives

$$\| \varphi_*(A) \| \leqslant \frac{1}{\sqrt{2\pi}} \| \varphi \|_{\mathscr{B}_N(\mathbf{R})}.$$

Lemma 3.4. *Let* $\varphi \in \mathscr{B}_N(\mathbf{R})$ *and* $\mathscr{F} \in \mathscr{B}_N(\mathbf{R})$, *where* $\mathscr{F}(x) = x\varphi(x)$. *Then* $\varphi_*(A)\, A = \mathscr{F}_*(A)$.

Proof. By definition, for any $h^* \in B_2^*$ and $q \in E$ we have

$$h^*[\varphi^*(A)\, Aq] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(t)\, h^*[U(t)\, Aq]\, dt,$$

$$h^*[\mathscr{F}_*(A)\, q] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\mathscr{F}}(t)\, h^*[U(t)\, q]\, dt,$$

where $\tilde{\varphi}$ and $\tilde{\mathscr{F}}$ are the Fourier transforms of the functions φ and $\mathscr{F}$.

Let us show that for any function $\tilde{g}$, which with its derivative belongs to the space $C_N(\mathbf{R})$, the following formula is valid:

$$i \int_{-\infty}^{\infty} \tilde{\varphi}(t)\, \tilde{g}'(t)\, dt = \int_{-\infty}^{\infty} \tilde{\mathscr{F}}(t)\, \tilde{g}(t)\, dt. \tag{3.5}$$

First of all, note that it is sufficient to prove the formula (3.5) for a case when $\tilde{g} \in C^\infty(\mathbf{R}) \cap C_N(\mathbf{R})$ because for $\tilde{g} \in C_N(\mathbf{R})$, $\tilde{g}' \in C_N(\mathbf{R})$ for any $\varepsilon > 0$ there is an infinitely differentiable function $\tilde{h}$ such that

$$\| \tilde{g} - \tilde{h} \|_{C_N(\mathbf{R})} < \varepsilon, \quad \| \tilde{g}' - \tilde{h}' \|_{C_N(\mathbf{R})} < \varepsilon.$$

Thus, let $\tilde{g} \in C^\infty(\mathbf{R}) \cap C_N(\mathbf{R})$, $\tilde{g}' \in C_N(\mathbf{R})$. Consider the sequence $\{\tilde{g}_n\} \subset C_0^\infty(\mathbf{R})$ which has the following properties:

(a) $\tilde{g}_n(t) = \tilde{g}(t)$ for $t \in (-n, n)$,

(b) $\| \tilde{g}_n - \tilde{g} \|_{C_N(\mathbf{R})} \leqslant c_1$, $\| \tilde{g}_n' - \tilde{g}' \|_{C_N(\mathbf{R})} \leqslant c_2$, $c_1, c_2 = \text{const}$.

Then, for any element $\tilde{\Phi} \in C_N^+(\mathbf{R})$ the following relationships are valid:

$$\int_{-\infty}^{\infty} \tilde{\Phi}(t)\,\tilde{g}(t)\,dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} \tilde{\Phi}(t)\,\tilde{g}_n(t)\,dt, \tag{3.6}$$

$$\int_{-\infty}^{\infty} \tilde{\Phi}(t)\,\tilde{g}'(t)\,dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} \tilde{\Phi}(t)\,\tilde{g}_n'(t)\,dt. \tag{3.7}$$

In fact, let $\tilde{\Phi} = \lim\limits_{j\to\infty} \tilde{\Phi}_j$, $\tilde{\Phi}_j \in F^{-1}\mathscr{B}_N^0(\mathbf{R})$. For any $\varepsilon > 0$ we shall find such a number j_0 that $\|\tilde{\Phi}_j - \tilde{\Phi}\| < \varepsilon$ for $j \geqslant j_0$. Let the function support Φ_{j_0} be in the interval $(-n_0, n_0)$. Then for $n \geqslant n_0$

$$\left| \int_{-\infty}^{\infty} \tilde{\Phi}(t)\,(\tilde{g}(t) - \tilde{g}_n(t))\,dt \right| =$$
$$= \left| \int_{-\infty}^{\infty} (\tilde{\Phi}(t) - \tilde{\Phi}_{j_0}(t))\,(\tilde{g}(t) - \tilde{g}_n(t))\,dt \right| \leqslant c\varepsilon, \quad c = \text{const}.$$

For any $\tilde{f} \in F^{-1}\mathscr{B}_N^0(\mathbf{R})$, $\tilde{h} \in C_0^\infty(\mathbf{R})$ *Parseval's equality* is valid:

$$\int_{-\infty}^{\infty} \tilde{f}(t)\,\tilde{h}(t)\,dt = \int_{-\infty}^{\infty} f(t)\,h(-t)\,dt, \tag{3.8}$$

where $f = F\tilde{f}$, $h = F\tilde{h}$. From the embedding $\mathscr{B}_N(\mathbf{R}) \subset C(\mathbf{R})$ and from the density $\mathscr{B}_N^0(\mathbf{R})$ in $\mathscr{B}_N(\mathbf{R})$ it follows that the formula (3.8) is valid for any $f \in C_N^+(\mathbf{R})$. This means that we can rewrite the formulas (3.6) and (3.7) in the following way:

$$\int_{-\infty}^{\infty} \tilde{\Phi}(t)\,\tilde{g}(t)\,dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} \Phi(x)\,g_n(-x)\,dx, \tag{3.9}$$

$$\int_{-\infty}^{\infty} \tilde{\Phi}(t)\,\tilde{g}'(t)\,dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} \Phi(x)\,G_n(-x)\,dx, \tag{3.10}$$

where $\Phi = F\tilde{\Phi}$, $g_n = F\tilde{g}_n$, $G_n = F\tilde{g}'$; here $G_n(x) = ixg_n(x)$. By setting $\tilde{\Phi} = \tilde{\varphi}$ in (3.10) and $\tilde{\Phi} = \tilde{\mathscr{F}}$ in (3.9) we obtain

$$\int_{-\infty}^{\infty} \tilde{\varphi}(t)\,\tilde{g}'(t)\,dt = -i \lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi(x)\,x g_n(-x)\,dx =$$
$$= -i \lim_{n\to\infty} \int_{-\infty}^{\infty} \mathscr{F}(x)\,g_n(-x)\,dx = -i \int_{-\infty}^{\infty} \tilde{\mathscr{F}}(t)\,\tilde{g}(t)\,dt$$

and this completes the proof of the formula (3.5).

By using formula (3.5) we obtain

$$h^*[\mathcal{F}_*(A)q]=\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\varphi}(t)\frac{d}{dt}h^*[U(t)q]\,dt=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\varphi}(t)h^*[AU(t)q]\,dt=h^*[\varphi_*(A)Aq].$$

Lemma 3.4 is proved.

Corollary. *Let* $\mathcal{F}_j \in \mathcal{B}_{N_k}(\mathbf{R})$, *where* $\mathcal{F}_j(x)=x^i\mathcal{F}_0(x)$, $j=0,1,\ldots,k$. *Then* $\mathcal{F}_{0*}(A)A^k=\mathcal{F}_{k*}(A)$.

The proof consists of successive applications of Lemma 3.4 to functions $\varphi=\mathcal{F}_0,\mathcal{F}_1,\ldots,\mathcal{F}_{k-1}$. Now estimate the norm of the operator $(A-z)^{-1}$.

Lemma 3.5. *The operator* $[A-(a+ib)]^{-1}$ *exists for* $b\neq 0$ *and the following estimate is valid:*

$$\|[A-(a+ib)]^{-1}\|_{B_1}^{B_2}\leqslant\frac{c}{|b|}+\frac{c'}{|b|^{N+1}},\tag{3.11}$$

where c and c' *are constants,* $\|\cdot\|_{B_1}^{B_2}$ *is the norm of the operator acting from* B_1 *to* B_2.

Proof. Let $b\neq 0$. Denote $z=a+ib$. Since $\operatorname{Im} z\neq 0$ the function $\varphi:\varphi(x)=(x-z)^{-1}$ belongs to $W_2^k(\mathbf{R})$ for any k, so $\varphi\in\mathcal{B}_N(\mathbf{R})$. Since $1\in\mathcal{B}_N(\mathbf{R})$ the function ψ:

$$\psi(x)=x(x-z)^{-1}=z(x-z)^{-1}+1$$

belongs to $\mathcal{B}_N(\mathbf{R})$. By virtue of the preceding lemma $\varphi_*(A)A=(z\varphi_*)(A)+1_E$, where 1_E is the identical image of E to itself.

Thus,

$$\varphi_*(A)(A-z)=1_E,$$

i.e., the operator $(A-z)$ has the inverse:

$$(A-z)^{-1}=\varphi_*(A)|_{R(A-z)}.$$

Here

$$\|(A-z)^{-1}\|_{B_1}^{B_2}\leqslant\|\varphi_*(A)\|_{B_1}^{B_2}\leqslant\frac{1}{\sqrt{2\pi}}\|\varphi\|_{\mathcal{B}_N(\mathbf{R})}.$$

Set $\varphi_0(x)=\dfrac{1}{x-i|b|}$. Then

$$\|\varphi\|_{\mathcal{B}_N(\mathbf{R})}=\|\varphi_0\|_{\mathcal{B}_N(\mathbf{R})}.$$

Next by setting

$$\tilde{\varphi}_0=F^{-1}\varphi_0,\quad \varphi_1(x)=\frac{1}{x-i},\quad \tilde{\varphi}_1=F^{-1}\varphi_1,$$

we obtain for any $g \in C_N(\mathbf{R})$

$$\int_{-\infty}^{\infty} \tilde{\varphi}_0(t)\, g(t)\, dt = \frac{1}{|b|} \int_{-\infty}^{\infty} \tilde{\varphi}_1(t)\, g\left(\frac{t}{|b|}\right) dt.$$

Since

$$\sup_{t \in \mathbf{R}} \frac{\left| g\left(\frac{t}{|b|}\right)\right|}{(1+|t|)^N} \leqslant \|g\|_{C_N(\mathbf{R})} \sup_{t \in \mathbf{R}} \left(\frac{1+|t|}{1+|bt|}\right)^N =$$

$$= \|g\|_{C_N(\mathbf{R})} \max\left\{1, \frac{1}{|b|^N}\right\},$$

we have

$$\|\varphi_0\|_{\mathscr{B}_N(\mathbf{R})} \leqslant \|\varphi_1\|_{\mathscr{B}_N(\mathbf{R})} \max\left\{\frac{1}{|b|}, \frac{1}{|b|^{N+1}}\right\},$$

hence we obtain (3.11). This completes the proof of the lemma. The following lemma is proved in the same way.

Lemma 3.6. *The operator* $[A-(a+ib)]^{-k}$, $k > 0$ *satisfies the estimate*

$$\|[A-(a+ib)]^{-k}\|_{B_1}^{B_2} \leqslant \frac{c_k}{|b|^k} + \frac{c_k^1}{|b|^{k+N}},$$

where c_k *and* c_k^1 *are constants.*

Proof. Obviously, for $\operatorname{Im} z \neq 0$ and $0 \leqslant j < k$ the function $\varphi_j(x) = x^j (x-z)^{-k}$ belongs to $\mathscr{B}_N(\mathbf{R})$. Since $1 \in \mathscr{B}_N(\mathbf{R})$, it follows that the function

$$\varphi_k(x) = x^k (x-z)^{-k} = 1 - \frac{(x-z)^k - x^k}{(x-z)^k} =$$

$$= 1 - \sum_{i=1}^{k} C_k^i (-1)^i z^i x^{k-i} (x-z)^{-k}$$

belongs to $\mathscr{B}_N(\mathbf{R})$. Therefore, using the corollary of Lemma 3.4 we have

$$\varphi_{0*}(A)(A-z)^k g = g$$

for any $g \in E$. Consequently, $(A-z)^{-k} = \varphi_{0*}(A)|_{R(A-z)^k}$, whence

$$\|(A-z)^{-k}\|_{B_1}^{B_2} \leqslant \|\varphi_{k*}(A)\|_{B_1}^{B_2} \leqslant \frac{1}{\sqrt{2\pi}} \|\varphi_0\|_{\mathscr{B}_N(\mathbf{R})} \leqslant$$

$$\leqslant \frac{1}{\sqrt{2\pi}} \|\bar{\varphi}_0\|_{\mathscr{B}_N(\mathbf{R})} \max\left\{\frac{1}{|b|^k}, \frac{1}{|b|^{N+k}}\right\},$$

where $\bar{\varphi}_0(x) = (x-i)^{-k}$. Thus the lemma is proved. Note that the statement of this lemma is not directly derived from Lemma 3.5.

The operator $\tilde{h}$: $\mathscr{B}_N(\mathbf{R}) \to B_2$ corresponds to each vector $h \in E$ by means of the formula

$$\tilde{h}\varphi = \varphi_*(A)\,h.$$

The operator $\tilde{h}$ is determined in $\mathscr{B}_N(\mathbf{R})$ and is bounded:

$$\|\tilde{h}\| = \sup_{\varphi\in\mathscr{B}_N(\mathbf{R}),\ \varphi\neq 0} \frac{\|\varphi_*(A)\,h\|_2}{\|\varphi\|_{\mathscr{B}_N(\mathbf{R})}} \leqslant \frac{1}{\sqrt{2\pi}}\|h\|_1. \tag{3.12}$$

Put the norm $\|\cdot\|_{\mathrm{mid}}$ in E by the formula

$$\|h\|_{\mathrm{mid}} \overset{\mathrm{def}}{=} \sqrt{2\pi}\,\|\tilde{h}\|.$$

We shall call the completion of E in this norm *an intermediate Banach space* B_{mid}.

Lemma 3.7. *The following inequalities are valid*:

$$\|\cdot\|_1 \geqslant \|\cdot\|_{\mathrm{mid}} \geqslant \|\cdot\|_2.$$

Then

$$B_1 \preceq B_{\mathrm{mid}} \preceq B_2.$$

Proof. From (3.12) it follows immediately that $\|\cdot\|_{\mathrm{mid}} \leqslant \|\cdot\|_1$. On the other hand,

$$\|h\|_{\mathrm{mid}} = \sup_{\varphi\in\mathscr{B}_N(\mathbf{R}),\ \varphi\neq 0} \frac{\sqrt{2\pi}\,\|\varphi_*(A)\,h\|_2}{\|\varphi\|} \geqslant \frac{\sqrt{2\pi}\,\|h\|_2}{\|1\|_{\mathscr{B}_N(\mathbf{R})}} = \|h\|_2,$$

since $\|1\|_{\mathscr{B}_N(\mathbf{R})} = \left\|\sqrt{2\pi}\,\delta_0\right\|_{C_N^+(\mathbf{R})} = \sqrt{2\pi}$. This completes the proof of the Lemma.

Lemma 3.8. *For any $t_0 \in \mathbf{R}$ the following equalities are valid*:

(1) $U(t_0) = (e^{-iAt_0})_*$;

(2) $\varphi_*(A)\,(e^{-iAt_0})_* = [e^{-iAt_0}\varphi(A)]_*, \quad \forall \varphi \in \mathscr{B}_N(\mathbf{R})$.

Proof. First of all, for any $h \in E$ we have

$$(e^{-iAt_0})_*\,h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2\pi}\,\delta_{t_0} U(t)\,h\,dt = U(t_0)\,h.$$

It remains to be proved that

$$\varphi_*(A)\,U(t_0)\,h = [e^{-iAt_0}\varphi(A)]_*\,h.$$

We have

$$\varphi_*(A)\,U(t_0)\,h=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\varphi}(t)\,U(t)\,U(t_0)\,h\,dt=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\varphi}(t)\,U(t_0+t)\,h\,dt.$$

Here we note that if the function $\mathscr{F}\in\mathscr{B}_N(\mathbf{R})$ is represented as $\mathscr{F}(x)=\varphi(x)\,e^{-it_0x}$, then $F^{-1}\mathscr{F}=(F^{-1}\varphi)*\sqrt{2\pi}\,\delta_{t_0}$, so for any $g\in C_N(\mathbf{R})$

$$\int_{-\infty}^{\infty}(F^{-1}\mathscr{F})(t)\,g(t)\,dt=\int_{-\infty}^{\infty}(F^{-1}\varphi)(t)\,g(t+t_0)\,dt.$$

For this reason

$$[e^{-iAt_0}\varphi(A)]_*\,h=\int_{-\infty}^{\infty}\tilde{\varphi}(t)\,U(t_0+t)\,h\,dt=\varphi_*(A)\,U(t_0)\,h,$$

this completes the proof.

Lemma 3.9. *Let* $\psi(x)=\sum_{k=1}^{r}a_ke^{-it_kx}$. *Then*

$$\sqrt{2\pi}\,\|\psi_*(A)\,h\|_{\mathrm{mid}}\leqslant\|\psi\|_{\mathscr{B}_N(\mathbf{R})}\,\|h\|_{\mathrm{mid}}.$$

Proof. The following inequalities are obvious (for $\psi\neq 0$):

$$\sup_{\substack{\varphi\in\mathscr{B}_N(\mathbf{R})\\ \varphi\neq 0}}\frac{\|\varphi_*(A)\,h\|_2}{\|\varphi\|_{\mathscr{B}_N(\mathbf{R})}}\geqslant\sup_{\substack{\varphi\in\mathscr{B}_N(\mathbf{R})\\ \varphi\neq 0}}\frac{\|[(\varphi\psi)_*(A)]\,h\|_2}{\|\varphi\cdot\psi\|_{\mathscr{B}_N(\mathbf{R})}}\geqslant$$

$$\geqslant\sup_{\substack{\varphi\in\mathscr{B}_N(\mathbf{R})\\ \varphi\neq 0}}\frac{\|(\varphi\cdot\psi)_*(A)\,h\|_2}{\frac{1}{\sqrt{2\pi}}\|\psi\|_{\mathscr{B}_N(\mathbf{R})}\,\|\varphi\|_{\mathscr{B}_N(\mathbf{R})}}.$$

Hence, it follows (also by Lemma 3.8) that

$$\frac{1}{\|\psi\|_{\mathscr{B}_N(\mathbf{R})}}\,\|\psi_*(A)\,h\|_{\mathrm{mid}}=\sqrt{2\pi}\sup_{\substack{\varphi\in\mathscr{B}_N(\mathbf{R})\\ \varphi\neq 0}}\frac{\|\varphi_*(A)\,\psi_*(A)\,h\|_2}{\|\psi\|_{\mathscr{B}_N(\mathbf{R})}\,\|\varphi\|_{\mathscr{B}_N(\mathbf{R})}}\leqslant$$

$$\leqslant\sup_{\substack{\varphi\in\mathscr{B}_N(\mathbf{R})\\ \varphi\neq 0}}\frac{\|\varphi_*(A)\,h\|_2}{\|\varphi\|_{\mathscr{B}_N(\mathbf{R})}}=\frac{1}{\sqrt{2\pi}}\,\|h\|_{\mathrm{mid}},$$

this completes the proof.

Lemma 3.10. *The function u in the definition of the generator satisfies the following inequality for any $t_1, t_2 \in \mathbf{R}$*

$$\| u(t_2) \|_{\mathrm{mid}} \leqslant (1+|t_1-t_2|)^N \| u(t_1) \|_{\mathrm{mid}}.$$

Proof. By virtue of Lemma 3.1 we have

$$u(t_0+t) = U(t)\, u(t_0).$$

Consequently, by Lemma 3.8 we can establish

$$u(t_0+t) = (e^{-iAt})_* \, u(t_0).$$

Hence according to Lemma 3.9 we obtain

$$\| u(t_0+t) \|_{\mathrm{mid}} \leqslant \| \delta_t \|_{C_N^+(\mathbf{R})} \| u(t_0) \|_{\mathrm{mid}} = (1+|t|)^N \| u(t_0) \|_{\mathrm{mid}}.$$

The lemma is proved.

Lemma 3.11. *For any $\psi \in \mathscr{B}_N(\mathbf{R})$, $h \in E$ the element $I_{\mathrm{mid}}[\psi; h]$ of the space B_{mid}^{**} defined by the formula*

$$I_{\mathrm{mid}}[\psi; h](h^*) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\psi}(t)\, h^*(U(t)\, h)\, dt, \quad h^* \in B_{\mathrm{mid}}^*, \quad \psi = F\tilde{\psi}$$

belongs to B_{mid}, where

$$\| I_{\mathrm{mid}}[\psi; h] \|_{\mathrm{mid}} \leqslant \frac{1}{\sqrt{2\pi}} \| \psi \|_{\mathscr{B}_N(\mathbf{R})} \| h \|_{\mathrm{mid}}.$$

Denote by $\psi_{\mathrm{mid}}(A)$ the continuation of the operator $h \to I_{\mathrm{mid}}[\psi; h]$ to the homomorphism $B_{\mathrm{mid}} \to B_{\mathrm{mid}}$. Obviously

$$\| \psi_{\mathrm{mid}}(A) \| \leqslant \frac{1}{\sqrt{2\pi}} \| \psi \|_{\mathscr{B}_N(\mathbf{R})}.$$

Lemma 3.12. *The mapping $\psi \to \psi_{\mathrm{mid}}(A)$ is a homomorphism of the algebra $\mathscr{B}_N(\mathbf{R})$ into the algebra of operators of space B_{mid}.*

Proof. We must prove that $\psi_{\mathrm{mid}}(A)\, \varphi_{\mathrm{mid}}(A) = (\psi\varphi)_{\mathrm{mid}}(A)$ for any $\varphi, \psi \in \mathscr{B}_N(\mathbf{R})$. For any $g^* \in B_{\mathrm{mid}}^*$ we have

$$g^*(\varphi_{\mathrm{mid}}(A)\, h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(t)\, g^*(U(t)\, h)\, dt.$$

Since $\psi_{\mathrm{mid}}(A)$ is the homomorphism,

$$g^*(\psi_{\mathrm{mid}}(A)\, \varphi_{\mathrm{mid}}(A)\, h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(t)\, g^*(\psi_{\mathrm{mid}}(A)\, U(t)\, h)\, dt.$$

By considering A as a generator with the producing pair of spaces (B_1, B_{mid}), and by applying Lemma 3.8 we obtain

$$\int_{-\infty}^{\infty} \tilde{\varphi}(t)\, g^*(\psi_{\text{mid}}(A)\, U(t)\, h)\, dt = \int_{-\infty}^{\infty} \tilde{\varphi}(t)\, g^*((\psi(A)\, e^{-iAt})_{\text{mid}}\, h)\, dt =$$

$$= \int_{-\infty}^{\infty} \tilde{\varphi}(t) \left(\int_{-\infty}^{\infty} \left(\tilde{\psi} * \delta_t\right)(t')\, g^*(U(t')\, h)\, dt' \right) dt.$$

On the other hand,

$$g^*((\psi \cdot \varphi)_{\text{mid}}(A)\, h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\tilde{\psi} * \tilde{\varphi})(t)\, g^*(U(t)\, h)\, dt.$$

The proof of the lemma will be completed if we prove the following formula: for any $\tilde{\varphi}$, $\tilde{\psi} \in C_N^+(\mathbf{R})$, $f \in C_N(\mathbf{R})$

$$\int_{-\infty}^{\infty} \tilde{\varphi}(t)\, dt \int_{-\infty}^{\infty} (\tilde{\psi} * \delta_t)(t')\, f(t')\, dt' = \int_{-\infty}^{\infty} (\tilde{\psi} * \tilde{\varphi})(t)\, f(t)\, dt. \tag{3.13}$$

Note that

$$\int_{-\infty}^{\infty} (\tilde{\psi} * \delta_t)(t')\, f(t)\, dt' = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\psi}(\tau)\, f(\tau + t)\, d\tau,$$

so (3.13) now becomes

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(t)\, dt \int_{-\infty}^{\infty} \tilde{\psi}(\tau)\, f(\tau + t)\, dt = \int_{-\infty}^{\infty} (\tilde{\psi} * \tilde{\varphi})(t)\, f(t)\, dt. \tag{3.14}$$

But (3.14) had been established in Sec. 1. Thus the lemma is proved.

Lemma 3.13. *There is a sequence* $\{\varphi_n\} \in C_0^\infty(\mathbf{R})$ *so that for any* $h \in B_{\text{mid}}$, $h^* \in B^*_{\text{mid}}$

$$\lim_{n \to \infty} h^*[(\varphi_{n_{\text{mid}}}(A) - 1)\, h] = 0.$$

Proof. Choose the sequence $\{\varphi_n\}$ in the following way:

$\varphi_1(x) = 1$ for $|x| < 1$, $\varphi_1(x) = 0$ for $|x| > 2$,

$\varphi_n(x) = \varphi_1(x/n)$.

Such a sequence $\{\varphi_n\}$ is bounded in $\mathscr{B}_N(\mathbf{R})$. In fact, let $\tilde{\varphi}_n$ be the Fourier transform of the function φ_n. Then

$$\tilde{\varphi}_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} \varphi_n(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} \varphi_1\left(\frac{x}{n}\right) dx =$$

$$= \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itxn} \varphi_1(x)\, dx = n\tilde{\varphi}_1(nt).$$

Next we have, for any function $f \in C_N(\mathbf{R})$:

$$\left| \int_{-\infty}^{\infty} \tilde{\varphi}_n(t) f(t)\, dt \right| = \left| \int_{-\infty}^{\infty} n\tilde{\varphi}_1(nt) f(t)\, dt \right| =$$

$$= \left| \int_{-\infty}^{\infty} \tilde{\varphi}_1(t) f\left(\frac{t}{n}\right) dt \right| \leqslant \| \tilde{\varphi}_1 \|_{C^+_N(\mathbf{R})} \sup_{t\in\mathbf{R}} \frac{\left| f\left(\frac{t}{n}\right)\right|}{(1+|t|)^N} =$$

$$= \| \varphi_1 \|_{C^+_N(\mathbf{R})} \sup_{t\in\mathbf{R}} \frac{|f(t)|}{(1+n|t|)^N} \leqslant$$

$$\leqslant \| \varphi_1 \|_{C^+_N(\mathbf{R})} \sup_{t\in\mathbf{R}} \frac{|f(t)|}{(1+|t|)^N} = \| \varphi_1 \|_{C^+_N(\mathbf{R})} \| f \|_{C_N(\mathbf{R})}.$$

Hence, it follows that $\| \varphi_n \|_{\mathscr{B}_N(\mathbf{R})} \leqslant \| \varphi_1 \|_{\mathscr{B}_N(\mathbf{R})}$. Set $\psi_n(x) = \frac{\varphi_n(x)}{x+i}$. Then $\psi_n(x) - \frac{1}{x+i} \to 0$ in any Sobolev space $W^l_2(\mathbf{R})$. For this reason $\psi_n(x) \to \frac{1}{x+i}$ in $\mathscr{B}_N(\mathbf{R})$, so that the sequence of operators $\psi_{n_{\mathrm{mid}}}(A)$ converges in norm to the operator $\left(\frac{1}{A+i}\right)_{\mathrm{mid}}$.

Next we have

$$\varphi_{n_{\mathrm{mid}}}(A) = \psi_{n_{\mathrm{mid}}}(A)(A+i)h \tag{3.15}$$

for any $h \in E$. Note that

$$\left(\frac{1}{A+i}\right)_{\mathrm{mid}} (A+i)h = h \tag{3.16}$$

for any $h \in E$. Consequently, for any $h \in E$ the following limiting relationship is valid:

$$\lim_{n\to\infty} \varphi_{n_{\mathrm{mid}}}(A)h = h. \tag{3.17}$$

From (3.17) we obtain for any $h \in E$, $h^* \in B^*_{\mathrm{mid}}$:

$$h^*[(\varphi_{n_{\mathrm{mid}}}(A) - 1)h] \to 0$$

for $n \to \infty$.

Now let h be an arbitrary element of B_{mid}, $h^* \in B^*_{\text{mid}}$. Since the sequence of operators $\{\varphi_{n_{\text{mid}}}(A) - 1\}$ is uniformly bounded by some constant M, for a given $\varepsilon > 0$ the element $\bar{h} \in E$ can be chosen so that $\|\bar{h} - h\|_{\text{mid}} < \varepsilon$ and we obtain

$$|h^* [(\varphi_{n_{\text{mid}}}(A) - 1) h]| \leqslant$$
$$\leqslant |h^* [(\varphi_{n_{\text{mid}}}(A) - 1)\bar{h}]| + M \|h^*\| \varepsilon \to M \|h^*\| \varepsilon$$

for $n \to \infty$, and this means that

$$\lim_{n\to\infty} h^* [(\varphi_{n_{\text{mid}}}(A) - 1) h] = 0,$$

since ε is arbitrary. This completes the proof of the lemma.

Corollary. *The set*

$$M = \bigcup_{\varphi \in C_0^\infty(\mathbf{R})} R(\varphi_{\text{mid}}(A))$$

is dense in B_{mid}.

Proof. Let us assume that the opposite is true, i.e., that $\bar{M} \neq B_{\text{mid}}$. Then from the Hahn-Banach theorem there follows the existence of a non-zero element $h^* \in B^*_{\text{mid}}$ such that $h^*(h) = 0$ for any $h \in M$. Let $h^*(h_0) \neq 0$. Consider a sequence $\{h_n\} = \{\varphi_{n_{\text{mid}}}(A) h_0\}$, where $\{\varphi_n\}$ is a sequence of functions of $C_0^\infty(\mathbf{R})$ featured in Lemma 3.13. We have $h^*(h_n) = 0$ for any n. On the other hand, by Lemma 3.13 we have

$$\lim_{n\to\infty} h^*(h_n) = h^*(h_0) \neq 0.$$

The resulting contradiction proves that M is dense in B_{mid}.

Lemma 3.14. *If* $\operatorname{Im} z \neq 0$ *then the operator* $[(A - z)^{-1}]_{\text{mid}}$ *is invertible.*

Proof. Let $h \in B_{\text{mid}}$ and let $[(A - z)^{-1}]_{\text{mid}} h = 0$. If $\{\varphi_n\}$ is a sequence of the functions of $C_0^\infty(\mathbf{R})$ considered in Lemma 3.13 then

$$0 = [(A - z)\varphi_n(A)]_{\text{mid}} [(A - z)^{-1}]_{\text{mid}} h = \varphi_{n_{\text{mid}}}(A) h$$

for all integers n. From Lemma 3.13

$$0 = \lim_{u\to\infty} h^* [\varphi_{n_{\text{mid}}}(A) h] = h^*(h)$$

for any $h^* \in B^*_{\text{mid}}$. Therefore $h = 0$, Q.E.D.

Lemma 3.15. *There exists a closure* $\bar{A}$ *of operator* A *on* B_{mid}.

Proof. Let $\operatorname{Im} z \neq 0$. Then the operator, which is inverse to $[(A - z)^{-1}]_{\text{mid}}$, is closed. By virtue of Lemma 3.5 this operator is

an extension of operator $A - z$. This means that operator $A = (A - z) + z$ has a closed extension, Q.E.D.

Operator $\overline{A}$ will be called *a generator*. Now we shall give a new interpretation of the formula

$$\varphi_{\text{mid}}(A)\, h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(t)\, U(t)\, h\, dt, \quad \tilde{\varphi} = F^{-1}\varphi. \tag{3.18}$$

Consider space $C_N(\mathbf{R}^n, B)$, which consists of functions continuous on $\mathbf{R}^n$ with values in a Banach space, having the finite norm

$$\| f \|_{C_N(\mathbf{R}^n,\, B)} = \sup_{x \in \mathbf{R}^n} \frac{\| f(x) \|}{(1 + |x|)^N}.$$

Example. For any $h \in E$ the function f:

$$f(t) = U(t)\, h$$

belongs to $C_N(\mathbf{R}, B_{\text{mid}})$, where

$$\| f \|_{C_N(\mathbf{R},\, B_{\text{mid}})} \leqslant \| h \|_{\text{mid}}.$$

Consider next for any $f \in C_N(\mathbf{R}^n, B)$ the operator $I_0[f] : F^{-1}\mathscr{B}_N^0(\mathbf{R}^n) \to B$ defined by the formulas

$$I_0[f]\, \delta_\xi = f(\xi),$$

$$I_0[f]\, \varphi = \int_{\mathbf{R}^n} \varphi(x)\, f(x)\, dx, \quad \varphi \in C_0^\infty(\mathbf{R}^n).$$

Problem. Prove that

$$\| I_0[f] \| \leqslant \| f \|_{C_N(\mathbf{R}^n,\, B)}.$$

Note. For any $\varphi \in C_0^\infty(\mathbf{R}^n)$ and any non-equal vectors $\xi_1, \ldots, \xi_k$ from $\mathbf{R}^n$ we have

$$\left\| \varphi + \sum_{i=1}^{k} a_i \delta_{\xi_i} \right\|_{C_N^+(\mathbf{R}^n)} = \int_{-\infty}^{\infty} |\varphi(x)|\, (1 + |x|)^N dx + \sum_{i=1}^{k} |a_i|\, (1 + |\xi_i|)^N.$$

Extend the operator $I_0[f]$ to the homomorphism $I[f] : C_N^+(\mathbf{R}^n) \to B$. We shall use the notation

$$I[f]\, \varphi = \int_{\mathbf{R}^n} \varphi(x)\, f(x)\, dx,$$

which provides the necessary interpretation of equality (3.18). Obviously, both definitions of the operator $\varphi_{\text{mid}}(A)$ are in fact identical.

Let $f \in C_N(\mathbf{R}^n, B)$ and let the function $\mathscr{F}$ defined by the formula $\mathscr{F}(x) = \hat{C}f(x)$, where $\hat{C}$ is a closed operator on B, belong to $C_N(\mathbf{R}^n, B)$. Then for any $\varphi \in F^{-1}\mathscr{B}_N^0(\mathbf{R}^n)$

$$\int\limits_{\mathbf{R}^n} \varphi(x)\hat{C}f(x)\,dx = \hat{C}\int\limits_{\mathbf{R}^n} \varphi(x) f(x)\,dx. \tag{3.19}$$

If φ is an arbitrary element of $C_N^+(\mathbf{R}^n)$ then, by choosing the sequence $\{\varphi_k\} \subset F^{-1}\mathscr{B}_N^0(\mathbf{R}^n)$, which converges to φ and going to the limit $k \to \infty$ in the equality

$$\int\limits_{\mathbf{R}^n} \varphi_k(x)\hat{C}f(x)\,dx = \hat{C}\int\limits_{\mathbf{R}^n} \varphi_k(x) f(x)\,dx,$$

we achieve that formula (3.19) is also valid in this case.

Lemma 3.16. *Let* $\operatorname{Im} z \neq 0$. *Then*

$$(\bar{A} - z)^{-1} = [(A - z)^{-1}]_{\text{mid}}.$$

Proof. We have to prove the following two formulas:

$$[(A - z)^{-1}]_{\text{mid}}(\bar{A} - z)h = h, \quad h \in D_{\bar{A}}, \tag{3.20}$$

$$(\bar{A} - z)[(A - z)^{-1}]_{\text{mid}} = 1. \tag{3.21}$$

Formula (3.20) is valid for $h \in E$ according to Lemma 3.5; if $h_n \to h$, $h_n \in E$, $h \in D_{\bar{A}}$, then in (3.20) the limit can be reached because $\bar{A} - z$ is a closed operator and $[(A - z)^{-1}]_{\text{mid}}$ is a homomorphism. This completes the proof of formula (3.20).

Now let $h \in E$ again. Denote by $\tilde{r}_z(t)$ the Fourier transform of the function $r_z(x) = (x - z)^{-1}$. We have

$$\begin{aligned} h &= [(A - z)^{-1}]_{\text{mid}}(A - z)h = \\ &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \tilde{r}_z(t)U(t)(A - z)h\,dt = \\ &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \tilde{r}_z(t)(A - z)U(t)h\,dt = \\ &= (\bar{A} - z)\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \tilde{r}_z(t)U(t)h\,dt = \\ &= (\bar{A} - z)[(A - z)^{-1}]_{\text{mid}}h. \end{aligned}$$

Here we have made use of formula (3.19). Thus, for $h \in E$

$$(\bar{A} - z)[(A - z)^{-1}]_{\text{mid}}h = h.$$

From the fact that $\bar{A}$ is closed and E is dense in B_{mid}, it follows that the equality is valid for any $h \in B_{\text{mid}}$, i.e., formula (3.21) is valid. This completes the proof of the lemma.

We shall summarize the main results obtained in this section in the form of the following theorem.

Theorem 3.1. (1) *The generator* A *as an operator on* B_{mid} *has a closure* $\bar{A}$.

(2) *There exists a homomorphism* M *of the Banach algebra* $\mathscr{B}_N(\mathbf{R})$ *into the algebra* $\operatorname{Op}(B_{\text{mid}})$ *of homomorphisms* $B_{\text{mid}} \to B_{\text{mid}}$, *besides the operator* $(\bar{A} - z)^{-1}$ *corresponding to the function*

$$x \to \frac{1}{x-z}, \quad \operatorname{Im} z \neq 0.$$

Notation. Let the homomorphism $\Phi : B_{\text{mid}} \to B_{\text{mid}}$ correspond to the function $\varphi \in \mathscr{B}_N(\mathbf{R})$. Denote $\Phi = \varphi(\bar{A})$. The image of the homomorphism M will be denoted by the symbol $\mathfrak{A}$.

The function $\varphi \in \mathscr{B}_N(\mathbf{R})$ will be called the *symbol of the operator* $\varphi(\bar{A})$.

Sec. 4. The Extension of the Class of Possible Symbols

Let A be a generator of degree N with the defining pair (B_1, B_2) of Banach spaces, every space containing a dense linear manifold $D = D_A$. In the previous section the operators $\varphi(\bar{A})$ were introduced, where $\varphi \in \mathscr{B}_N(\mathbf{R})$. By definition we have assumed $\varphi(\bar{A}) = \overline{\varphi_{\text{mid}}(A)}$, where $\varphi_{\text{mid}}(A)$ is an operator defined on E. Now we shall put forward a formula for $\varphi(\bar{A})$ which is more evident. Namely,

$$\varphi(\bar{A}) h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(t) e^{-i\bar{A}t} h \, dt$$

for any $h \in B_{\text{mid}}$. In fact, let $\{h_n\}$ be a sequence of elements of E which converges to h. Then

$$\| e^{-i\bar{A}t} h - U(t) h_n \| = \| e^{-i\bar{A}t}(h - h_n) \| \leqslant (1 + |t|)^N \| h - h_n \|.$$

Consequently, the functions $U(t) h_n$ converge to $e^{-i\bar{A}t} h$ in $C_N(\mathbf{R}, B_{\text{mid}})$. For this reason

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(t) e^{-i\bar{A}t} h \, dt = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(t) u(t) h_n \, dt =$$
$$= \lim_{n\to\infty} \varphi_{\text{mid}}(A) h_n = \varphi(\bar{A}) h,$$

Q.E.D.

Theorem 4.1. *Let* $\varphi \in \mathscr{B}_N(\mathbf{R})$ *and let*

$$\psi(x) = x^k \varphi(x),$$

where k *is an integer also belonging to* $\mathscr{B}_N(\mathbf{R})$. *Then*

$$\psi(\bar{A}) = \bar{A}^k \varphi(\bar{A}) = \overline{\varphi(\bar{A})\, A^k} = \overline{\varphi(\bar{A})\, \bar{A}^k}.$$

To obtain the proof we shall need the lemma that follows.

Lemma 4.1. *For any* $h \in D_{\bar{A}}$ *the following formula is valid:*

$$e^{-i\bar{A}t}\bar{A}h = \bar{A}e^{-i\bar{A}t}h.$$

Proof. Let $\{h_n\}$ be a sequence of elements of E which converges to h, where $\bar{A}h = \lim\limits_{n\to\infty} Ah_n$.

$$e^{-i\bar{A}t}\bar{A}h = \lim_{n\to\infty} e^{-i\bar{A}t}Ah_n = \lim_{n\to\infty} Ae^{-i\bar{A}t}h_n = \bar{A}e^{-i\bar{A}t}h,$$

Q.E.D.

Proof of Theorem 4.1. For any $h \in E$ the following equality is valid: $\psi_{\mathrm{mid}}(A)\, h = \varphi_{\mathrm{mid}}(A)\, A^k\, h$, i.e.

$$\psi_{\mathrm{mid}}(A)\, h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(t)\, U(t)\, A^k h\, dt. \tag{4.1}$$

By virtue of (3.19) the above equality may be written as follows:

$$\psi_{\mathrm{mid}}(A)\, h = \frac{1}{\sqrt{2\pi}} \bar{A}^k \int_{-\infty}^{\infty} \tilde{\varphi}(t)\, u(t)\, h\, dt = \bar{A}^k \varphi_{\mathrm{mid}}(A)\, h.$$

Here we have made use of the law of closedness of the degree of a closed operator. Besides, the fact has been taken into account that $(\bar{A} + i)^{-1}$ is the homomorphism $B_{\mathrm{mid}} \to B_{\mathrm{mid}}$. We have proved that the operator $\psi(\bar{A})$ coincides with the operator $\bar{A}^k \varphi(\bar{A})$ on the set E which is dense in B_{mid}. Since $\varphi(\bar{A})$ and $\psi(\bar{A})$ are homomorphisms and the operator $\bar{A}^k$ is closed, operator $\bar{A}^k \varphi(\bar{A})$ coincides everywhere with operator $\psi(\bar{A})$.

Further, the equality (4.1) means that the operator $\varphi(\bar{A})\, A^k$ is a restriction of the homomorphism $\psi(\bar{A})$ of the dense set E. For this reason the following formula is valid:

$$\psi(\bar{A}) = \overline{\varphi(\bar{A})\, A^k}.$$

It remains to prove the formula

$$\psi(\bar{A}) = \overline{\varphi(\bar{A})\, \bar{A}^k}.$$

In other words, it is necessary to show that the operator $\varphi(\bar{A})\bar{A}^k$ has a closed extension, since it coincides with the homomorphism $\psi(\bar{A})$ on the dense set E. Let $h_n \in D_{\bar{A}^k}$, $h_n \to 0$ and the sequence $\varphi(\bar{A})^k \bar{A}\, h_n$ converges. We have

$$\varphi(\bar{A})\bar{A}^k h_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(t)\, e^{-i\bar{A}t} \bar{A}^k h_n\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(t)\, \bar{A}^k e^{-i\bar{A}t} h_n\, dt =$$

$$= \frac{1}{\sqrt{2\pi}} \bar{A}^k \int_{-\infty}^{\infty} \tilde{\varphi}(t)\, e^{-i\bar{A}t} h_n\, dt = \frac{1}{\sqrt{2\pi}} \bar{A}^k \varphi(A)\, h_n \to 0$$

for $n \to \infty$ due to the fact that the operator $\bar{A}$ is closed, Q.E.D.

Thus, we have shown that the formula

$$[fg](\bar{A}) = f(\bar{A})\, g(\bar{A})$$

is valid not only in the case when $f, g \in \mathscr{B}_N(\mathbf{R})$ but also when f is a polynomial,

$$g \in \mathscr{B}_N(\mathbf{R}) \quad \text{and} \quad fg \in \mathscr{B}_N(\mathbf{R}).$$

We shall adhere to the following notation: for the sake of simplicity the closure $\bar{A}$ of the generator A in B_{mid} will be denoted by A. We shall mention this specifically only in cases where some misunderstanding may arise.

Lemma 4.2. *If $\varphi \in \mathscr{B}_N(\mathbf{R})$ and if $\varphi(x)$ does not turn zero at any point of* $\mathbf{R}$, *then the operator $\varphi(A)$ is invertible (i.e. $\varphi(A)h \neq 0$ for $h \neq 0$).*

Proof. Let $\{e_n\}$ be such a sequence of functions in $C_0^\infty(\mathbf{R})$ that

$$\lim_{n\to\infty} h^*(e_n(A)h) = h^*(h)$$

for any $h \in B_{\text{mid}}$, $h^* \in B^*_{\text{mid}}$. The existence of such sequences has been established in Lemma 3.13. If $\varphi(A)h = 0$ then

$$\psi_n(A)\,\varphi(A)\,h = 0,$$

where $\psi_n(x) = e_n(x)/\varphi(x)$. But $\psi_n(A)\,\varphi(A) = e_n(A)$. Consequently for any $h^* \in B^*_{\text{mid}}$,

$$h^*(h) = \lim_{n\to\infty} h^*(e_n(A)h) = 0.$$

Hence it follows that $h = 0$ and the lemma is proved.

Theorem 4.2. *For any $f \in \mathscr{B}_N(\mathbf{R})$ and a polynomial P there exists a closure of the operator $f(A)\,P(A)$ on B_{mid}.*

Proof. Let $h_n \to 0$ and let $f(A)\, P(A)\, h_n \to q$ for $n \to \infty$. Show that $q = 0$. Denote by φ a function in S which does not turn to zero anywhere. We have

$$\lim_{n\to\infty} \varphi(A)\, f(A)\, P(A)\, h_n = \varphi(A)\, q.$$

But $\varphi f/P \in \mathscr{B}_N(\mathbf{R})$. And therefore by virtue of Theorem 4.1 we have

$$\varphi(A)\, q = \lim_{n\to\infty} \varphi(A)\, f(A)\, P(A)\, h_n = \lim_{n\to\infty} P(A)\, \varphi(A)\, f(A)\, h_n = 0.$$

Hence it follows that $q = 0$, due to the invertibility of the operator $\varphi(A)$, and the theorem is proved.

Theorem 4.3. *Let $f \in \mathscr{B}_N(\mathbf{R})$ and let P be a polynomial. Then the operator $P(A)\, f(A)$ is an extension of the operator $f(A)\, P(A)$.*

Proof. If h belongs to the domain of the operator $P(A)$, then

$$f(A)\, P(A)\, h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(t)\, e^{-iAt} P(A)\, h\, dt,$$

where $\tilde{f}$ is the Fourier transform of the function f. Hence by using Lemma 4.1 and the fact that the operator $P(A)$ is closed we obtain

$$f(A)\, P(A)\, h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(t)\, P(A)\, e^{-iAt} h\, dt =$$

$$= \frac{1}{\sqrt{2\pi}} P(A) \int_{-\infty}^{\infty} \tilde{f}(t)\, e^{-iAt} h\, dt = P(A)\, f(A)\, h.$$

Therefore the operator $P(A)\, f(A)$ is an extension of the operator $f(A)\, P(A)$.

Now let h be an arbitrary element of the domain of the operator $\overline{f(A)\, P(A)}$. Then there exists such a sequence $\{h_n\}$ of elements of the domain of the operator $P(A)$ that

$$\lim_{n\to\infty} h_n = h, \quad \lim_{n\to\infty} f(A)\, P(A)\, h_n = \overline{f(A)\, P(A)}\, h.$$

Since $f(A)$ is a homomorphism and $f(A)\, P(A)$ is a restriction of the operator $P(A)\, f(A)$ we obtain

$$\lim_{n\to\infty} f(A)\, h_n = f(A)\, h, \quad \lim_{n\to\infty} P(A)\, f(A)\, h_n = \overline{f(A)\, P(A)}\, h.$$

Hence due to the closed nature of the operator $P(A)$ it follows that

$$P(A)\, f(A)\, h = \overline{f(A)\, P(A)}\, h,$$

Q.E.D.

Lemma 4.3. *Let f_1, f_2, f_3 be functions belonging to $\mathscr{B}_N(\mathbf{R})$ and let P and Q be polynomials, where*

$$f_1(x)\,P(x) = f_2(x)\,Q(x) + f_3(x).$$

Then for any $h \in D_{P(A)f_1(A)} \cap D_{Q(A)f_2(A)}$ the following equality is valid:

$$P(A)\,f_1(A)\,h = Q(A)\,f_2(A)\,h + f_3(A)\,h.$$

Proof. Let a function $\varphi \in S$ not turn zero at any point. Then according to Lemma 4.2 the operator $\varphi(A)$ is invertible; it suffices to show that

$$\varphi(A)\,[P(A)\,f_1(A) - Q(A)\,f_2(A) - f_3(A)]\,h = 0.$$

The functions φP and φQ belong to $\mathscr{B}_N(\mathbf{R})$, besides

$$\varphi(A)\,P(A) \subset [P\varphi](A), \quad \varphi(A)\,Q(A) \subset [Q\varphi](A).$$

Therefore

$$\varphi(A)\,[P(A)\,f_1(A) - Q(A)\,f_2(A) - f_3(A)]\,h = \\ = [(P\varphi)(A)\,f_1(A) - (Q\varphi)(A)\,f_2(A) - \varphi(A)\,f_3(A)]\,h = 0,$$

since the operator $[P\varphi](A)\,f_1(A) - [Q\varphi](A)\,f_2(A) - \varphi(A)\,f_3(A)$ has the symbol $\varphi(Pf_1 - Qf_2 - f_3) = 0$, Q.E.D.

Now we shall define the increasing functions of a generator.

Definition. *Let* $g(x) = f(x)\,P(x) + f_1(x)$, *where* $f \in \mathscr{B}_N(\mathbf{R})$, $f_1 \in \mathscr{B}_N(\mathbf{R})$ *and where P is a polynomial. Set*

$$g(A)\,h \overset{\text{def}}{=} \overline{f(A)\,P(A)}\,h + f_1(A)\,h.$$

Note. Element h of B_{mid} belongs to the domain of the operator $g(A)$ if and only if there exists such a representation of the function g

$$g(x) = f(x)\,P(x) + f_1(x),$$

where f, $f_1 \in \mathscr{B}_N(\mathbf{R})$ and P is a polynomial, that

$$h \in D_{\overline{f(A)P(A)}}.$$

From Lemma 4.3 and Theorem 4.3 it follows that the definition is correct in the sense that the operator $g(A)$ does not depend on the type of representation of the function g in the form $fP + f_1$. In the case when $g \in \mathscr{B}_N(\mathbf{R})$ the definition is obviously in line with the definition of the homomorphism $g(A)$ mentioned in the preceding section. In the case when g is a polynomial the definition is also in line with the definition of a polynomial of an operator. In fact, let

$$g(x) = f(x)\,P(x) + f_1(x),$$

where f, $f_1 \in \mathscr{B}_N(\mathbf{R})$ and where P and g are polynomials. First of all, note that the degree of P is not lower than the degree of g since the functions f and f_1 are bounded. Therefore, the domain of the operator $P(A)$ is contained in the domain of the operator $g(A)$. It must be shown that the domain of the operator $\overline{f(A)P(A)}$ does not exceed the domain of the operator $g(A)$. Let h be an arbitrary element of the domain of the operator $\overline{f(A)P(A)}$. Then there exists such a sequence $\{h_n\}$ of elements of the domain of the operator $P(A)$ that $\lim\limits_{n\to\infty} h_n = h$. Hence the sequence $\{f(A)P(A)h_n\}$ is fundamental. But $h_n \in D_{g(A)}$. Consequently, according to Theorem 4.3 and to Lemma 4.3 the following equality is valid:

$$g(A)h_n = f(A)P(A)h_n + f_1(A)h_n.$$

Hence the sequence $\{g(A)h_n\}$ converges. Since the operator $g(A)$ is closed, $h \in D_{g(A)}$, Q.E.D.

Suppose now that the closed operator A is invertible.

Theorem 4.4. *Let* $\varphi \in \mathscr{B}_N(\mathbf{R})$ *and let*

$$\psi(x) = x^{-k}\varphi(x),$$

where k is an integer, also belong to $\mathscr{B}_N(\mathbf{R})$. *Then*

$$\psi(A) = A^{-k}\varphi(A).$$

Proof. We have

$$\varphi(x) = x^k\psi(x).$$

Therefore

$$\varphi(A) = A^k\psi(A), \quad \psi(A) = A^{-k}\varphi(A).$$

Now define the operators which correspond to symbols with singularities at zero.

Definition. *Let* $g(x) = f(x)x^{-k}$, *where* $f \in \mathscr{B}_N(\mathbf{R})$ *and k is an integer. Set*

$$g(A) \stackrel{\text{def}}{=} A^{-k}f(A).$$

Verify the correctness of the given definition. Let $f_1, f_2 \in \mathscr{B}_N(\mathbf{R})$, $x^{-k_1}f_1(x) = x^{-k_2}f_2(x)$. Show that

$$A^{-k_1}f_1(A) = A^{-k_2}f_2(A).$$

Let $k_2 > k_1$. We have $f_2(x) = x^{k_2-k_1}f_1(x)$. Consequently $f_2(A) = A^{k_2-k_1}f_1(A)$. By multiplying both members of this equality by A^{-k_2} from the left we obtain the relationship in question.

Lemma 4.4. *Let* $f \in \mathscr{B}_N(\mathbf{R})$, $h \in D_{A^{-k}}$. *Then*
$f(A)A^{-k}h = A^{-k}f(A)h$.

Proof. We have

$$f(A)h = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(t)e^{-iAt}h\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(t)e^{-iAt}A^kA^{-k}h\,dt =$$

$$= \frac{A^k}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(t)e^{-iAt}A^{-k}h\,dt = A^kf(A)A^{-k}h.$$

By multiplying both members of the obtained equality by A^{-k} we assert the proof of the lemma.

Definition. *Let* $g(x) = f_1(x)P(x) + f_2(x)x^{-k} + f_3(x)$, *where* f_1, f_2, f_3 *are functions belonging to* $\mathscr{B}_N(\mathbf{R})$, P *is a polynomial*, k *is an integer. Set*

$$g(A)h = g_1(A)h + g_2(A)h + f_3(A)h,$$

where $g_1(x) = f_1(x)P(x)$, $g_2(x) = f_2(x)x^{-k}$, h *is an arbitrary element of* $D_{g_1(A)} \cap D_{g_2(A)}$.

Verify the correctness of this definition. Let

$$g(x) = g_1(x) + g_2(x) + f_3(x) = g_4(x) + g_5(x) + f_6(x),$$

where

$$g_1(x) = f_1(x)P(x), \quad g_2(x) = f_2(x)x^{-k},$$
$$g_4(x) = f_4(x)Q(x), \quad g_5(x) = f_5(x)x^{-l},$$

$f_j \in \mathscr{B}_N(\mathbf{R})$, $j = 1, 2, 3, 4, 5, 6$, P and Q are polynomials, k and l are integers. First of all, note that the functions g_2 and g_5 may be modified in a bounded neighborhood of zero with a view to obtain functions belonging to $\mathscr{B}_N(\mathbf{R})$. Since

$$g_1 - g_4 = g_5 - g_2 + f_6 + f_3$$

and since the function $g_1 - g_4$ coincides with some function belonging to $\mathscr{B}_N(\mathbf{R})$ in a neighborhood of zero we obtain

$$g_1(x) - g_4(x) = f_7(x),$$

where $f_7 \in \mathscr{B}_N(\mathbf{R})$. Therefore

$$g_1(A) = g_4(A) + f_7(A).$$

Further, the following equality is valid:

$$g_2(x) = g_5(x) + f_8(x),$$

where $f_8 = f_3 + f_6 - f_7 \in \mathscr{B}_N(\mathbf{R})$. Let

$$h \in D_{A^{-k}f_2(A)} \cap D_{A^{-l}f_5(A)}.$$

Set

$$q = [g_2(A) - g_5(A) - f_8(A)]\,h = [A^{-k}f_2(A) - A^{-l}f_5(A) - f_8(A)]\,h.$$

It is required to prove that $q = 0$. We shall prove this statement in the following way: let $\varphi \in S$ be a function which does not turn zero anywhere and let $\varphi_m(x) = x^m\varphi(x)$, $m = 1, 2, \ldots$. We shall show that the operator $\varphi_m(A)$ is invertible and that $\varphi_m(A)\,q = 0$ for a sufficiently large m. We have $\varphi_m(A) = A^m\varphi(A)$. For this reason

$$(\varphi_m(A)\,u = 0) \Rightarrow (A^m\varphi(A)\,u = 0) \Rightarrow$$
$$\Rightarrow (\varphi(A)\,u = 0) \Rightarrow (u = 0).$$

Here we have made use of the invertibility of operators A and $\varphi(A)$. The invertibility of operator $\varphi_m(A)$ is proved.

Let $m \geqslant \max(k, l)$. Then the functions $x^{-k}\varphi_m(x)$ and $x^{-l}\varphi_m(x)$ belong to the space $\mathscr{B}_N(\mathbf{R})$. By making use of Theorem 4.4 and of Lemma 4.4 we obtain

$$\varphi_m(A)\,q = [A^{m-k}\varphi(A)\,f_2(A) - A^{m-l}\varphi(A)\,f_5(A) - A^m\varphi(A)\,f_8(A)]\,h = 0,$$

Q.E.D.

Example. Let $\sqrt{x} \overset{\text{def}}{=} i\sqrt{|x|}$ for $x < 0$. Then the function $x^{N+1}\sqrt{x}$ is $N+1$ times continuously differentiable along a real straight path. Let $1 = e_1(x) + e_2(x)$, where $e_1 \in C_0^\infty(\mathbf{R})$, $e_1(x) = 1$ for sufficiently small values of x. Set $f_1(x) = e_1(x)\,x^{N+1}\sqrt{x}$, $f_2(x) = e_2(x)/\sqrt{x}$. Then $f_1, f_2 \in \mathscr{B}_N(\mathbf{R}^1)$ and

$$\sqrt{x} = g_1(x) + g_2(x),$$

where $g_1(x) = x^{-N-1}f_1(x)$, $g_2(x) = xf_2(x)$. Consequently, for any $h \in D_{g_1(A)} \cap D_{g_2(A)}$ the following formula is valid:

$$\sqrt{A}\,h = A^{-N-1}f_1(A)\,h + Af_2(A)\,h.$$

Problem. We have defined the infinitely differentiable functions of a commutative operator which are increasing at a rate not exceeding the degree of the argument. Show that they form an algebra with μ-structure, where the algebra A is an algebra of commutative unbounded operators, and the set M consists of generators and real functions of them.

Sec. 5. Homomorphism of Asymptotic Formulas. The Method of Stationary Phase

Let A be a closure in B_{mid} of a generator of degree N and let k be an integer. From Theorem 4.1 it follows: if $\varphi, \psi \in \mathscr{B}_N(\mathbf{R})$ and $\varphi(x) - \psi(x) = O(x^{-k})$ in a sense that the function $x^k[\varphi(x) - \psi(x)]$ belongs to $\mathscr{B}_N(\mathbf{R})$, then for any $g \in B_{\text{mid}}$ the vector $\varphi(A)g - \psi(A)g$ belongs to the domain of the operator A^k. This is a homomorphism.

To illustrate the point we shall consider the so-called method of the stationary phase. Let $\varphi \in C_0^\infty(\mathbf{R})$, let $\mathscr{F} \in C_0^\infty(\mathbf{R})$ be a real function, and let the equation $\mathscr{F}'(\xi) = 0$ have a unique solution ξ_0 on the support of the function φ, where $\mathscr{F}''(\xi_0) \neq 0$. Consider the integral

$$I(x) = \int_{-\infty}^{\infty} \varphi(\xi) e^{ix\mathscr{F}(\xi)} d\xi. \tag{5.1}$$

The method of the stationary phase provides an asymptotic expansion of the function $I(x)$ in negative powers of x:

$$I(x) = e^{ix\mathscr{F}(\xi_0)} \sum_{j=0}^{n} a_j(\varphi, \mathscr{F}) x^{-j-\frac{1}{2}} + R_{n+1}(x),$$

where $R_{n+1}(x) = O\left(x^{-n-\frac{3}{2}}\right)$. We reproduce here the derivation of the formula of the method of the stationary phase to obtain the estimate of the remainder R_{n+1} in the required form.

Now make the substitution of variables in the integral (5.1)

$$[\mathscr{F}(\xi) - \mathscr{F}(\xi_0)] = t^2 \operatorname{sgn}[\mathscr{F}''(\xi)]$$

and denote $\omega = |x|$, $\sigma = \operatorname{sgn}[x\mathscr{F}''(\xi_0)]$. We obtain

$$I(x) = e^{ix\mathscr{F}(\xi_0)} \int_{-\infty}^{\infty} e^{i\omega\sigma t^2} \psi(t)\, dt, \tag{5.2}$$

where $\psi(t) = \varphi(\xi(t))\, d\xi/dt$, $\psi \in C_0^\infty(\mathbf{R})$. Transform the formula (5.2) in the following way:

$$e^{-ix\mathscr{F}(\xi_0)} I(x) = \psi(0) \int_{-\infty}^{\infty} e^{i\omega\sigma t^2} dt + \int_{-\infty}^{\infty} e^{i\omega\sigma t^2} [\psi(t) - \psi(0)]\, dt =$$

$$= \sqrt{\frac{\pi}{\omega}}\, e^{i\sigma\pi/4} \psi(0) - \frac{1}{2i\omega\sigma} \int_{-\infty}^{\infty} e^{i\omega\sigma t^2} \psi_1(t)\, dt,$$

where $\psi_1(t) = \frac{d}{dt} \frac{\psi(t) - \psi(0)}{t}$. Here we have used integration by parts. It is easy to see that outside the support of function $\psi(t)$

the following equality is valid:

$$\psi_1(t) = \frac{\psi(0)}{t^2}$$

By transforming the integral $\int\limits_{-\infty}^{\infty} e^{i\omega\sigma t^2}\psi_1(t)\,dt$ in the same way as the integral $\int\limits_{-\infty}^{\infty} e^{i\omega\sigma t^2}\psi(t)\,dt$ we obtain the formula

$$e^{-ix\mathcal{F}(\xi_0)}I(x) = \sqrt{\frac{\pi}{\omega}}\, e^{i\sigma\pi/4}\left(\psi(0) + \frac{i}{2\omega\sigma}\psi_1(0)\right) + \left[\frac{i}{2\omega\sigma}\right]^2 \int\limits_{-\infty}^{\infty} e^{i\omega\sigma t^2}\psi_2(t)\,dt,$$

where $\psi_2(t) = \frac{d}{dt}\frac{\psi_1(t)-\psi_1(0)}{t}$. Continuing the process we obtain

$$e^{-ix\mathcal{F}(\xi_0)}I(x) = \sqrt{\frac{\pi}{\omega}}\, e^{i\sigma\pi/4}\sum_{j=0}^{n}\left(\frac{i}{2\omega\sigma}\right)^j\psi_j(0) + \left(\frac{i}{2\omega\sigma}\right)^{n+1}\int\limits_{-\infty}^{\infty} e^{i\omega\sigma t^2}\psi_{n+1}(t)\,dt, \tag{5.3}$$

where

$$\psi_{j+1}(t) = \frac{d}{dt}\frac{\psi_j(t)-\psi_j(0)}{t}, \qquad j = 1, 2, \ldots, \quad \psi_0 = \psi_1.$$

To prove the formula (5.3) we must verify that for any integer j

$$\lim_{t\to\infty}\frac{\psi_j(t)-\psi_j(0)}{t} = 0. \tag{5.4}$$

By using the explicit formula for $\psi_1(t)$, where t is large it is easy to establish by induction that for $t \notin \operatorname{supp}\psi$ we have $\psi_j(t) = t^{-2}P(t^{-1})$, where P is a polynomial in j. The formula (5.4) is thereby derived.

Thus we obtain the following expansion:

$$\sqrt{x}\, e^{-ix\mathcal{F}(\xi_0)}\int\limits_{-\infty}^{\infty} e^{ix\mathcal{F}(\xi)}\varphi(\xi)\,d\xi = \sqrt{\pi}\, e^{i\pi/4\,\operatorname{sgn}\mathcal{F}''(\xi_0)} \times \sum_{j=0}^{n}\left(\frac{i}{2x\operatorname{sgn}\mathcal{F}''(\xi_0)}\right)^j\psi_j(0) + r_{n+1}(x), \tag{5.5}$$

$$r_{n+1}(x) = \sqrt{x}\left(\frac{i\operatorname{sgn}\mathcal{F}''(\xi_0)}{2x}\right)^{n+1}\int\limits_{-\infty}^{\infty} e^{ixt^2\operatorname{sgn}\mathcal{F}''(\xi_0)}\psi_{n+1}(t)\,dt, \tag{5.6}$$

where for negative x

$$\sqrt{x} \overset{\text{def}}{=} \sqrt{|x|}\, e^{i\pi/2 \operatorname{sgn} \mathscr{F}''(\xi_0)}.$$

We cannot replace the variable x directly by the generator A in formula (5.5) because the function $\frac{1}{x^j}$ does not belong to $\mathscr{B}_N(\mathbf{R})$. Let us correct the expansion (5.5) in the following way. Let rad x be an infinitely differentiable function which coincides with $\sqrt{x}$ outside a finite neighborhood of zero, and let $\rho_j(x)$ be an infinitely differentiable function which coincides with x^{-j} outside a finite neighborhood of zero. Then

$$(\operatorname{rad} x)\, e^{-ix\mathscr{F}(\xi_0)} \int_{-\infty}^{\infty} e^{ix\mathscr{F}(\xi)} \varphi(\xi)\, d\xi = \sqrt{\pi}\, e^{i\pi/4 \operatorname{sgn} \mathscr{F}''(\xi_0)} \times$$

$$\times \sum_{j=0}^{n} \left(\frac{i \operatorname{sgn} \mathscr{F}''(\xi_0)}{2} \right)^j \rho_j(x)\, \psi_j(0) + \bar{r}_{n+1}(x), \tag{5.7}$$

where $\bar{r}_{n+1}(x) = r_{n+1}(x)$ for sufficiently large $|x|$.

Note that all terms in (5.7) except $\bar{r}_{n+1}(x)$ are obviously infinitely differentiable. Hence, it follows that the function $\bar{r}_{n+1}(x)$ is also infinitely differentiable.

Now evaluate the remainder term $r_{n+1}(x)$ at infinity. For the sake of certainty set $\operatorname{sgn} \mathscr{F}''(\xi_0) = 1$. Continuing the expansion (5.5) we obtain for any integer $m \geqslant n+1$:

$$x^{n+1} r_{n+1}(x) = \sqrt{\pi}\, e^{i\pi/4} \left(\frac{i}{2}\right)^{n+1} \psi_{n+1}(0) +$$

$$+ \sqrt{\pi}\, e^{i\pi/4} \sum_{j=n+2}^{m} \left(\frac{i}{2}\right)^j x^{n+1-j} \psi_j(0) +$$

$$+ \sqrt{x} \left(\frac{i}{2}\right)^{m+1} x^{n-m} \int_{-\infty}^{\infty} e^{ixt^2} \psi_{m+1}(t)\, dt. \tag{5.8}$$

The first term in the right-hand member of (5.8) is a constant and this is why it belongs to $\mathscr{B}_N(\mathbf{R})$. Next all terms in $\sum_{j=n+2}^{m}$ in the right-hand side of (5.8) are square-integrable at infinity and all their derivatives have the same property. Finally, consider the function

$$f(x) = \sqrt{x}\, x^{n-m} \int_{-\infty}^{\infty} e^{ixt^2} \psi_{m+1}(t)\, dt \overset{\text{def}}{=} \sqrt{x}\, x^{n-m} g(x).$$

The function g is bounded. Estimate its derivatives. We have

$$g'(x) = \int_{-\infty}^{\infty} it^2 e^{ixt^2} \psi_{m+1}(t)\, dt = -\frac{1}{2x} \int_{-\infty}^{\infty} e^{ixt^2} [t\psi_{m+1}(t)]'\, dt.$$

We have carried out integration here by parts and have made use of the fact that $\lim\limits_{t\to\infty} t\psi_{m+1}(t) = 0$. For calculating $g''(x)$ we note that the function $[t\psi_{m+1}(t)]'$ and the function $\psi_{m+1}(t)$ can be represented in the form of the product of t^{-2} by a polynomial in t^{-1} in such a way that

$$g''(x) = \frac{1}{(2x)^2} \int_{-\infty}^{\infty} e^{ixt^2} \frac{d}{dt}\, t\, \frac{d}{dt}\, t\psi_{m+1}(t)\, dt.$$

By calculating the higher derivatives in the same way we verify that the function $g(x)$ is infinitely differentiable and $g^{(k)}(x) = O(|x|^{-k})$. Hence, it follows that for sufficiently large m the function is integrable at infinity for $k \leqslant N + 1$.

From the obtained estimates it follows that the function $x^{n+1}\bar{r}_{n+1}(x)$ can be represented in the form of the sum of a constant and a function in $W_2^{N+1}(\mathbf{R})$. Consequently, it belongs to $\mathscr{B}_N(\mathbf{R})$. Obviously the function $\bar{r}_{n+1}$ itself belongs also to $\mathscr{B}_N(\mathbf{R})$. Thus for any $g \in B_{\text{mid}}$

$$\bar{r}_{n+1}(A)\, g \in D_A^{n+1}.$$

Now consider the left-hand member of formula (5.7). This function belongs to $\mathscr{B}_N(\mathbf{R})$ because the right-hand member has this property. Set

$$\operatorname{rad} x \cdot I(x) \stackrel{\text{def}}{=\!=} K(x).$$

The following expansion is valid:

$$K(A) = \sqrt{\pi}\, e^{i\left(A\mathscr{F}(\xi_0) + \frac{\pi}{4}\operatorname{sgn}\mathscr{F}''(\xi_0)\right)} \times$$
$$\times \sum_{j=0}^{n} \left[\frac{i\operatorname{sgn}\mathscr{F}''(\xi_0)}{2}\right]^j \psi_j(0)\, \rho_j(A) + \bar{r}_{n+1}(A),$$

where the range of the operator $\bar{r}_{n+1}(A)$ is contained within the domain of the operator A^{n+1}. Now set

$$g(x, \xi) = \frac{1}{(x+i)^2}\, \varphi(\xi)\, e^{ix\mathscr{F}(\xi)}, \quad L(x) = \frac{1}{(x+i)^2}\, I(x).$$

The function $g(x, \xi)$ generates the function $G(\xi)$ with values in $W_2^{N+1}(\mathbf{R}) \subset \mathscr{B}_N(\mathbf{R})$. The function G is continuous. Indeed, the following estimate is valid:

$$\frac{\partial^{j+1} g(x, \xi)}{\partial\xi\, \partial x^j} = O\left(\frac{1}{(x^2+1)^{1/2}}\right),$$

which is uniform in ξ. Consequently

$$L(A) = \int_{-\infty}^{\infty} g(A, \xi)\, d\xi.$$

Next, it is not difficult to verify that $I \in \mathscr{B}_N(\mathbf{R})$. This means that the latter equality may be rewritten in the form

$$(A+i)^{-2} I(A) = \int_{-\infty}^{\infty} g(A, \xi)\, d\xi, \quad I(A) = (A+i)^2 \int_{-\infty}^{\infty} g(A, \xi)\, d\xi.$$

Show that $K(A) = \operatorname{rad} A \cdot I(A)$. Let $h \in E$. Then for any integer k and any $\chi \in \mathscr{B}_N(\mathbf{R})$ we have

$$\chi(A)(A+i)^k h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\chi}(t)(A+i)^k U(t)\, h\, dt = (A+i)^k \chi(A)\, h.$$

Hence, in particular, it follows that $\chi(A)\, h$ belongs to the domain of any degree of the operator A. Set

$$\operatorname{rad}_k(x) = \frac{\operatorname{rad} x}{(x+i)^k};$$

for sufficiently large k we have $\operatorname{rad}_k \in \mathscr{B}_N(\mathbf{R})$. For any $h \in E$, $\operatorname{rad} A \times \times I(A)\, h = \operatorname{rad}_k(A)(A+i)^k I(A)\, h = \operatorname{rad}_k(A)\, I(A)(A+i)^k h$. On the other hand,

$$(A+i)^{-k} K(A) = \operatorname{rad}_k(A)\, I(A);$$

for this reason

$$\operatorname{rad} A \cdot I(A)\, h = (A+i)^{-k} K(A)(A+i)^k h = K(A)\, h.$$

Now let h be an arbitrary element of $B_{\min}$, $h_j \in E$, $h_j \to h$ for $j \to \infty$. Then

$$K(A)\, h_j \to K(A)\, h, \quad I(A)\, h_j \to I(A)\, h \text{ for } j \to \infty,$$

consequently,

$$K(A)\, h = \lim_{j\to\infty} \operatorname{rad} A \cdot I(A)\, h_j = \operatorname{rad} A \cdot I(A)\, h,$$

Q.E.D. We can now write the formula

$$K(A) = \operatorname{rad} A \cdot (A+i)^2 \int_{-\infty}^{\infty} g(A, \xi)\, d\xi.$$

Hence it follows that for any $h \in E$

$$K(A)\,h = \operatorname{rad} A \cdot (A+i)^2 \int_{-\infty}^{\infty} g(A,\xi)\,h\,d\xi =$$

$$= \operatorname{rad} A \int_{-\infty}^{\infty} g(A,\xi)\,(A+i)^2 h\,d\xi = \operatorname{rad} A \int_{-\infty}^{\infty} \varphi(\xi)\, e^{iA\mathcal{F}(\xi)} h\,d\xi.$$

To extend the equality obtained for $h \in E$

$$K(A)\,h = \operatorname{rad} A \int_{-\infty}^{\infty} \varphi(\xi)\, e^{iA\mathcal{F}(\xi)} h\,d\xi$$

for the case of an arbitrary $h \in B_{\text{mid}}$ we shall first show that the function

$$\xi \to \varphi(\xi)\, e^{iA\mathcal{F}(s)} h \tag{5.9}$$

with values in B_{mid} is continuous along the real straight path. It is sufficient to show that the mapping

$$t \to e^{iAt} h, \qquad t \in \mathbf{R}$$

is continuous. Due to the boundedness of the operator e^{iAt} this corresponds to the following statement: for any $\varepsilon > 0$ we may find a positive number δ so that for $|\tau| < \delta$ we have

$$\| e^{iA\tau} h - h \| < \varepsilon.$$

Choose such $h_0 \in E$ that the equality

$$\| (e^{iA\tau} - 1)(h - h_0) \| < \frac{\varepsilon}{2} \tag{5.10}$$

is valid for any τ of a fixed bounded neighborhood of zero. Set $q_0 = (A + i)^2 h_0$. Then

$$\| (e^{iA\tau} - 1)\, h_0 \| = \| (A + i)^{-2} (e^{iA\tau} - 1)\, q_0 \| \to 0$$

for $\tau \to 0$, because the function G with values in $W_2^{N+1}(\mathbf{R})$, defined by the formula

$$[G(\tau)](x) = \frac{1}{(x+i)^2}\,[e^{ix\tau} - 1]$$

is continuous. Choose $\delta > 0$ so that for $|\tau| < \delta$ inequality (5.10) is valid, and the inequality

$$\| (e^{iA\tau} - 1)\, h_0 \| < \frac{\varepsilon}{2}$$

is also valid.

Then for $|\tau| < \delta$

$$\|(e^{iA\tau}-1)h\| \leqslant \|(e^{iA\tau}-1)h_0\| + \|(e^{iA\tau}-1)(h-h_0)\| < \varepsilon.$$

Thus the continuity of the function (5.9) is proved. Consequently, the integral

$$\int_{-\infty}^{\infty} \varphi(\xi)\, e^{iA\mathscr{F}(\xi)} h\, d\xi$$

understood as the limit of integral sums in the norm of B_{mid}, exists.

Further, if $h_n \to h$ for $n \to \infty$, then $\|\varphi(\xi)\, e^{iA\mathscr{F}(\xi)}(h_n - h)\|_{B_{\mathrm{mid}}} \to$ $\to 0$ strives uniformly in ξ to zero as $n \to \infty$ since

$$\|e^{iA\mathscr{F}(\xi)}\| \leqslant (1 + |\mathscr{F}(\xi)|)^N.$$

Let h be an arbitrary element of B_{mid} and $\{h_m\}$ be a sequence of elements of E which converges to h. Then

$$\lim_{m\to\infty} I(A)\, h_m = I(A)\, h,$$

$$\lim_{m\to\infty} \int_{-\infty}^{\infty} \varphi(\xi)\, e^{iA\mathscr{F}(\xi)} h_m\, d\xi = \int_{-\infty}^{\infty} \varphi(\xi)\, e^{iA\mathscr{F}(\xi)} h\, d\xi.$$

But $\int_{-\infty}^{\infty} \varphi(\xi)\, e^{iA\mathscr{F}(\xi)} h_m\, d\xi = I(A)\, h_m$. For this reason

$$I(A)\, h = \int_{-\infty}^{\infty} \varphi(\xi)\, e^{iA\mathscr{F}(\xi)} h\, d\xi;$$

hence it follows that

$$K(A)\, h = \operatorname{rad} A \int_{-\infty}^{\infty} \varphi(\xi)\, e^{iA\mathscr{F}(\xi)} h\, d\xi.$$

Denote

$$a_j(\varphi, \mathscr{F}) = \sqrt{\pi}\, e^{i\frac{\pi}{4}\operatorname{sgn}\mathscr{F}''(\xi_0)} \left[\frac{i \operatorname{sgn}\mathscr{F}''(\xi_0)}{2}\right]^j \psi_j(0),$$

where the functions ψ_j are defined from the functions φ and $\mathscr{F}$ described above. The numbers $a_j(\varphi, \mathscr{F})$ depend only on values of the functions φ, $\mathscr{F}$ and their derivatives at the point ξ_0 up to some order dependent on j.

Thus we obtain the following statement.

Theorem 5.1. *Let* $\varphi \in C_0^\infty(\mathbf{R})$ *and let* $\mathscr{F}$ *be an infinitely differentiable real function on* $\mathbf{R}$; *it has a unique critical point* ξ_0 *on the support of the function* φ, *where* $\mathscr{F}''(\xi_0) \neq 0$. *Next let* rad x *be an infinitely differentiable function which coincides outside a bounded neighborhood*

of zero with the function

$$x^{\frac{1}{2}} \overset{\text{def}}{=} \begin{cases} \sqrt{x} & \text{for } x \geqslant 0, \\ \sqrt{|x|}\, e^{i\pi/2 \operatorname{sgn} \mathscr{F}''(\xi_0)} & \text{for } x < 0, \end{cases}$$

and, finally, let $\rho_j(x)$ *be infinitely differentiable functions which coincide outside a bounded neighborhood of zero with functions* x^{-j}.

Then for any integer $n \geqslant 0$ *the following expansion is valid*:

$$\operatorname{rad} A \int_{-\infty}^{\infty} \varphi(\xi)\, e^{iA\mathscr{F}(\xi)} h\, d\xi =$$

$$= e^{iA\mathscr{F}(\xi)} \sum_{j=0}^{n} a_j(\varphi, \mathscr{F})\, \rho_j(A)\, h + q_{n+1}, \tag{5.11}$$

where q_{n+1} *belongs to the domain of the operator* A^{n+1}.

Note. Let the operator A be invertible. Then in (5.11) we may replace $\rho_j(A)\,h$ by $A^{-j}h$ on condition that $h \in D_{A^{-n}}$. To prove this statement it is sufficient to show that for any $h \in D_{A^{-j}}$

$$[\rho_j(A) - A^{-j}]\, h \in D_{A^{n+1}}.$$

Denote $[\rho_j(A) - A^{-j}]\, h = q$. Then $A^j q = [A^j \rho_j(A) - 1]\, h$. The function $x^j \rho_j(x)$ belongs to $\mathscr{B}_N(\mathbf{R})$ and is identical to unity in the neighborhood of infinity. Therefore for any integer m the function $x^m(x^j\rho_j(x) - 1)$ belongs to $\mathscr{B}_N(\mathbf{R})$ so that the range of the operator $A^j\rho_j(A) - 1$ is contained in D_{A^m}. This means that for any m we have $q \in D_{A^{j+m}}$; in particular, $q \in D_{A^{n+1}}$.

Sec. 6. The Spectrum of a Generator

Let A be a generator of degree N.

Definition. *The minimal open subset of a real axis with the following property*

$$(\varphi \in C_0^\infty(\mathbf{R}) \text{ and } \operatorname{supp} \varphi \subset \rho(A)) \Rightarrow (\varphi(\overline{A}) = 0)$$

is called the resolvent set $\rho(A)$ *of the operator* A. *The set* $\sigma(A) = \mathbf{R} \diagdown \rho(A)$ *is called the spectrum of the operator* A.

Problem. Prove that $\sigma(A) \neq \varnothing$.

Theorem 6.1. *Let* $\lambda \in \rho(A)$. *Then an operator* $(\overline{A} - \lambda)^{-1}$ *exists which is defined everywhere on* B_{mid} *and satisfies the estimate*

$$\|(\overline{A} - \lambda)^{-1}\| \leqslant f(d),$$

where d is the distance between the point λ and the spectrum of the operator A and f is some non-increasing (numerical) function.

Proof. Let φ be an infinitely differentiable function and $\varphi(x) = (x-\lambda)^{-1}$ for $|x-\lambda| \geqslant \varepsilon$, where $\varepsilon < d$. Show that

$$\varphi(\overline{A}) = (\overline{A}-\lambda)^{-1}, \quad \|(\overline{A}-\lambda)^{-1}\| \leqslant f(d). \tag{6.1}$$

From the equality

$$\varphi(x)(x-\lambda) = 1 - \chi(\lambda),$$

where $\operatorname{supp}\chi \subset [\lambda-\varepsilon, \lambda+\varepsilon] \subset \rho(A)$, it follows that

$$(\overline{A}-\lambda)\,\varphi(\overline{A}) = 1,$$

$$\varphi(\overline{A})(A-\lambda)h = h, \quad \forall h \in D_A.$$

The latter two equalities show that

$$\varphi(\overline{A}) = (\overline{A}-\lambda)^{-1}.$$

Estimate the norm of the operator $(\overline{A}-\lambda)^{-1} = \varphi(\overline{A})$.

$$\|(\overline{A}-\lambda)^{-1}\| \leqslant \frac{1}{\sqrt{2\pi}}\|\varphi\|_{\mathscr{B}_N(\mathbf{R})} \leqslant \frac{1}{\sqrt{2\pi}}\|\varphi\|_{W_2^{N+1}(\mathbf{R})}.$$

The norm of the function φ in $W_2^{N+1}(\mathbf{R})$ depends on ε and on the method of continuing the function $(x-\lambda)^{-1}$ into the interval $|x-\lambda| < \varepsilon$. Let Φ_ε be the set of all possible φ, where ε is fixed. Then

$$\|(\overline{A}-\lambda)^{-1}\| \leqslant \frac{1}{\sqrt{2\pi}} \inf_{\substack{0\leqslant\varepsilon<d \\ \varphi\in\Phi_\varepsilon}} \|\varphi\|_{W_2^{N+1}(\mathbf{R})} = f(d).$$

The theorem is proved.

Theorem 6.2. *Let $\lambda \in \mathbf{R}$. If $(\overline{A}-\lambda)^{-1}$ is a homomorphism, then $\lambda \in \rho(A)$.*

Proof. Let $0 < \delta < \dfrac{1}{\|(\overline{A}-\lambda)^{-1}\|}$. Show that $\varphi(\overline{A}) = 0$ for any function $\varphi \in C_0^\infty(\mathbf{R})$ which satisfies the condition $\operatorname{supp}\varphi \subset (\lambda-\delta, \lambda+\delta)$. Let φ be the described function. Then

$$\|\varphi(\overline{A})\| = \|(\overline{A}-\lambda)^{-k}(\overline{A}-\lambda)^k\varphi(\overline{A})\| \leqslant$$
$$\leqslant \frac{1}{\sqrt{2\pi}}\|(\overline{A}-\lambda)^{-1}\|^k \|\psi_k\|_{W_2^{N+1}(\mathbf{R})},$$

where $\psi_k(x) = (x-\lambda)^k\varphi(x)$.

It is easy to obtain the following estimate

$$\|\psi_k\|_{W_2^{N+1}(\mathbf{R})} = O(\delta^k).$$

Consequently,

$$\| \varphi (\bar{A}) \| = O [(\delta \| (\bar{A} - \lambda)^{-1} \|)^k] \to 0 \quad \text{for} \quad k \to \infty.$$

This means that $\| \varphi (\bar{A}) \| = 0$ and the theorem is proved.

From Theorems 6.1 and 6.2 it follows that the spectrum of a generator coincides with the spectrum of its closure in B_{mid} (in the common sense), i.e., $(\lambda \in \rho (A)) \Leftrightarrow [(\bar{A} - \lambda)^{-1}$ is a homomorphism].

Theorem 6.3. *For any function* $f \in \mathscr{B}_N (\mathbf{R})$ *with the support in* $\rho (A)$ *the the following equality is valid*:

$$f (\bar{A}) = 0.$$

Proof. First of all, suppose that the support of the function f is compact. Let $\{f_n\}$ be a sequence of infinitely differentiable functions which converges to f in $\mathscr{B}_N (\mathbf{R})$, g is a function belonging to $C_0^\infty (\mathbf{R})$ with the support in $\rho (A)$, which is equal to 1 on supp f. Then $gf_n \in C_0^\infty (\mathbf{R})$, $\operatorname{supp} (gf_n) \subset \rho (A)$ and $gf_n \to f$ in $\mathscr{B}_N (\mathbf{R})$. Since $g (\bar{A}) f_n (\bar{A}) = 0$, it follows that $f (\bar{A}) = 0$.

Reject the assumption that the set supp f is compact. Let $\{\varphi_n\}$ be a sequence of functions belonging to $C_0^\infty (\mathbf{R})$ such that for any $h^* \in B^*$ and for any $h \in B$ the following relationship is valid:

$$\lim_{n \to \infty} h^* (\varphi_n (\bar{A}) h) = h^* (h).$$

Set $f_n (x) = f (x) \varphi_n (x)$. Then supp $f_n \subset \rho (A)$, so that $f_n (\bar{A}) = 0$. For any $h^* \in B^*$ and any $h \in B$ we have

$$0 = \lim_{n \to \infty} h^* (f_n (\bar{A}) h) = \lim_{n \to \infty} h^* (\varphi_n (\bar{A}) f (\bar{A}) h) = h^* (f (\bar{A}) h).$$

Hence it follows that $f (\bar{A}) = 0$.

It is clear that Theorem 6.3 remains valid if $\mathscr{B}_N (\mathbf{R})$ is replaced by $\mathscr{B}_N (\sigma (A))$, whereby the estimate from above of the norm of the operator $\varphi (\bar{A})$ will be improved, and the kernel of the homomorphism M will shrink. However, this mapping is not a monomorphism (i.e., not one-to-one) either, even when $\mathscr{B}$ is finite-dimensional and N is the exact degree of the operator A. The fact is that different Jordanian blocks of a matrix A may generate groups of homomorphisms with an arbitrary order of growth.

Example. Let λ_1, λ_2 be different real numbers and let

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

Then

$$e^{-iAt} = \begin{pmatrix} e^{-i\lambda_1 t} & 0 & 0 \\ 0 & e^{-i\lambda_2 t} & -ite^{-i\lambda_2 t} \\ 0 & 0 & e^{-i\lambda_2 t} \end{pmatrix},$$

so that A is a generator of first degree but not a generator of zero degree. For any function $\varphi \in \mathscr{B}_1(\mathbf{R})$ we have

$$\varphi(A) = \begin{pmatrix} \varphi(\lambda_1) & 0 & 0 \\ 0 & \varphi(\lambda_2) & \varphi'(\lambda_2) \\ 0 & 0 & \varphi(\lambda_2) \end{pmatrix}.$$

The spectrum $\sigma = \sigma(A)$ of the operator A consists of two points: λ_1 and λ_2. Let $\varphi \in C_0^\infty(\mathbf{R})$, $\lambda_2 \notin \operatorname{supp} \varphi$, $\varphi(\lambda_1) = 0$, $\varphi'(\lambda_1) \neq 0$. Then $\varphi(A) = 0$, but $\| \varphi \|_{\mathscr{B}_1(\sigma)} > 0$ (here the norm of the class of equivalency, which the function φ belongs to, is denoted by $\| \varphi \|_{\mathscr{B}_1(\sigma)}$ as usual). Despite the fact that the homomorphism $M_\sigma: \mathscr{B}_N(\sigma(A)) \to \mathfrak{A}_N$ is not a monomorphism, we obtain a sufficiently good estimate for $\| \varphi(A) \|$ from above.

Definition. *An eigenelement of the operator $\overline{A}$ at the point λ is called such a vector $g \in D_{\overline{A}}$ that $(\overline{A} - \lambda) g = 0$. The associated element of degree k of the operator $\overline{A}$ at the point is called such a vector $g \in D_{\overline{A}^k}$ that $(\overline{A} - \lambda)^k g = 0$ (but $(\overline{A} - \lambda)^{k-1} g \neq 0$). The eigen and associated elements of the operator $\overline{A}$ will be called the* e.a. *elements of the operator $\overline{A}$.*

If the operator $\overline{A}$ has only isolated points of the spectrum and the number of linearly independent e.a. elements corresponding to each point of the spectrum is finite, then the operator $\overline{A}$ is said to possess a *discrete spectrum.*

Problem. Let h be an eigenvector of the operator $\overline{A}$ at the point λ. Prove that $e^{-i\overline{A}t} h = e^{-i\lambda t} h$, where $h \in D_{\overline{A}}$.

Lemma 6.1. *Let λ be an isolated point of the spectrum of the generator A of degree N. If g is an* e.a. *element of the operator $\overline{A}$ at point λ, then $(\overline{A} - \lambda)^{N+1} g = 0$.*

Proof. Let g be an e.a. element of the operator $\overline{A}$ at the point λ. Then for some k

$$(\overline{A} - \lambda)^k g = 0. \tag{6.2}$$

Show that if $k > N + 1$, then for (6.2) it follows that $(\overline{A} - \gamma)^{k-1} g = 0$. Suppose that $k > N + 1$. Let $\{\chi_n\}$ be such a sequence of infinitely differentiable functions that $\chi_n(\lambda) = 1$, and that $\operatorname{supp} \chi_n$

is contained in the interval $\left(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}\right)$. Therefore, the support of a function χ_n contains only one point of the spectrum of the operator $\bar{A}$ for sufficiently large n. Besides, let

$$\lim_{n\to\infty} \| \psi_n \|_{\mathscr{B}_N(\mathbf{R})} = 0,$$

where $\psi_n(x) = (x - \lambda)^{k-1} \chi_n(x)$. It is easily seen that such a sequence exists. We have

$$\| \chi_n(A) (\bar{A} - \lambda)^{k-1} g \| \to 0 \quad \text{for } n \to \infty. \tag{6.3}$$

On the other hand, the vector

$$h = (\bar{A} - \lambda)^{k-1} g$$

is an eigenelement of the operator $\bar{A}$ since

$$(\bar{A} - \lambda) h = (\bar{A} - \lambda)^k g = 0.$$

This means that

$$\chi_n(\bar{A}) h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\chi}_n(t) e^{-i\bar{A}t} h \, dt =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\chi}_n(t) e^{-i\lambda t} h \, dt = \chi_n(\lambda) h = h,$$

i.e.,

$$\chi_n(\bar{A}) (\bar{A} - \lambda)^{k-1} g = (\bar{A} - \lambda)^{k-1} g.$$

From (6.3) it follows that $(\bar{A} - \lambda)^{k-1} g = 0$. By induction we obtain that $(\bar{A} - \lambda)^{N+1} g = 0$.

Theorem 6.4. *Let the spectrum of the generator A consist of isolated points λ_i, $i = 1, 2, \ldots$. Then the set of all* e.a. *elements of the operator $\bar{A}$ is complete, i.e., the linear span of the set is dense in B_{mid}.*

Proof. Let χ^i be an arbitrary function of $C_0^\infty(\mathbf{R})$ with a support containing only one point λ_i of the spectrum of the operator $\bar{A}$ and let χ^i be a function such that $\chi^i(x) = 1$ in some neighborhood of a point λ_i. Then $(\bar{A} - \lambda_i)^{N+1} \chi^i(\bar{A}) = 0$. Indeed, let $\{\chi_n^i\}$ at a fixed i be the sequence considered in Lemma 6.1 (for $k = N + 2$, $\lambda = \lambda_i$), where $\chi_n^i(x) = 1$ in some neighborhood (dependent on n) of the point λ_i. With sufficiently large n we have

$$\chi_n^i(\bar{A}) = \chi^i(\bar{A}),$$

and

$$\| (\bar{A}-\lambda_i)^{N+1} \chi_n^i (\bar{A}) \| \to 0 \quad \text{for } n \to \infty,$$

Q.E.D.

In Sec. 3 it was proved that the linear span of the set of all vectors of the form $\varphi(\bar{A})h$, $\varphi \in C_0^\infty(\mathbf{R})$, $h \in B_{\text{mid}}$ is dense in B_{mid}. This means that for the proof of the theorem it suffices to verify that for any $h \in B_{\text{mid}}$, $\varphi \in C_0^\infty(\mathbf{R})$ the vector $\varphi(\bar{A})h$ can be represented in the form of a finite sum $\sum\limits_i g_i$, where g_i satisfies the equality $(\bar{A} - \lambda_i)^{N+1} g_i = 0$. Let $\{\Omega_i\}$ be the open covering of the support of the function φ, where $\lambda_i \notin \Omega_i$ for $i \neq j$, $\lambda_i \in \Omega_i$. And let $\{\chi^i\}$ be a C^∞-partition of unity which is subordinate to the covering* $\{\Omega_i\}$.

Then $\varphi(\bar{A})h = \sum \varphi(\bar{A})\chi^i(\bar{A})h$. Set $g_i = \varphi(\bar{A})\chi^i(\bar{A})h$. We have

$$(\bar{A}-\lambda_i)^{N+1} g_i = \varphi(\bar{A})(\bar{A}-\lambda_i)^{N+1}\chi^i(\bar{A})h = 0.$$

The theorem is proved.

From the proof of Theorem 6.4 it also follows that for every isolated point λ_i of the spectrum of the generator A there exists a nonzero eigenvector of the operator $\bar{A}$ at this point (i.e., λ_i is an eigenvalue of the operator $\bar{A}$). In fact it suffices to verify that there exists a nonzero e.a. element of the operator $\bar{A}$ at the point λ_i. Let $\varphi \in C_0^\infty(\mathbf{R})$ be such a function with a support containing exactly one point λ_i of the spectrum of the operator $\bar{A}$ that $\varphi(\bar{A}) \neq 0$ (such a function exists in accordance with the definition of a generator). Then there exists $h \in B_{\text{mid}}$, such that $g \overset{\text{def}}{=\!=} \varphi(\bar{A})h \neq 0$. If χ^i is the function considered in the proof of Theorem 6.4 then $\chi^i(\bar{A}) \times \varphi(\bar{A}) = \varphi(\bar{A})$ so that $g = \chi^i(\bar{A})\varphi(\bar{A})h$. Hence it follows that $(\bar{A} - \lambda_i)^{N+1} g = 0$. If λ is an eigenvalue of the operator $\bar{A}$ then $\| f(\bar{A}) \| \geqslant | f(\lambda) |$ for any $f \in \mathscr{B}_N(\mathbf{R})$. In fact if g is a nonzero eigenelement of the operator $\bar{A}$ at the point λ, then $f(\bar{A})g = f(\lambda)g$. Thus, in the case when the spectrum of the operator consists of isolated points we obtain the following estimate from below for $\| f(\bar{A}) \|$:

$$\| f(\bar{A}) \| \geqslant \sup_{\lambda \in \sigma(A)} | f(\lambda) |. \tag{6.4}$$

* This means that $\chi^i \in C^\infty$, $\chi^i = 0$ outside Ω_i, $0 \leqslant \chi^i \leqslant 1$ and $\sum\limits_i \chi^i \equiv 1$ on supp φ. In treatises on mathematical analysis it is proved that such a set of functions exists.

Sec. 7. Regular Operators

In this section we extend the results of the preceding sections to the case of n-parameter groups.

Let E be a vector space and let $\{U(t)\}$, $t \in \mathbf{R}^n$ be an n-parameter group of homomorphisms $E \to E$:

$$U(t+\tau) = U(t)\, U(\tau).$$

Next, let E have two norms $\|\ \|_1$ and $\|\ \|_2$, moreover, $\|\ \|_1 \geqslant c \|\ \|_2$, $c = \text{const}$, and let

$$\frac{\partial U(t)\, h}{\partial t_j} = -iA_j U(t)\, h,$$

where the derivative is understood as $\|\ \|_1$ and where A_j: $E \to E$ are linear operators. We shall call $A = (A_1, \ldots, A_n)$ *a generating set of operators* of degree $N = N_1, \ldots, N_n$ (with respect to the group $\{U(t)\}$ and norms $\|\ \|_1$, $\|\ \|_2$), if

$$\| U(t)\, h \|_2 \leqslant c_1 \prod_i^n (1 + |t_i|)^{N_i} \| h \|_1.$$

Theorem 7.1. *If A is the generating set of degree N with respect to the group $\{U(t)\}$ and norms $\|\ \|_1$, $\|\ \|_2$, then in E there exists such a norm $\|\ \|_{\text{mid}}$ that A is the generating set of degrees N with respect to the same group and norms $\|\cdot\|_{\text{mid}}$, $\|\cdot\|_{\text{mid}}$.*

Proof. Define $\|\cdot\|_{\text{mid}}$ by the formula

$$\| h \|_{\text{mid}} = \sup_{t \in \mathbf{R}^n} \frac{\| U(t)\, h \|_2}{\prod_i (1 + |t_i|)^{N_i}}. \tag{7.1}$$

Obviously the axioms of the norm have been carried out and $\| h \|_{\text{mid}} \geqslant \| h \|_2$. From (3.3) it follows that

$$\| h \|_{\text{mid}} \leqslant \sup_{t \in \mathbf{R}^n} \frac{c_1 (1 + |t|)^N \| h \|_1}{(1 + |t|)^N} = c_1 \| h \|_1.$$

For this reason the derivative of any function with values in E with the norm of the form $\|\ \|_1$ is simultaneously the derivative with the norm of the form $\|\ \|_{\text{mid}}$. Further,

$$\frac{\| U(t)\, h \|_{\text{mid}}}{(t + |t|)^N} = \sup_{t \in \mathbf{R}^n} \frac{\| U(t+\tau)\, h \|_2}{(1 + |\tau|)^N (1 + |t|)^N} \leqslant$$

$$\leqslant \sup_{t,\, \tau \in \mathbf{R}^n} \left\{ \frac{\| U(t+\tau)\, h \|_2}{(1 + |t+\tau|)^N} \cdot \frac{(1 + |t+\tau|)^N}{(1 + |t|)^N (1 + |\tau|)^N} \right\}.$$

Since the latter factor does not exceed 1 it follows that

$$\| U(t)h \|_{\mathrm{mid}} \leqslant (1+|t|)^N \| h \|_{\mathrm{mid}},$$

Q.E.D.

Let B_{mid} be the completion of E in the norm $\| \cdot \|_{\mathrm{mid}}$. The extension of the operator $U(t)$ to the homomorphism $B_{\mathrm{mid}} \to B_{\mathrm{mid}}$ will be denoted by $e^{-i\bar{A}t}$, where $At = A_1 t_1 + \ldots + A_n t_n$. By $\bar{A} = (\bar{A}_1, \ldots, \bar{A}_n)$ we shall denote the set of closures of the operators A_i in B_{mid}. Note that the operators $\bar{A}_i$ exist (see Theorem 3.1).

Definition. *Let* $\varphi \in \mathscr{B}_N(\mathbf{R}^n)$ *and let* A *be a generating set of degree* N. *Define the homomorphism* $\varphi(\bar{A}) : B_{\mathrm{mid}} \to B_{\mathrm{mid}}$ *by means of the formula*

$$\varphi(\bar{A})h = (2\pi)^{-n/2} \int_{\mathbf{R}^n} \tilde{\varphi}(t) e^{-i\bar{A}t} h \, dt,$$

where $\tilde{\varphi} = F^{-1}\varphi$. *The function* φ *will be called the symbol of the operator* $\varphi(\bar{A})$.

Obviously the following estimate is valid:

$$\| \varphi(\bar{A}) \| \leqslant c \| \varphi \|_{\mathscr{B}_N(\mathbf{R}^n)}, \quad c = \mathrm{const},$$

where $\mathscr{B}_N(\mathbf{R}^n) = \mathscr{B}_{N_1, \ldots, N_n}(\mathbf{R}^n)$.

Thus the mapping $M : \varphi \to \varphi(A)$ is a homomorphism of the Banach space $\mathscr{B}_N(\mathbf{R}^n)$ into the Banach space $\mathrm{Op}(B_{\mathrm{mid}})$ of the homomorphism $B_{\mathrm{mid}} \to B_{\mathrm{mid}}$.

Theorem 7.2. *The homomorphism* M *is a homomorphism of algebras.*

The proof is analogous to the proof of Theorem 3.1.

The following theorem is well known.

Theorem 7.3. *Let* (A_1, A_2) *be a generating set in a separable Hilbert space* H *which consists of self-adjoint operators. Then*

$$\| f(A_1, A_2) \| \leqslant \| f \|_{C(\mathbf{R}^2)}.$$

Definition. *The minimal open set in* $\mathbf{R}^n$ *with such a property that*

$$(\varphi \in C_0^\infty(\mathbf{R}^n) \text{ and } \operatorname{supp} \varphi \subset \rho(A)) \Rightarrow (\varphi(\bar{A}) = 0)$$

will be called a resolvent set $\rho(A)$ *of the generating set* A. *The set* $\sigma(A) = \mathbf{R}^n \setminus \rho(A)$ *will be called a spectrum of the generating set.*

Definition. *The operator of the form* $T = A_1 + iA_2$, *where* (A_1, A_2) *is a generating set of degree* $\vec{N} = \{N, 0\}$, *and*

$$e^{-i(\bar{A}_1 t_1 + \bar{A}_2 t_2)} = e^{-i\bar{A}_1 t_1} e^{-i\bar{A}_2 t_2} = e^{-i\bar{A}_1 t_2} e^{-i\bar{A}_1 t_2}$$

will be called the regular operator of degree N.

Lemma 7.1. *There exists a closure $\overline{T}$ of a regular operator T in B_{mid}.*

Proof. Let $f(x) = \frac{1}{|x|^2+1}$, $x \in \mathbf{R}^2$ and let $T = A_1 + iA_2$, where $A = (A_1, A_2)$ is the generating set. The operator $f(\overline{A})$ has an inverse. In fact, let $\{\varphi_n\}$ be such a sequence of functions of $C_0^\infty(\mathbf{R}^2)$ that

$$\lim_{n\to\infty} h^*(\varphi_n(\overline{A})\,h) = h^*(h)$$

for any $h \in B_{\text{mid}}$, $h^* = B^*_{\text{mid}}$. If $f(\overline{A})\,h = 0$, then

$$\psi_n(\overline{A})\,f(\overline{A})\,h = 0,$$

where $\psi_n(x) = \frac{\varphi_n(x)}{f(x)} = \varphi_n(x)\,(|x|^2+1)$.

But $\psi_n(x)\,f(x) = \varphi_n(x)$, for this reason $\psi_n(\overline{A})\,f(\overline{A}) = \varphi_n(\overline{A})$. Therefore for any $h^* \in B^*_{\text{mid}}$

$$h^*(\varphi_n(\overline{A})\,h) = 0 = h^*(h),$$

hence it follows that $h = 0$.

Consider now the operator $f(\overline{A})\,T$. This operator has a closed extension $g(\overline{A})$, where $g(x) = \frac{x_1+ix_2}{|x|^2+1}$. Thus

$$(x_n \to 0 \text{ and } f(\overline{A})\,Tx_n \to z) \Rightarrow (z = 0).$$

Let $x_n \to 0$ and $Tx_n \to y$. Then $f(\overline{A})\,Tx_n \to f(\overline{A})\,y = 0$. Since the operator $f(\overline{A})$ is invertible it follows that $y = 0$, and the lemma is proved.

Theorem 7.4. *The spectrum of a closure $\overline{T}$ of a regular operator $T = A_1 + iA_2$ in B_{mid} coincides with the spectrum of a generating set.*

The proof is analogous to the proof of Theorems 6.1 and 6.2.

Theorem 7.5. *Let T be a regular operator and let the spectrum of its closure $\overline{T}$ in B_{mid} consist of isolated points. Then the system of* e.a. *elements of the operator T is complete in B_{mid}.*

The proof is analogous to the proof of Theorem 6.4.

Lemma 7.2. *Let T be an operator in a normed space B, and $\{B_\alpha\}$ be a family of vector subspaces of B such that*

(a) *B_α are finite-dimensional,* $\dim B_\alpha \leqslant N + 1$, *where any system $\{g_\alpha\}$ is such that when $g_\alpha \in B_\alpha$, $g \neq 0$, it is linearly independent;*

(b) *B_α are invariant with respect to T.*

Then shrinkage T_0 of the operator T on $E = \bigcup B_\alpha$ is a regular operator of degree N (with respect to some pair of norms).

Proof. Let T_α be the shrinkage of the operator T on B_α : $T_\alpha = TP_\alpha$ where $P_\alpha : E \to B_\alpha$ is the projector from E onto B_α. Without restriction of generality one may say that the normal Jordanian form of the matrix T_α consists of one Jordanian block (otherwise one may represent B_α as the direct sum of invariant subspaces of the operator T_α). Thus, let T_α be reduced to the form:

$$\begin{pmatrix} \lambda_\alpha & 1 & 0 & \dots & 0 \\ 0 & \lambda_\alpha & 1 & 0 \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \lambda_\alpha \end{pmatrix}.$$

Consider $(N+1) \times (N+1)$ matrices A'_α and A''_α which have the following forms respectively in the same representation:

$$\begin{pmatrix} \operatorname{Re}\lambda_\alpha & 1 & 0 & \dots & 0 \\ 0 & \operatorname{Re}\lambda_\alpha & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \operatorname{Re}\lambda_\alpha & \dots & 1 \\ 0 & \dots & \dots & \dots & \operatorname{Re}\lambda_\alpha \end{pmatrix}; \quad \begin{pmatrix} \operatorname{Im}\lambda_\alpha & \dots & 0 \\ 0 & \operatorname{Im}\lambda_\alpha \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \operatorname{Im}\lambda_\alpha \end{pmatrix}.$$

It is obvious that A'_α is a generator of degree N and A''_α is a scalar operator so that A''_α commutes with A'_α, and A''_α is a generator of zero degree. The generating set (A'_α, A''_α) generates the group

$$e^{-iA_\alpha t} = e^{-iA'_\alpha t' - iA''_\alpha t''}.$$

In fact, let μ be a real number and A be a Jordanian block of the size $N+1$:

$$A = \begin{pmatrix} \mu & 1 & 0 & \dots & 0 \\ 0 & \mu & 1 & 0 \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \mu & 1 \\ 0 & \dots & \dots & \dots & \mu \end{pmatrix}.$$

Then the vector-function $u(t) = e^{-iAt}h$, $h \in \mathbf{C}^{N+1}$ represents a solution of the Cauchy problem

$$\frac{du}{dt} = -iAu, \qquad u(0) = h,$$

or, in the extended form,

$$\frac{du_1}{dt} = -i\mu u_1 - iu_2,$$

$$\dots\dots\dots\dots$$

$$\frac{du_N}{dt} = -i\mu u_N - iu_{N+1},$$

$$\frac{du_{N+1}}{dt} = -i\mu u_{N+1},$$

$$u_j(0) = h_j, \quad j = 1, \dots, N+1,$$

where u_j, h_j are the components of vectors u and h. By solving the equations successively from the lower one upwards we obtain

$$u_j(t) = \sum_{k=0}^{N+1-j} e^{-\mu t} P_{j,k}(t)\, h_{j+k},$$

where $P_{j,k}(t)$ are polynomials of degree k (it is not difficult to calculate them precisely). Hence, it follows that

$$\| e^{-iA_\alpha t} h \|_B \leqslant c_\alpha (1 + |t|)^N \| h \|_B,$$

where c_α is a constant dependent on α. Besides, $T_\alpha = A'_\alpha + iA''_\alpha$. This decomposition corresponds to the decomposition of the operator T_0:

$$T_0 = A' + iA'',$$

where $A' = \sum A'_\alpha P_\alpha$, $A'' = \sum A''_\alpha P_\alpha$.

The pair (A', A'') induces the group

$$U(t', t'')\, h = \sum e^{-i(A'_\alpha t' + A''_\alpha t'')} P_\alpha h.$$

Introduce in E the norm $\| \ \|_1$:

$$\| h \|_1 \overset{\text{def}}{=} \sum c_\alpha \| P_\alpha h \|_B.$$

It is clear that $\| \ \|_1 \geqslant \| \ \|_B$.

The following estimate is valid:

$$\| U(t', t'')\, h \|_B \leqslant (1 + |t'|)^N \| h \|_1.$$

Thus (A', A'') is a generating set of degree $(N, 0)$ with respect to norms $\| \ \|_1$, $\| \ \|_B$, and the theorem is proved.

From this theorem the theorem given below follows directly.

Theorem 7.6. *Let T be a closed operator in the Banach space B with a discrete spectrum having a complete system of* e.a. *elements, the operator T being without any associated elements of the order exceeding N. Then the restriction of the operator T to a linear manifold dense in B is a regular operator.*

Sec. 8. The Generalized Eigenfunctions and Associated Functions

We shall start with a physical analogy. In quantum mechanics the state of a system of particles with n degrees of freedom can be described by a function of the space $L_2(\mathbf{R}^n)$ while operators on $L_2(\mathbf{R}^n)$ correspond to physical values. The logic of quantum mechanics provides the possibility of the decomposition of an arbitrary element of $L_2(\mathbf{R}^n)$ in "eigenfunctions" of an operator corresponding to any physical value. (The inverted commas are used here because the so-called eigenfunctions of physics do not necessarily belong to $L_2(\mathbf{R}^n)$.)

For example, let $n = 1$ and let A be a multiplication operator by the independent variable of $L_2(\mathbf{R}^n)$. This operator corresponds to one of the most important physical values, namely, the coordinate of a particle. The equations

$$(A - \lambda)\psi = 0$$

for eigenfunctions of the operator A have no ordinary solutions. In physics one assumes, however, that δ_λ is *a generalized eigenfunction of a multiplication operator* by an independent variable, and so one writes

$$x\delta(x - \lambda) = \lambda\delta(x - \lambda). \tag{8.1}$$

The equality (8.1) may be interpreted in the following way. Consider the closure $\overline{A - \lambda}$ of the operator $A - \lambda$ as an operator which acts from the space $W_2^{-s}(\mathbf{R})$ to the space $L_2(\mathbf{R})$, where s is sufficiently large. Then

$$\overline{(A-\lambda)}\,\delta_\lambda = 0.$$

Note that the equality $\bar{A}\delta_\lambda = \lambda\delta_\lambda$ is not valid, since $\delta_\lambda \notin L_2(\mathbf{R})$. The statement that $\overline{A - \lambda} \neq \bar{A} - \lambda$ is justified because the multiplication operator by a scalar is not bounded as an operator acting from $W_2^{-s}(\mathbf{R})$ to $L_2(\mathbf{R})$.

Next (in physics) the following expansion is used:

$$\psi(x) = \int \psi(\lambda)\,\delta(x-\lambda)\,d\lambda, \tag{8.2}$$

or, in a more general form, the following expansion is written:

$$\psi(x) = \int c(\lambda)\,\psi_\lambda(x)\,d\lambda, \tag{8.3}$$

where $\{\psi_\lambda(x)\}$ is the family of generalized eigenfunctions of the operator of some physical value. Since $\psi_\lambda(x)$ is featured in (8.3) as a function of the parameter λ, it is quite natural to assume that the operator $(\overline{A - \lambda})$ acts on the space of functions of the argument λ, i.e., to consider λ as a multiplication operator by an independent variable.

In the present section we shall obtain the generalization of (8.2) for the case of the expansion of an arbitrary vector of B_{mid} in generalized associated eigenfunctions of a regular operator.

Let us proceed to the definitions and notation. Consider $\mathrm{Hom}(\mathscr{B}_N(\mathbf{R}^n), \mathbf{C})$. By definition, $\mathrm{Hom}(\mathscr{B}_N(\mathbf{R}^n), \mathbf{C})$ is the space of bounded functionals in $\mathscr{B}_N(\mathbf{R}^n)$, i.e., the space $\mathscr{B}_N^*(\mathbf{R}^n)$ for whose elements the following notation was agreed upon:

$$f[g] = \int f(\lambda)\,g(\lambda)\,d\lambda, \quad f \in \mathrm{Hom}\,[\mathscr{B}_N(\mathbf{R}^n), \mathbf{C}], \quad g \in \mathscr{B}_N. \tag{8.4}$$

As it was agreed above, this integral which has meaning only for continuous $f(\lambda)$ is formally written down for any functional f, $f(\lambda)$ being called a generalized function.

Let $G(\lambda)$ be a continuous function of λ with values in B. Then the integral

$$\int_{\mathbf{R}^n} G(\lambda) f(\lambda)\, d\lambda, \quad f(\lambda) \in \mathscr{B}_N(\mathbf{R}^n)$$

determines the element of B so that the operator G

$$Gf \overset{\text{def}}{=\!=} \int_{\mathbf{R}^n} G(\lambda) f(\lambda)\, d\lambda$$

is an element of the space Hom $(\mathscr{B}_N(\mathbf{R}^n), B)$. For any element $G \in$ Hom $(\mathscr{B}_N(\mathbf{R}^n), B)$ we agree to write formally

$$Gf = \int G(\lambda) f(\lambda)\, d\lambda.$$

By analogy with (8.4) we shall call $G(\lambda)$ *a generalized function of* λ *with values in* B. In accordance with this definition, ordinary generalized functions are generalized functions with values in the complex plane.

Let T be a regular operator of degree N and B_{mid} be the corresponding intermediate Banach space, and let $\lambda = \lambda_1 + i\lambda_2 \in \mathbf{C}$.

Consider the operator

$$T^{(k)} = (T - \lambda)^k,$$

which acts from B_{mid} to B_{mid}. We shall consider it as an operator acting from the space of generalized functions of λ with values in B_{mid} to the space of continuous functions of λ with values in B_{mid} so that the values of the operator $(T - \lambda)^k$ should belong to B_{mid} for any fixed λ.

Thus, we shall consider $(T - \lambda)^k \overset{\text{def}}{=\!=} T^{(k)}$ as an operator acting from Hom $(\mathscr{B}_N(\mathbf{R}^2), B_{\text{mid}})$ to $C(\mathbf{R}^2, B_{\text{mid}})$, where $\lambda = \lambda_1 + i\lambda_2 = (\lambda_1, \lambda_2)$. We shall make the meaning of this statement more precise. Denote by means of $T_0^{(k)}$

$$\text{Hom}(\mathscr{B}_N(\mathbf{R}^2), B_{\text{mid}}) \to C(\mathbf{R}^2, B_{\text{mid}})$$

the operator defined by such continuous functions with values in B_{mid} that the mapping

$$\lambda \to (\overline{T - \lambda})^k f(\lambda)$$

is a continuous function in $\mathbf{R}^2$ with values in B_{mid}. The operator is acting according to the formula

$$T_0^k f(\lambda) = (\overline{T - \lambda})^k f(\lambda).$$

We shall extend the operator $T_0^{(k)}$ up to the operator $T^{(k)}$ as follows. Let $\{f_n\}$ be a sequence of functions belonging to $D_{T_0^{(k)}}$ such that there exists a distribution f with values in B_{mid} possessing the property to make for any function $\psi \in C_0^\infty(\mathbf{R}^2)$

$$\lim_{n\to\infty}\int_{\mathbf{R}^n} f_n(\lambda)\,\psi(\lambda)\,d\lambda = \int_{\mathbf{R}^2} f(\lambda)\,\psi(\lambda)\,d\lambda.$$

Let $\lim\limits_{n\to\infty} T_0^{(k)} f_n = g$. Then set $T^{(k)} f = g$.

Verify the correctness of the given definition. Let $f_n \in D_{T_0^{(k)}}$ and for any $\psi \in C_0^\infty(\mathbf{R}^2)$

$$\lim_{n\to\infty}\int_{\mathbf{R}^2} f_n(\lambda)\,\psi(\lambda)\,d\lambda = 0.$$

Then allow for the existence of $\lim\limits_{n\to\infty} T_0^{(k)} f_n = g$.

We must show that $g = 0$. For any $\psi \in C_0^\infty(\mathbf{R}^2)$:

$$\lim_{n\to\infty}\int_{\mathbf{R}^2} \overline{(T-\lambda)^k}\, f_n(\lambda)\,\psi(\lambda)\,d\lambda = \int_{\mathbf{R}^2} \quad (\lambda)\,\psi(\lambda)\,d\lambda.$$

Let P be a smooth sufficiently rapidly decreasing function, which is strictly positive, and let $T = A_1 + iA_2$, where $(A_1, A_2) = A$ is the generating set, which determines the regular operator T. Set

$$q(x) = P(x)(x_1 + ix_2 - \lambda)^k = \sum_{j=0}^{k} q_j(x)\,\lambda^k.$$

Then the operator $[P(A)]^{-1} q(A)$ is a closed extension of the operator $(T - \lambda)^k$, the operator $[P(A)]^{-1}$ being closed as an operator inverse to the homomorphism. For this reason

$$\lim_{n\to\infty}\int_{\mathbf{R}^2} \overline{(T-\lambda)^k}\, f_n(\lambda)\,\psi(\lambda)\,d\lambda =$$

$$= [P(A)]^{-1} \lim_{n\to\infty} \sum_{j=0}^{k} \int_{\mathbf{R}^2} q_j(A)\,\lambda^k f_n(\lambda)\,\psi(\lambda)\,d\lambda = 0$$

for any $\psi \in C_0^\infty(\mathbf{R}^2)$. Hence, it follows that $g = 0$.

Definition. *The kernel of the operator $T^{(k)}$ will be called the system of generalized* e.a. *functions of the order $\leqslant k$ of the operator T.*

Definition. *The system $\mathscr{E}$ of generalized* e.a. *functions of the order $\leqslant k$ of the operator T will be called complete if for any vector $h \in B_{\text{mid}}$ there exists a generalized function $G \in \mathscr{E}$ and a function $f \in \mathscr{B}_N(\mathbf{R}^2)$,*

such that

$$h=\int_{\mathbf{R}^2} f(\lambda)\, G(\lambda)\, d\lambda.$$

Theorem 8.1. *The system of the generalized* e.a. *functions of the order* $\leqslant N+2$ *of a regular operator* T *of degree* N *is complete.*

Lemma. *Let* $\varphi_n(x)$, $x\in\mathbf{R}^2$ *be the following* δ*-formed sequence*

$$\varphi_n(x)=n^2\varphi(nx),\quad \varphi\in C_0^\infty(\mathbf{R}),\quad \int_{\mathbf{R}^2}\varphi(x)\,dx=1.$$

Then

$$\lim_{n\to\infty}\|(x_1+Cix_2)^{N+2}\varphi_n(x)\|_{\mathscr{B}_{N,0}(\mathbf{R}^2)}=0.$$

Proof. Denote $\psi_n(x)=(x_1+Cix_2)^{N+2}\varphi_n(x)$. We have $\psi_n(x)=n^{-N}\psi_1(nx)$. Consequently, $\tilde{\psi}_n(p)=n^{-(N+2)}\tilde{\psi}_1\left(\frac{p}{n}\right)$, where $\tilde{\psi}_n=F^{-1}\psi_n$. Let f be an arbitrary continuous function in $\mathbf{R}^2$ which satisfies the condition $\sup\limits_{p\in\mathbf{R}^2}\frac{|f(p)|}{(1+|p_1|)^N}\leqslant 1$. Then

$$\left|\int_{\mathbf{R}^2}\tilde{\psi}_n(p)f(p)\,dp\right|=n^{-N-1}\left|\int_{\mathbf{R}^2}\tilde{\psi}_1(p)f(np)\,dp\right|\leqslant$$

$$\leqslant n^{-N-1}\|\psi_1\|_{\mathscr{B}_{N,0}(\mathbf{R}^2)}\sup_{p\in\mathbf{R}^2}\frac{|f(np)|}{(1+|p_1|)^N}\leqslant$$

$$\leqslant n^{-N-1}\|\psi_1\|_{\mathscr{B}_{N,0}(\mathbf{R}^2)}\sup_{t\in\mathbf{R}}\frac{(1+n|t|)^N}{(1+|t|)^N}=n^{-1}\|\psi_1\|_{\mathscr{B}_{N,0}(\mathbf{R}^2)}.$$

Hence, it follows that

$$\|\psi_n\|_{\mathscr{B}_{N,0}(\mathbf{R}^2)}\leqslant n^{-1}\|\psi_1\|_{\mathscr{B}_{N,0}(\mathbf{R}^2)}\to 0$$

for $n\to\infty$.

And the lemma is proved.

Proof of Theorem 8.1. Consider in the space $\mathscr{B}_{N,0}(\mathbf{R}^2)$ the operator $\check{T}$ with the domain $D=C_0^\infty(\mathbf{R}^2)$ acting according to the formula $\check{T}f(x)=(x_1+ix_2)f(x)$.

It is not difficult to see that $\check{T}$ is regular, its space B_{mid} coinciding with $\mathscr{B}_{N,0}(\mathbf{R}^2)$. Now show that the identity operator

$$I\in\mathrm{Op}(\mathscr{B}_{N,0}(\mathbf{R}^2))=\mathrm{Hom}(\mathscr{B}_{N,0}(\mathbf{R}^2),\ \mathscr{B}_{N,0}(\mathbf{R}^2))$$

belongs to the kernel of the operator $\check{T}^{(N+1)}$. Let φ be a monotonously non-increasing infinitely differentiable function defined in $[0,\infty)$

and vanishing with all derivatives in $[1, \infty)$ and let $\int\limits_0^1 \xi\varphi(\xi)\, d\xi = \frac{1}{2\pi}$.

Set

$$\varphi_n(x) = n^2\varphi(n\,|\,x\,|).$$

Every $\lambda \in \mathbf{R}^2$ will have a corresponding mapping

$$\Phi_n(\lambda) : x \to \varphi_n(x - \lambda).$$

Obviously, Φ_n is a continuous function with values in $\mathscr{B}_{N,0}(\mathbf{R}^2)$. Further, let $\{e_n\}$ be a sequence of functions in $C_0^\infty(\mathbf{R}^2)$ such that $e_n(x) = 1$ for $|\,x\,| < n$ and $0 \leqslant e_n(x) \leqslant 1$. Set

$$I_n(\lambda) = e_n(\lambda)\,\Phi_n(\lambda).$$

Let ψ be an arbitrary function in $C_0^\infty(\mathbf{R}^2)$. Consider the sequence of integrals

$$\int\limits_{\mathbf{R}^2} I_n(\lambda)\,\psi(\lambda)\,d\lambda = \int\limits_{\mathbf{R}^2} \Phi_n(\lambda)\,e_n(\lambda)\,\psi(\lambda)\,d\lambda.$$

For sufficiently large n

$$\int\limits_{\mathbf{R}^2} [I_n(\lambda) - I(\lambda)]\,\psi(\lambda)\,d\lambda = \int\limits_{\mathbf{R}^2} \Phi_n(\lambda)\,\psi(\lambda)\,d\lambda - \psi \overset{\text{def}}{=} \psi_n - \psi.$$

Show that $\|\psi_n - \psi\|_{W_2^{N+1}(\mathbf{R}^2)} \to 0$ for $n \to \infty$. This will mean in particular that $\psi_n \to \psi$ in $\mathscr{B}_{N,0}(\mathbf{R}^2)$. We have

$$\psi_n(x) = \int\limits_{\mathbf{R}^2} \varphi_n(x - \lambda)\,\psi(\lambda)\,d\lambda = \int\limits_{\mathbf{R}^2} \varphi_n(y)\,\psi(x - y)\,dy. \tag{8.5}$$

From (8.5) it is seen that $\psi_n(x)$ uniformly converges to $\psi(x)$ along with all its derivatives. Since the supports of functions ψ_n belong to some compact set independent of n it follows that $\psi_n \to \psi$ in any Sobolev space.

Now consider the sequence of functions

$$G_n = \check{T}_0^{(N+2)} I_n;\quad G_n(\lambda) = \overline{(\check{T} - \lambda)^{N+2}}\, e_n(\lambda)\,\Phi_n(\lambda)$$

or in the amplified form

$$G_n(\lambda)(x) = (x_1 + ix_2 - \lambda_1 - i\lambda_2)^{N+2}\, e_n(\lambda)\,\varphi_n(x - \lambda).$$

It is evident that

$$\|G_n(\lambda)\|_{\mathscr{B}_{N,0}(\mathbf{R}^2)} \leqslant \|G_n(0)\|_{\mathscr{B}_{N,0}(\mathbf{R}^2)} = \|h_n\|_{\mathscr{B}_{N,0}(\mathbf{R}^2)},$$

where $h_n(x) = (x_1 + ix_2)^{N+2}\,\varphi_n(x)$. It is easy to verify that $\|h_n\| \to$ $\to 0$ for $n \to \infty$. Hence, it follows that $\|G_n(\lambda)\|_{\mathscr{B}_{N,0}(\mathbf{R}^2)} \to 0$

uniformly in λ for $n \to \infty$ so that

$$\check{T}^{(N+2)} I = 0.$$

Thus, the system of generalized e.a. functions of the order $\leqslant N + 2$ of the operator $\check{T}$ is complete. Actually for any $f \in \mathscr{B}_{N,0}(\mathbf{R}^2)$

$$f = \int_{\mathbf{R}^2} f(\lambda) I(\lambda) \, d\lambda.$$

Let T now be an arbitrary regular operator $T = A_1 + iA_2$, where $A = (A_1, A_2)$ is a generating set and let h be a vector in B_{mid}. Consider a mapping

$$\Phi_n: \lambda \to \varphi_{n,\lambda}(\bar{A}) h, \text{ where } \varphi_{n,\lambda}(x) = \varphi_n(x - \lambda).$$

Φ_n is obviously a continuous function with values in B_{mid}. Set $K_n(\lambda) = e_n(\lambda) \Phi_n(\lambda)$.

Let ψ, as before, serve as an arbitrary function belonging to $C_0^\infty(\mathbf{R}^2)$. Consider a sequence of integrals

$$\int_{\mathbf{R}^2} K_n(\lambda) \psi(\lambda) \, d\lambda = \int_{\mathbf{R}^2} \Phi_n(\lambda) e_n(\lambda) \psi(\lambda) \, d\lambda = g_n.$$

Let K be the following distribution with values in B_{mid}

$$\int_{\mathbf{R}^2} K(\lambda) f(\lambda) \, d\lambda = f(\bar{A}) h. \tag{8.6}$$

Show that

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} \int_{\mathbf{R}^n} \Phi_n(\lambda) \psi(\lambda) \, d\lambda = \int_{\mathbf{R}^n} K(\lambda) \psi(\lambda) \, d\lambda = \psi(\bar{A}) h.$$

We have

$$\int_{\mathbf{R}^2} \Phi_n(\lambda) \psi(\lambda) \, d\lambda = \left[\int_{\mathbf{R}^2} \varphi_{n,\lambda}(\bar{A}) \psi(\lambda) \, d\lambda \right] h = \psi_n(\bar{A}) h,$$

where $\psi_n(x) = \int_{\mathbf{R}^2} \varphi_n(x - \lambda) \psi(\lambda) \, d\lambda$ so that

$$\psi_n \to \psi \text{ in } \mathscr{B}_{N,0}(\mathbf{R}^2) \text{ and } \psi_n(\bar{A}) h \to \psi(\bar{A}) h$$

in B_{mid}.

Now consider a sequence of distributions $g_n = T_0^{(N+2)} K_n$:

$$g_n(\lambda) = (T - \lambda)^{N+2} e_n(\lambda) \Phi_n(\lambda) = \overline{(T - \lambda)^{N+2}} e_n(\lambda) \varphi_{n,\lambda}(\bar{A}) h.$$

It is obvious that

$$\| g_n(\lambda) \|_{B_{\mathrm{mid}}} \leqslant \| \overline{(T-\lambda)^{N+1}\, \varphi_{n,\lambda}(\bar{A})} \| = \| \psi_{n,\lambda}(\bar{A})\, h \|,$$

where

$$\psi_{n,\lambda}(x) = (x_1 + ix_2 - \lambda_1 - i\lambda_2)^{N+2}\, \varphi_n(x-\lambda).$$

For this reason

$$\| g_n(\lambda) \|_{B_{\mathrm{mid}}} \leqslant \| \psi_{n,0} \|_{\mathscr{B}_{N,0}(\mathbf{R}^2)} \| h \|_{B_{\mathrm{mid}}} \to 0 \text{ for } n \to \infty,$$

which means that $T^{N+1} K = 0$, so K belongs to the family of generalized e.a. functions.

In accordance with (8.6) the vector h may be represented in the form

$$h = \int_{\mathbf{R}^2} K(\lambda) \cdot 1\, d\lambda$$

and the theorem is proved.

Sec. 9. Self-Adjoint Operators as Transformers in the Schmidt Space*

Let A and G be self-adjoint operators in the separable Hilbert space H, and let T be *a Schmidt operator*, i.e., $T \in B_2(H)$, where $B_2(H)$ is a set of operators which have a bounded Schmidt norm. One may compare operators A and G in the space $B_2(H)$ (let us denote them by $\vec{A}$ and $\overleftarrow{G}$) acting according to the rule

$$\vec{A}T \stackrel{\mathrm{def}}{=} AT, \quad T \in D_{\vec{A}} = \{S \in B_2(H):\ AS \in B_2(H)\},$$

$$\overleftarrow{G}T \stackrel{\mathrm{def}}{=} TG, \quad T \in D_{\overleftarrow{G}} = \{S \in B_2(H):\ \overline{SG} \in B_2(H)\}.$$

Theorem 9.1. *The operators $\vec{A}$ and $\overleftarrow{G}$ are self-adjoint in the Hilbert space $B_2(H)$ and commute in some dense set.*

Proof. First of all show that $D_{\vec{A}} \cap D_{\overleftarrow{G}}$ is dense in $B_2(H)$. Let $E_n(A)$ be an operator with the symbol $\theta(x+n)\,\theta(n-x)$ ($E'_\lambda(\Delta)$ for $\lambda = -n$, $\Delta = 2n$) and let $\{e_i^{(n)}\}$ be the complete orthonormal system in $E_n(A)H$, where $\{e_i^{(n+1)}\}$ contains $\{e_i^{(n)}\}$ as a subsystem. Let $E_n(G)$ and $\{g_i^{(n)}\}$ be similar projectors and orthonormal systems for the operator G. Then the set $\{e_i\}$ of all vectors of the type $e_i^{(n)}$ forms a complete orthonormal system in H, and the same is valid for vectors $g_i^{(n)}$. Indeed, any vector in H may be approximated in

* We shall cite some known results here which are necessary for the succeeding sections.

norm by linear combinations of vectors of the form $\varphi(A)h$, where $\varphi \in C_0^\infty(\mathbf{R})$ and a vector of the form $\varphi(A)h$ belongs to $E_n(A)H$ for sufficiently large n.

We have for any $T \in B_2(H)$:

$$\| TE_n(G) - T \|^2_{B_2(H)} = \| T \|^2_{B_2(H)} - \sum_i \| Tg_i^{(n)} \|^2 \to 0$$

for $n \to \infty$. Similarly for $n \to \infty$

$$\| E_n(A)T - T \|^2_{B_2(H)} \to 0,$$

because this is a sequence of remainders of the convergent series $\sum_{i,j} |(Te_j, e_i)|^2$. Thus the set K of operators of the form $E_n(A) \times \times TE_m(G)$, $T \in B_2(H)$ is dense in $B_2(H)$. But the set D is contained in $D_{\vec{A}} \cap D_{\overleftarrow{G}}$ because the operators $AE_n(A)$ and $E_m(G)G$ are bounded. This completes the proof of the density $D_{\vec{A}} \cap D_{\overleftarrow{G}}$ in $B_2(H)$.

Let $\{T_j\}_{j=1}^\infty$ be such a complete orthonormal system in $B_2(H)$ that $T_j \in D_{\vec{A}} \cap D_{\overleftarrow{G}}$, $j = 1, 2, \ldots$. It is obvious that $\vec{A}\overleftarrow{G}T_j = \overleftarrow{G}\vec{A}T_j$ so that $\vec{A}$ and $\overleftarrow{G}$ commute in a set which is dense in $B_2(H)$. This set is a set of linear combinations of the form $\sum_{j=1}^{n} a_jT_j$.

Prove that $\vec{A}$ and $\overleftarrow{G}$ are self-adjoint operators. Let $T \in D_{\vec{A}}$, i.e. $AT \in B_2(H)$. Then for $S \in D_{\vec{A}}$ we have $(\vec{A}S, T)_2 = (AS, T)_2 = (S, AT)_2 = (S, \vec{A}T)_2$, i.e., $T \in D_{\vec{A}^*}$ and $\vec{A}^*T = \vec{A}T$. Prove the inverse inclusion: $D_{\vec{A}^*} \subset D_{\vec{A}}$.

Let $T \in D_{\vec{A}^*}$, i.e., there exists such an operator $W \in B_2(H)$ that $(AS, T)_2 = (S, W)_2$ for any $S \in D_{\vec{A}}$. Choose in H such a complete orthonormal system $\{e_j\}$ that $e_j \in D_A$, $j = 1, 2, \ldots$. Set $T_{ij}u = (u, e_i)e_j$. Then $T_{ij}e_k = \delta_{ik}e_j$ so that

$$\| T_{ij} \|_2^2 = \sum_{k=1}^{\infty} \| \delta_{ik}e_j \|^2 = 1.$$

It is easy to see that operators T_{ij}, T_{kl} with non-coincident pairs of subscripts are orthogonal in $B_2(H)$. Thus $\{T_{ij}\}$ is an orthonormal system in $B_2(H)$. It is easy to verify that this system is complete in $B_2(H)$. We have $\| AT_{ij} \|_2 = \| Ae_j \| < \infty$ so that $T_{ij} \in D_{\vec{A}}$. Next we have the relationships

$$(AT_{ij}, T)_2 = (T_{ij}, W)_2, \; i, j = 1, 2, \ldots.$$

By definition of the scalar product in $B_2(H)$ the latter equalities may be rewritten as follows

$$(Ae_j, Te_i) = (e_j, We_i);$$

Hence, it follows that

$$ATe_i = We_i, \quad i = 1, 2, \ldots,$$

i.e., $AT = W \in B_2(H)$. This completes the proof of the self-adjointness of the operator $\vec{A}$.

For the proof of the self-adjointness of the operator $\overleftarrow{G}$ consider a complete orthonormal system $\{e'_j\}$ in H which consists of vectors belonging to $D_{\overleftarrow{G}}$, and let

$$T'_{ij}u = (u, e'_i)\, e_j, \quad \forall u \in H.$$

Determine the adjoint operator $(T'_{ij}G)^*$ of the product $T'_{ij}G$. Since the range of the operator T'_{ij} is contained in $D_{\overleftarrow{G}}$ we have

$$(T'_{ij}G)^* = GT'^*_{ij} = GT'_{ji}.$$

For this reason, if $T \in D_{\overleftarrow{G}^*}$ then there exists such an operator $W \in B_2(H)$ that

$$(T'_{ij}G, T)_2 = (T^*, (T'_{ij}G)^*)_2 = (T^*, GT'_{ij})_2 = (T'_{ij}, W)_2.$$

Hence

$$(e'_j, TGe'_i) = (e'_j, We'_i), \quad i, j = 1, 2, \ldots,$$

i.e., the operator TG has a closure which is equal to $W \in B_2(H)$. This means that $T \in D_{\overleftarrow{G}}$. Prove the inverse inclusion $D_{\overleftarrow{G}} \subset D_{\overleftarrow{G}^*}$ and the equality $\overleftarrow{G}^*S = \overleftarrow{G}S$ for $S \in D_{\overleftarrow{G}}$. This will complete the proof of the statement that $\overleftarrow{G}$ is a self-adjoint operator.

Let $T \in D_{\overleftarrow{G}}$, i.e. $\overline{TG} \in B_2(H)$, where $(TG)^* \in B_2(H)$. Determine the action of the operator $(TG)^*$ on the vector e'_i. We have, for any $u \in H$:

$$(\overline{TG}u, e'_i) = (u, (TG)^*\, e'_i).$$

For this reason for $u \in D_{\overleftarrow{G}}$ we get:

$$(Gu, T^*e'_i) = (u, (TG)^*\, e'_i).$$

Hence, it follows that $T^*e'_i \in D_{\overleftarrow{G}}$ and $(TG)^*\, e'_i = GT^*e'_i$. Taking this into account for any $S \in D_{\overleftarrow{G}}$ we obtain

$$(S, \overleftarrow{G}T)_2 = (S, \overline{TG})_2 = ((TG)^*, S^*)_2 =$$
$$= \sum_k ((TG)^*\, e'_k, S^*e'_k) = \sum_k (\overleftarrow{G}T^*e_k, S^*e_k);$$

$$(\overleftarrow{G}S,\ T)_2 = (\overline{SG},\ T)_2 = (T^*,\ (SG)^*)_2 =$$
$$= \sum_k (T^* e'_k,\ (SG)^* e'_k) = \sum_k (T^* e'_k,\ (\overleftarrow{G}S^*)\, e'_k).$$

Hence it follows that $T \in D_{\overleftarrow{G^*}}$ and $\overleftarrow{G^*}T = \overleftarrow{G}T$. The statement is proved.

The operators $\overrightarrow{A}$ and $\overleftarrow{G}$ form the groups $\{e^{-i\overrightarrow{A}t}\}$ and $\{e^{-i\overleftarrow{G}t}\}$ which act according to the formula

$$e^{-i\overrightarrow{A}t}T = e^{-iAt}T, \quad e^{-i\overleftarrow{G}t}T = Te^{-iGt}. \tag{9.1}$$

For example, let us prove the second equality. Consider the two functions

$$u_1(t) = [e^{-i\overleftarrow{G}t}T]^* h,$$
$$u_2(t) = (Te^{-i\overleftarrow{G}t})^* h, \quad T^*h \in D_{\overleftarrow{G}}.$$

We have $u_1(0) = u_2(0) = T^*h$. Next

$$\frac{du_1}{dt} = (-i\overleftarrow{G}e^{-i\overleftarrow{G}t}T)^* h = i\,[(e^{-i\overleftarrow{G}t}T)\,G]^* h = iG\,(e^{-i\overleftarrow{G}t}T)^* h = iGu_1(t);$$
$$\frac{du_2}{dt} = \frac{d}{dt}\, e^{iGt}T^*h = iGe^{iGt}T^*h = iGu_2(t),$$

so the functions u_1 and u_2, satisfying the same equality $\frac{du}{dt} = iGu$ and one and the same initial condition, are identical. Since the set of operators T satisfying the condition $T^*h \in D_{\overleftarrow{G}}$ for the set of vectors h, dense in H, is dense in $B_2(H)$, it follows that the formula

$$e^{-i\overleftarrow{G}t}T = Te^{-iGt}$$

is valid for any $T \in B_2(H)$.

From (9.1) it follows that the set K of operators of the form $E_n(A) \times \times TE_m(G)$, which is dense in $B_2(H)$ (where $E_n(A)$, $E_m(G)$ are the above-considered projectors), is invariant with respect to the groups $\{e^{-i\overrightarrow{A}t}\}$, $\{e^{-i\overleftarrow{G}t}\}$, where

$$e^{-i\overrightarrow{A}t}e^{-i\overleftarrow{G}\tau} = e^{-i\overleftarrow{G}\tau}e^{-i\overrightarrow{A}t}.$$

Consider *the two-parameter group* $\{e^{-i\overrightarrow{A}t - i\overleftarrow{G}\tau}\}$:

$$e^{-i\overrightarrow{A}t - i\overleftarrow{G}\tau} \overset{\text{def}}{=\!=} e^{-i\overrightarrow{A}t}e^{-i\overleftarrow{G}\tau}.$$

For any $T \in K$ we have

$$\frac{\partial}{\partial t}\, e^{-i\overrightarrow{A}t - i\overleftarrow{G}\tau}T = -i\overrightarrow{A}e^{-i\overrightarrow{A}t - i\overleftarrow{G}\tau}T,$$
$$\frac{\partial}{\partial \tau}\, e^{-i\overrightarrow{A}t - i\overleftarrow{G}\tau}T = -i\overleftarrow{G}e^{i\overrightarrow{A}t - i\overleftarrow{G}\tau}T.$$

Thus $(\overrightarrow{A}, \overleftarrow{G})$ is a generating set of operators.

Theorem 9.2. *Let* $f \in \mathscr{B}_0(\mathbf{R}^2)$, $\tilde{f} = F^{-1}f$. *Then*

$$\left\| \frac{1}{2\pi} \int\limits_{\mathbf{R}^2} \tilde{f}(t, \tau)\, e^{-iAt} T e^{-iG\tau}\, dt\, d\tau \right\|_2 \leqslant \| f \|_{C(\mathbf{R}^2)} \| T \|_2.$$

The proof follows from the formula

$$\frac{1}{2\pi} \int\limits_{\mathbf{R}^2} \tilde{f}(t, \tau)\, e^{-iAt} T e^{-iG\tau}\, dt\, d\tau =$$

$$= \frac{1}{2\pi} \int\limits_{\mathbf{R}^2} \tilde{f}(t, \tau)\, e^{-i\overrightarrow{A}t - i\overleftarrow{G}\tau} T\, dt\, d\tau = f(\overrightarrow{A}, \overleftarrow{G})\, T.$$

Problem. We have defined infinitely differentiable functions of commutative operators, whose growth rate does not exceed the degree of the argument. Show that they form an algebra with the μ-structure, where the algebra $\mathscr{A}$ is an algebra of commutative unbounded operators, and that the set M consists of generators and their real functions.

II. CALCULUS OF NONCOMMUTATIVE OPERATORS

In this chapter we shall develop a μ-structure for the case when the set M consists of two noncommutative vector generators A_1, and A_2*. Without additional assumptions it is impossible to extend this development to a case when M consists of a greater number of noncommutative unbounded generators. For this reason the main "working" formulas of the theory of μ-structures (the formula of commutation and the K-formula) should be specifically proved with additional assumptions even for the case of two noncommutative operators because these formulas contain their commutator as well.

These difficulties, of course, do not exist for bounded operators. In the latter case the given patterns of functions of ordered operators coincide with μ-structures.

By relying on the logic of Chapter I we present an independent construction of functions of ordered operators here and it is only in Sec. 9 that we shall construct the growing symbols leading us to μ-structures.

Sec. 1. Preliminary Definitions

Let $\{B_\tau\}$ be a family of Banach spaces satisfying the following conditions (τ varying over the totality of integers):

(a) there exists a linear manifold D, which is dense in B_τ for any τ;

(b) $(\tau < \tau') \Rightarrow B_\tau \prec B_\tau$, i.e., $\|\cdot\|_{B_\tau} \geqslant \|\cdot\|_{B_{\tau'}}$.

The family $\{B_\tau\}$ will be called a *Banach scale* and the norm in B_τ will be denoted by $\|\cdot\|_\tau$.

The linear mapping $A : D \to D$ will be called a *generator of degree s with step k* on the Banach scale B_τ if, for any τ, the operator A is a generator of degree s with the defining pair of spaces $(B_\tau, B_{\tau+k})$.

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* Moreover, we shall concern ourselves with a slight generalization of the μ-structure since we shall define only the operations μ: $(x_1 \to \overset{1}{A_1}, x_2 \to \overset{2}{A_2})$ and μ: $(x_1 \to \overset{1}{A_2}, x_2 \to \overset{2}{A_1})$.

Denote by $\{U_A(t)\}$ the group of homomorphisms generated by A, and by $U_{A,\tau}(t)$ the closure of the operator $U_A(t): B_\tau \to B_{\tau+k}$.

Let $\{T_\tau\}$ be a family of homomorphisms $B_\tau \to B_{\tau+l}$, such that $T_\tau D \subset D$ and $T_\tau h = T_{\tau'} h$ for $h \in D$ for any τ and τ'. The family $T = \{T_\tau\}$ will be called a *translator* (or *translating operator*) with step l of the scale $\{B_\tau\}$. In cases when it will lead to no misunderstanding, we shall write Th instead of $T_\tau h$.

Let $A_1, \ldots, A_N$ be generators on the scale $\{B_\tau\}$ with steps correspondingly $k_1, \ldots, k_N$ and degrees $s_1, \ldots, s_N$, and let $T^{(1)}, \ldots, T^{(N-1)}$ be translators with steps $l_1, \ldots, l_{N-1}$, respectively. Now, to the operators $A_1, \ldots, A_N$, $T^{(1)}, \ldots, T^{(N-1)}$ we shall add superscriptions in the following way:

$$\overset{1}{A_1}, \ \overset{2}{T^{(1)}}, \ \overset{3}{A_2}, \ \ldots, \ \overset{2N-2}{T^{(N-1)}}, \ \overset{2N-1}{A_N} \tag{1.1}$$

Here we assume that operator A_1 acts first, operator $T^{(1)}$ acts second and operator A_N acts as $(2N-1)$th. The set (1.1) will be called a *vector-operator*. Note that in general, some components of the vector-operator may commute and be equal.

Let the vector-operator (1.1) correspond to the following homomorphism $U_\tau(t)$ depending on the parameter $t \in \mathbf{R}^N$:

$$U(t_1, t_2, \ldots, t_N) = \\ = U_{A_N}(t_N)\, T^{(N-1)} U_{A_{N-1}}(t_{N-1}) \ldots U_{A_2}(t_2)\, T^{(1)} U_{A_1}(t_1) \tag{1.2}$$

(here and below we shall skip the subscript τ in the notation $U_\tau(t)$ whenever possible).

Theorem 1.1. *For any $h \in B_\tau$ the function*

$$t \to U(t)\, h, \quad t \in \mathbf{R}^N, \quad U(t)\, h \in B_{\tau+k_1+\ldots+k_N+l_1+\ldots+l_{N-1}}$$

is continuous.

Proof. At $N = 1$ the statement is obvious. The proof is made by induction. To avoid the cumbersome details of the derivation, however, we shall only show the transition from $N = 1$ to $N = 2$, because the transition from an arbitrary N to $N + 1$ does not differ in principle from this easier case.

Let $N = 2$ and denote $T^{(1)}$ by T. We have

$$\begin{aligned} &U(t_1+\delta_1, t_2+\delta_2)\, h - U(t_1, t_2)\, h = \\ &= U_{A_2}(t_2+\delta_2)\, T U_{A_1}(t_1+\delta_1)\, h - U_{A_2}(t_2)\, T U_{A_1}(t_1)\, h = \\ &= U_{A_2}(t_2+\delta_2)\, T\, [U_{A_1}(t_1+\delta_1) - U_{A_1}(t_1)]\, h + \\ &+ [U_{A_2}(t_2+\delta_2) - U_{A_2}(t_2)]\, T U_{A_1}(t_1)\, h. \end{aligned}$$

It is obvious that each of the two latter terms tends to zero in the norm of the space $B_{\tau+k_1+\ldots+l_{N-1}}$ whenever δ_1 and $\delta_2 \to 0$.

Expand the spaces $\mathscr{B}_N(\mathbf{R}^n)$ constructed above. Denote by $C_N(\mathbf{R}^n)$ the Banach space of all continuous functions with the finite norm

$$\|f\|_{C_N(\mathbf{R}^n)} = \sup \frac{|f(x)|}{(1+|x|)^N}$$

and by $C_N^*(\mathbf{R}^n)$ the space conjugated to $C_N(\mathbf{R}^n)$. Fix a linear subspace of functionals suitable for our purpose. We shall introduce the following preliminary definition.

Definition. *We shall say that a functional* $L \in C_N^*(\mathbf{R}^n)$ *is equal to zero on an open manifold* $U \subset \mathbf{R}^n$ *if for any function* $f \in C_N(\mathbf{R}^n)$ *such that* $\operatorname{supp} f \subset U$ *the relation is valid*: $L(f) = 0$.

The smallest closed subspace $F \subset \mathbf{R}^n$ such that L is equal to zero on $\mathbf{R}^n - F$ will be called the *support of the functional* L and denoted $\operatorname{supp} L$.

Denote by $\mathscr{L}_N$ a linear subspace of $C_N^*(\mathbf{R}^n)$ consisting of all functionals with compact supports. These functionals will be called *finite functionals*.

Let $\psi_n \in C_0^\infty(\mathbf{R}^n)$ and the following conditions be satisfied: $0 \leqslant \psi_n(x) \leqslant 1$ for all $x \in \mathbf{R}^n$ and $\psi_n(x) = 1$ for $x \in \{x : |x| \leqslant n\}$. It is easily seen that for any $L \in \mathscr{L}_N$ there exists n_0 such that for all $f \in C_N(\mathbf{R}^n)$ the following relation is valid:

$$L((1-\psi_n) f) = 0 \quad \text{for any } n \geqslant n_0.$$

Hence

$$L(f) = L(\psi_{n_0} f) + L((1-\psi_{n_0}) f) = L(\psi_{n_0} f)$$

and

$$\|\psi_{n_0} f\|_{C_N(\mathbf{R}^n)} \leqslant \|f\|_{C_N(\mathbf{R}^n)}.$$

Consequent constructions are based upon this characteristic property of finite functionals. Note that the δ-function and functions from $C_0^\infty(\mathbf{R}^n)$ belong to $\mathscr{L}_N$. Therefore, as before, we shall denote by $C_N^+(\mathbf{R}^n)$ the strong closure $\mathscr{L}_N$ in $C_N^*(\mathbf{R}^n)$. Below we shall consider $C_N^+(\mathbf{R}^n)$ only in this sense and not as a strong closure of the subspace spanned by δ-functions and functions from $C_0^\infty(\mathbf{R}^n)$.

An important property of the functionals of $C_N^+(\mathbf{R}^n)$ is given in the lemma that follows.

Lemma 1.1. *Let* $T \in C_N^+(\mathbf{R}^n)$, *then for any* $\varepsilon > 0$ *there exists an integer* $A > 0$ *such that for all* $n > A$ *and all* $f \in C_N(\mathbf{R}^n)$ *the following inequality is valid*:

$$|T((1-\psi_n) f)| < \varepsilon \|f\|_{C_N(\mathbf{R}^n)}. \tag{1.3}$$

Proof. If $T \in \mathscr{L}_N$ then the statement is obvious. But if $T \in C_N^+(\mathbf{R}^n)$ and is not a finite functional, then there exists a sequence $\{T_m\}_{m \geqslant 0} \subset$

$\subset \mathscr{L}_N$ strongly convergent to T, i.e. for any $\varepsilon > 0$ there exists $M > 0$ such that for all $m > M$ the following inequality is valid:

$$\| T - T_m \|_{C_N^*(\mathbf{R}^n)} < \varepsilon.$$

Let $m_0 > M$, then

$$| T((1-\psi_n) f) - T_{m_0}((1-\psi_n) f) | < \varepsilon \| f \|_{C_N(\mathbf{R}^n)}.$$

Next, there exists an integer $A > 0$ such that $\operatorname{supp} T_{m_0} \cap \cap \operatorname{supp} (1 - \psi_n) f = \varnothing$ whenever $n > A$. Hence

$$| T((1-\psi_n) f) | \leqslant \varepsilon \| f \|_{C_N(\mathbf{R}^n)}$$

for any $n > A$. Lemma 1.1 is proved.

Corollary. For any function $f \in C_N(\mathbf{R}^n)$ there exists a sequence $\{\varphi_n\}_{n \geqslant 0} \subset C_0^\infty(\mathbf{R}^n)$ such that

$$\lim_{n\to\infty} L(\varphi_n) = L(f) \tag{1.4}$$

for any $L \in C_N^+(\mathbf{R}^n)$. If $f \neq 0$ then there exists $T_f \in C_N^+(\mathbf{R}^n)$ such that $T_f(f) \neq 0$.

Proof. Let $L \in C_N^+(\mathbf{R}^n)$. For any $f \in C_N(\mathbf{R}^n)$ we have

$$L(f) = L(\psi_n f) + L((1-\psi_n) f). \tag{1.5}$$

The function $\psi_n f$ is finite and continuous for any $n > 0$. Consequently, for any n there exists $\varphi_n \in C_0^\infty(\mathbf{R}^n)$ such that $\| \psi_n f - \varphi_n \|_{C_N(\mathbf{R}^n)} \leqslant \leqslant 1/n$.

From Lemma 1.1 it follows that for any $\varepsilon > 0$ there exists $M > 0$ such that $| L((1-\psi_n) f) | < \varepsilon \| f \|_{C_N}$ for $n > M$. By virtue of (1.5) we obtain

$$| L(f) - L(\varphi_n) | \leqslant \| L \|_{C_N^*(\mathbf{R}^n)} \cdot 1/n + \varepsilon \| f \|_{C_N(\mathbf{R}^n)}. \tag{1.6}$$

Relation (1.4) follows from this inequality.

If $\bar{f} \neq 0$ then there exists $x_0 \in \mathbf{R}^n$ such that $\bar{f}(x_0) \neq 0$. The functional δ_{x_0} defined by the formula

$$\delta_{x_0}(f) = f(x_0) \quad \text{for any} \quad f \in C_N(\mathbf{R}^n)$$

belongs to $\mathscr{L}_N$ and $\delta_{x_0}(\bar{f}) = \bar{f}(x_0) \neq 0$. Hence the proof is obtained.

We shall introduce the operation of convolution of functionals from $C_N^+(\mathbf{R}^n)$ with functions from $C_N(\mathbf{R}^n)$. But first we define some continuous linear mappings τ_h and "$\vee$" of the space $C_N(\mathbf{R}^n)$ into $C_N(\mathbf{R}^n)$ (which will be convenient for future usage) in the following way: for any $f \in C_N(\mathbf{R}^n)$ let

$$\tau_h f(x) \stackrel{\text{def}}{=} f(x-h)$$

and

$$\check{f}(x) \stackrel{\text{def}}{=} f(-x).$$

Definition. *The function F defined by the formula*

$$F(h) = T((\tau_h f)^{\vee}),$$

where $T \in C_N^+(\mathbf{R}^n)$ and $f \in C_N(\mathbf{R}^n)$, will be called the convolution of the functional $T \in C_N^+(\mathbf{R}^n)$ with the function $f \in C_N(\mathbf{R}^n)$ and denoted by

$$T * f(h) \stackrel{\text{def}}{=} T((\tau_h f)^{\vee}). \tag{1.7}$$

The function $(\tau_h f)^{\vee} \in C_N(\mathbf{R}^n)$ for any $h \in \mathbf{R}^n$. Hence the convolution is defined on $\mathbf{R}^n$ everywhere.

From (1.7) we have

$$|T * f(h)| \leqslant \|T\|_{C_N^*} \cdot \|f\|_{C_N} (1 + |h|)^N. \tag{1.8}$$

Moreover, the convolution belongs to $C_N(\mathbf{R}^n)$. In fact let $T \in C_N^+(\mathbf{R}^n)$, $f \in C_N(\mathbf{R}^n)$ and h_0 be a point in $\mathbf{R}^n$. For any h satisfying the inequality $|h - h_0| < \delta$ the following estimate is valid:

$$\|\tau_h f\|_{C_N(\mathbf{R}^n)} \leqslant \|f\|_{C_N(\mathbf{R}^n)} (1 + |h_0| + \delta)^N.$$

Consequently, the family of functions $(\tau_h f)_{|h-h_0|<\delta}$ is uniformly bounded and then from Lemma 1.1 it follows that for any $\varepsilon > 0$ there exists $M > 0$ such that for all $n > M$ the following inequality is valid:

$$|T((1 - \psi_n)(\tau_h f))^{\vee}| < \varepsilon/2. \tag{1.9}$$

The function $h \to \psi_n (\tau_h f)^{\vee}$ considered as a function with values in $C_N(\mathbf{R}^n)$ is continuous due to the fact that $\psi_n \in C_0^\infty(\mathbf{R}^n)$, and therefore for $\frac{\varepsilon}{2\|T\|_{C_N^*}} > 0$ there exists $\delta > 0$, such that

$$\|\psi_n((\tau_h - \tau_{h_0}) f)^{\vee}\|_{C_N(\mathbf{R}^n)} \leqslant \frac{\varepsilon}{2\|T\|_{C_N^*}}. \tag{1.10}$$

From (1.5), (1.9) and (1.10) it follows that for any $\varepsilon > 0$ there exists $\delta > 0$, such that

$$|T * f(h) - T * f(h_0)| \leqslant \varepsilon$$

whenever $|h - h_0| < \delta$.

The continuity of the convolution is thereby proved.

Moreover, from (1.8) we see that the convolution of any functional from $C_N^+(\mathbf{R}^n)$ with functions from $C_N(\mathbf{R}^n)$ is a continuous linear

mapping from $C_N(\mathbf{R}^n)$ into $C_N(\mathbf{R}^n)$, which is invariant with respect to a shift, i.e., for any $h' \in \mathbf{R}^n$ we obtain

$$\tau_{h'}(T * f(h)) = T * (\tau_{h'} f)(h).$$

The opposite assertion is also correct: for any continuous linear mapping A of the space $C_N(\mathbf{R}^n)$ into $C_N(\mathbf{R}^n)$, which is invariant with respect to a shift, i.e., for any $h \in \mathbf{R}^n$ and $f \in C_N(\mathbf{R}^n)$ the following equality is fulfilled:

$$A\tau_h f = \tau_h A f; \tag{1.11}$$

there exists a functional $T \in C_N^*(\mathbf{R}^n)$, such that, for any $f \in C_N(\mathbf{R}^n)$, we obtain

$$Af = T * f.$$

Prove the assertion. Note that

$$T(\check{f}) = T * f(0). \tag{1.12}$$

Since A is continuous, the mapping $\check{f} \to Af(0)$ yields a continuous linear functional from $C_N^*(\mathbf{R}^n)$ which will be denoted by T.

Thus

$$T(\check{f}) = Af(0).$$

But from (1.12) we obtain $T * f(0) = Af(0)$. By replacing f by $\tau_h f$ and taking into account (1.11) we further obtain

$$T * f(h) = Af(h) \quad \text{for any} \quad h \in \mathbf{R}^n.$$

The assertion is proved. Note that T may not belong to $C_N^+(\mathbf{R}^n)$. This assertion, however, allows us to define the operation of the convolution of functionals of the space $C_N^+(\mathbf{R}^n)$.

Let $T_1, T_2 \in C_N^+(\mathbf{R}^n)$. Then it is obvious that the mapping

$$Af = T_1 * (T_2 * f)$$

is continuous and invariant with respect to a shift.

Thus, as proved above, there exists $T \in C_N^*(\mathbf{R}^n)$ such that for any $f \in C_N(\mathbf{R}^n)$

$$T * f = T_1 * (T_2 * f). \tag{1.13}$$

Now we may give the following definition.

Definition. *The functional T defined by relation* (1.13) *will be called a convolution of the functionals $T_1, T_2 \in C_N^+(\mathbf{R}^n)$ and denoted by*

$$T \overset{\text{def}}{=\!=} T_1 * T_2.$$

By using (1.12) and (1.13) we obtain the relation

$$T * f(0) = T\left(\check{f}\right) = T_1((T_2 * f)^{\vee}) \tag{1.14}$$

from which it follows that $T \in C_N^+(\mathbf{R}^n)$ due to the fact that $f \to (T_2 * f)^{\vee}$ is a continuous linear mapping from $C_N(\mathbf{R}^n)$ into $C_N(\mathbf{R}^n)$ and $T_1 \in C_N^+(\mathbf{R}^n)$.

Thus, we have proved the theorem that follows.

Theorem. *The space $C_N^+(\mathbf{R}^n)$ is invariant relative to the operation of convolution.*

Some important properties of the operation of convolution are given by the theorem that follows.

Theorem 1.2. *The operation of the convolution of functionals of the space $C_N^+(\mathbf{R}^n)$ is commutative and continuous in factors.*

Proof. The continuity of the convolution follows directly from (1.14). In fact,

$$\left| T\left(\check{f}\right)\right| \leqslant \| T_1 \|_{C_N^*} \| T_2 * f \|_{C_N} \leqslant \| T_1 \|_{C_N^*} \| T_2 \|_{C_N^*} \| f \|_{C_N}.$$

Hence

$$\| T_1 * T_2 \|_{C_N^*(\mathbf{R}^n)} \leqslant \| T_1 \|_{C_N^*(\mathbf{R}^n)} \| T_2 \|_{C_N^*(\mathbf{R}^n)} \tag{1.15}$$

which means the continuity of the convolution in factors.

Now show the commutativity of the convolution, i.e., prove that for any $f \in C_N(\mathbf{R}^n)$ the equality

$$T_1 * T_2(f) = T_2 * T_1(f) \tag{1.16}$$

holds.

If we show that (1.16) is fulfilled for any $f \in C_0^\infty(\mathbf{R}^n)$, then from Lemma 1.1 it follows that (1.16) is also correct for any $f \in C_N(\mathbf{R}^n)$. For this purpose we shall need the lemma that follows.

Lemma 1.2. *Let $\varphi_\varepsilon \in C_0^\infty(\mathbf{R}^n)$ and have the following properties: $\varphi_\varepsilon(x) \geqslant 0$ for any $x \in \mathbf{R}^n$ and $\int \varphi_\varepsilon(x)\,dx = 1$, $\varphi_\varepsilon(x) = 0$ for $|x| \geqslant \varepsilon > 0$; then for any $\varphi \in C_0^\infty(\mathbf{R}^n)$ the following relation is valid:*

$$\lim_{\varepsilon \downarrow 0} T * (\varphi_\varepsilon * \varphi) = T * \varphi.$$

Proof. From the obvious equalities we obtain

$$T * \varphi_\varepsilon * \varphi - T * \varphi = T * (\varphi_\varepsilon * \varphi - \varphi).$$

By virtue of the continuity of the convolution we have

$$\| T * (\varphi_\varepsilon * \varphi - \varphi) \|_{C_N(\mathbf{R}^n)} \leqslant \| T \|_{C_N^+(\mathbf{R}^n)} \| \varphi_\varepsilon * \varphi - \varphi \|_{C_N(\mathbf{R}^n)}. \tag{1.17}$$

But $\lim\limits_{\varepsilon \downarrow 0} \| \varphi_\varepsilon * \varphi - \varphi \|_{C_N(\mathbf{R}^n)} = 0$. This completes the proof of the lemma.

We shall now proceed with the proof of the theorem.

Let $\{T_1^n\}_{n>0}$ be a sequence of finite functionals convergent to T_1, and $\{T_2^m\}_{m\geqslant 0}$ be a similar sequence convergent to T_2. By considering (1.17) we obtain

$$\lim_{\varepsilon\downarrow 0}(T_1^n * \varphi_\varepsilon) * (T_2^m * \varphi_\varepsilon)(f) = T_1 * T_2(f).$$

But $T_1^n * \varphi_\varepsilon$ and $T_2^m * \varphi_\varepsilon$ are finite continuous functions, and for this reason we have

$$(T_1^n * \varphi_\varepsilon) * (T_2^m * \varphi_\varepsilon)(f) = (T_2^m * \varphi_\varepsilon) * (T_1^n * \varphi_\varepsilon)(f).$$

Thus, the commutativity of the convolution is proved, and at the same time Theorem 1.2 is also proved.

We shall establish another important property of the functionals of space $C_n^+(\mathbf{R}^n)$. Let $f \in C_N(\mathbf{R}^n)$. With no loss of generality one may set $n = 2$ here. Bring the function $F(x_1, x_2)$ in correspondence with any function $f(x_1, x_2)$ according to the formula

$$F(x_1\, x_2) = \int_0^{x_2} f(x_1,\ \xi)\, d\xi. \tag{1.18}$$

Show that the mapping $I_2 : f(x_1, x_2) \to F(x_1, x_2)$ given by (1.18) is a continuous linear mapping from $C_N(\mathbf{R}^n)$ into $C_{N+1}(\mathbf{R}^n)$.

In fact,

$$\left|\int_0^{x_2} f(x_1 \xi)\, d\xi\right| \leqslant \|f\|_{C_N(\mathbf{R}^2)} |x_2| (1 + |x|)^N.$$

Consequently,

$$\|F(x_1,\ x_2)\|_{C_{N+1}(\mathbf{R}^2)} \leqslant \|f\|_{C_N(\mathbf{R}^2)}. \tag{1.19}$$

Since the linearity of the mapping I_2 is obvious the assertion is proved. Define for any functional $T \in C_{N+1}^+(\mathbf{R}^2)$ the functional T^0 given by the formula:

$$T^0(f) = T(I_2(f)) \quad \text{for any} \quad f \in C_N(\mathbf{R}^2). \tag{1.20}$$

Since I_2 and T are continuous mappings, it follows that $T^0 \in C_N^*$. Besides, $T \in C_{N+1}^+(\mathbf{R}^n)$, hence $T^0 \in C_N^+(\mathbf{R}^n)$. By considering (1.19) it is not difficult to note that the following inequality is valid:

$$\|T^0\|_{C_N^*(\mathbf{R}^n)} \leqslant \|T\|_{C_{N+1}^*(\mathbf{R}^n)}.$$

Thus by means of relation (1.20) we have defined a continuous linear mapping $\hat{I}_2 : C_{N+1}^* \to C_N^+$ which has the following property: if $f(x_1, x_2)$ is a continuous function from $C_N(\mathbf{R}^2)$ such that

$f'_{x_2}(x_1, x_2) \in C_N(\mathbf{R}^2)$, then for any $T \in C^+_{N+1}(\mathbf{R}^n)$ we have

$$\hat{I}_2(T)(f'_{x_2}(x_1, x_2)) = T(f(x_1, x_2) - f(x_1, 0)).$$

Formulate the obtained result in the general form.

Theorem 1.3. *There exists a continuous linear mapping $\hat{I}_i: C^+_{N+1}(\mathbf{R}^n) \to C^+_N(\mathbf{R}^n)$ which has the following property: if $f \in C_N(\mathbf{R}^n)$ and $f'_{x_2} \in C_N(\mathbf{R}^n)$, then for any $T \in C^+_{N+1}(\mathbf{R}^n)$ we have*

$$\hat{I}_i(T)(f'_{x_i}) = T(f(x_1, \ldots, x_i, \ldots, x_n) - f(x_1, \ldots, 0, \ldots, x_n)),$$

where $\hat{I}_i$ is defined by the formula identical to (1.20).

Now pass on to the construction of spaces $\mathscr{B}_N(\mathbf{R}^n)$. For this purpose we shall define the Fourier transform of the functionals of the spaces $C^+_N(\mathbf{R}^n)$.

Definition. *The mapping $\mathscr{F}$ which brings the function*

$$\widetilde{L}(t) \overset{\text{def}}{=} \frac{1}{(2\pi)^{n/2}} L(e^{-ixt}) \tag{1.21}$$

in correspondence with each $L \in C^+_N(\mathbf{R}^n)$ will be called the Fourier transform of the functionals of the space $C^+_N(\mathbf{R}^n)$. It is obvious that the Fourier transform is a linear mapping. We shall establish the properties of the Fourier transform of the functionals of the space $C^+_N(\mathbf{R}^n)$. By $C^+(\mathbf{R}^n)$ denote the space $C^+_N(\mathbf{R}^n)$ for $N = 0$.

Lemma 1.3. *The Fourier transform of the functionals of the space $C^+(\mathbf{R}^n)$ has no kernel and maps $C^+(\mathbf{R}^n)$ into $C(\mathbf{R}^n)$.*

Proof. Let $T \in C^+(\mathbf{R}^n)$ and $T \neq 0$, and let $T(e^{-itx}) = 0$ for any $t \in \mathbf{R}^n$. Then there exists $f \in C(\mathbf{R}^n)$ such that $T(f) \neq 0$. Further, according to Lemma 1.1 there exists a sequence $\{\varphi_n\}_{n>0} \subset C_0^\infty(\mathbf{R}^n)$ such that $\lim\limits_{n\to\infty} T(\varphi_n) = T(f)$. Next for any $\varphi_n \in C_0^\infty(\mathbf{R}^n)$ there exists $\widetilde{\varphi}_n \in S$ such that $\varphi_n(x) = \frac{1}{(2\pi)^{n/2}} \int e^{-itx}\widetilde{\varphi}_n(t)\,dt$. The convergence of this integral is uniform.

Consequently,

$$T(\varphi_n) = \frac{1}{(2\pi)^{n/2}} \int T(e^{-itx})\,\widetilde{\varphi}_n(t)\,dt = 0.$$

Hence $\lim\limits_{n\to\infty} T(\varphi_n) = 0$. The obtained contradiction illustrates the fact that the Fourier transform of the space $C^+(\mathbf{R}^n)$ has no kernel. Now show the continuity of $\widetilde{T}$. For any $n > 0$ the mapping $t \to \psi_n e^{-itx}$ (ψ_n being defined by Lemma 1.1) is obviously continuous as a mapping from $\mathbf{R}^n$ into $C(\mathbf{R}^n)$. Then

$$T(e^{-itx}) = T(\psi_n e^{-itx}) + T((1 - \psi_n)\,e^{-itx}).$$

For any $\varepsilon > 0$ there exists $A > 0$ such that if $n > A$, then $|T((1-\psi_n)e^{-itx})| < \varepsilon/3$ for any $t \in \mathbf{R}^n$. At the same time there exists $\delta > 0$ such that $|T(\psi_n(e^{-itx} - e^{-it_0x}))| < \varepsilon/3$ for any t satisfying the inequality $|t - t_0| < \delta$ for some $n > A$.

Hence $|T(e^{-itx} - e^{-it_0x})| < \varepsilon$ whenever $|t - t_0| < \delta$. Lemma 1.3 is thereby proved.

Now let $N > 0$. Then the mapping $t \to e^{-itx}$ is continuous as a mapping from $\mathbf{R}^n$ into $C_N(\mathbf{R}^n)$. Moreover, for any polynomial $p(x_1, \ldots, x_n)$ with a degree no greater than $N-1$ the mapping $t \to pe^{-itx}$ from $\mathbf{R}^n$ into $C_N(\mathbf{R}^n)$ is continuous and the function

$$F_p(t) = T(e^{-itx}p(x))$$

is also continuous for any $T \in C_N^*(\mathbf{R}^n)$. This assertion follows directly from the definition of the norm in $C_N(\mathbf{R}^n)$. If the degree of the polynomial $p(x_1, \ldots, x_n)$ is equal to N, however, then, generally speaking, $F_p(t)$ will not be a continuous function.

If $\mathscr{D}_t^\alpha$ is a differential operator of degree no greater than $N-1$, then by virtue of the continuity of the mapping $t \to \mathscr{D}_t^\alpha e^{-itx}$ we have

$$T(\mathscr{D}_t^\alpha e^{-itx}) = \mathscr{D}_t^\alpha T(e^{-itx}) = T((-ix)^\alpha e^{-itx})$$

for any $T \in C_N^*(\mathbf{R}^n)$. This means that the Fourier transform of any functional $T \in C_N^*(\mathbf{R}^n)$ is an $N-1$ continuously differentiable function limited with its derivatives, i.e.,

$$\mathscr{F}(C_N^*) \subset C^{(N-1)}.$$

Besides, functionals from $C_N^+(\mathbf{R}^n)$ are characterized by the lemma that follows.

Lemma 1.3*. *The Fourier transform of the functionals of the space $C_N^+(\mathbf{R}^n)$ has no kernel and maps $C_N^+(\mathbf{R}^n)$ into $C^{(N)}(\mathbf{R}^n)$.*

Proof. The triviality of the kernel in the Fourier transform is proved in the same way as in Lemma 1.3.

We have already seen that $\mathscr{F}(C_N^+(\mathbf{R}^n)) \subset C^{(N-1)}(\mathbf{R}^n)$; therefore, it remains to be proved that for any polynomial p of degree N the function $F_p(t) = T(e^{-itx}p(x))$ is continuous for any $T \in C_N^+(\mathbf{R}^n)$.

The mapping from $C_N(\mathbf{R}^n)$ into $C(\mathbf{R}^n)$ described by the relation $f \to \dfrac{f}{(1+|x|)^N}$ is an isometric isomorphism of $C_N(\mathbf{R}^n)$ and $C(\mathbf{R}^n)$. Hence for any $L \in C_N^+(\mathbf{R}^n)$ from the formula

$$L(pe^{-ixt}) = L\left((1+|x|)^N \frac{pe^{-ixt}}{(1+|x|)^N}\right) =$$
$$= (L \circ (1+|x|)^N)\left(\frac{pe^{-ixt}}{(1+|x|)^N}\right) = \hat{L}\left(\frac{p(x)e^{-ixt}}{(1+|x|)^N}\right)$$

it follows that $\hat{L} \in C^+(\mathbf{R}^n)$. As in Lemma 1.3 it remains to be shown that $\hat{L}\left(\frac{p(x)\,e^{-ixt}}{(1+|x|)^N}\right)$ is a continuous function, and this will complete the proof of Lemma 1.3*.

Denote by $\mathscr{B}_N(\mathbf{R}^n)$ the Fourier transform of the space $C_N^+(\mathbf{R}^n)$ into $C^{(N)}(\mathbf{R}^n)$. As has been proved, the Fourier transform realizes an isomorphism between $C_N^+(\mathbf{R}^n)$ and $\mathscr{B}_N(\mathbf{R}^n)$. Therefore, we may introduce the norm in $\mathscr{B}_N(\mathbf{R}^n)$ as follows:

$$\|\varphi\|_{\mathscr{B}_N(\mathbf{R}^n)} \overset{\text{def}}{=} \|\mathscr{F}^{-1}\varphi\|_{C_N^*(\mathbf{R}^n)}. \tag{1.22}$$

Since $C_N^+(\mathbf{R}^n)$ is closed in $C_N^*(\mathbf{R}^n)$ it follows that $\mathscr{B}_N(\mathbf{R}^n)$ becomes a Banach space with respect to the norm (1.22).

Let $\mathscr{D}_t^\alpha$ be a differential operator with degree no greater than N, then for any function $\varphi \in \mathscr{B}_N(\mathbf{R}^n)$ we have

$$\mathscr{D}_t^\alpha \varphi = \frac{1}{(2\pi)^{n/2}}\,\mathscr{F}^{-1}(\varphi)\left((-ix)^\alpha e^{-itx}\right).$$

Hence

$$|\mathscr{D}_t^\alpha \varphi(t)| \leqslant \frac{1}{(2\pi)^{n/2}}\,\|\mathscr{F}^{-1}(\varphi)\|_{C_N^*(\mathbf{R}^n)}.$$

For this reason the convergence in $\mathscr{B}_N(\mathbf{R}^n)$ leads to the convergence in the norm $C^{(N)}(\mathbf{R}^n)$. Thus $\mathscr{B}_N(\mathbf{R}^n)$ is continuously embedded in $C^{(N)}(\mathbf{R}^n)$.

Let $T_1, T_2 \in C_N^+(\mathbf{R}^n)$. By definition of the convolution of functionals and by virtue of Theorem 1.2 there exists the Fourier transform of the convolution

$$\mathscr{F}(T_2 * T_1) = \mathscr{F}(T_1 * T_2) = \frac{1}{(2\pi)^{n/2}}\,T_1\left(T_2\left(e^{-i(x+y)t}\right)\right) = (2\pi)^{n/2}\,\widetilde{T}_1\widetilde{T}_2.$$

Next from (1.15) we obtain

$$\|\widetilde{T}_1\cdot\widetilde{T}_2\|_{\mathscr{B}_N(\mathbf{R}^n)} = \frac{1}{(2\pi)^{n/2}}\,\|T_2 * T_1\|_{C_N^*(\mathbf{R}^n)} \leqslant$$
$$\leqslant \frac{1}{(2\pi)^{n/2}}\,\|T_2\|_{C_N^*}\cdot\|T_1\|_{C_N^*}.$$

This provides the proof of the theorem that follows.

Theorem 1.4. *The space $\mathscr{B}_N(\mathbf{R}^n)$ is a Banach algebra with respect to the ordinary operation of multiplication of functions and is continuously embedded in the Banach algebra $C^{(N)}(\mathbf{R}^n)$.*

Let the function $f \in C^{(1)}(\mathbf{R}^n)$. Introduce the notation for the difference derivative of the function f with respect to x_i:

$$\frac{\delta f}{\delta x_i}(x_1, \ldots, x_i; x_i', x_{i+1}, \ldots, x_n) \overset{\text{def}}{=\!=}$$

$$= \frac{f(x_1, \ldots, x_i, \ldots, x_n) - f(x_1, \ldots, x_i', \ldots, x_n)}{x_i - x_i'}.$$

With no loss of generality in what follows below, we set $n = 1$. In complete analogy with Theorem 1.3 we may obtain the following result: let $f \in C_N(\mathbf{R}^2)$, then the mapping $I : f \to \int\limits_0^x f(\xi, x - \xi)\, d\xi$ is continuous and maps linearly $C_N(\mathbf{R}^2)$ into $C_{N+1}(\mathbf{R})$, and for any $T \in C_{N+1}^+(\mathbf{R})$ the functional L defined by the relation

$$L(f) = T(I(f)) \tag{1.23}$$

belongs to $C_N^+(\mathbf{R}^2)$. Thus we have defined a continuous linear mapping from $C_{N+1}^+(\mathbf{R})$ into $C_N^+(\mathbf{R}^2)$ provided by the formula (1.23).

Put in (1.23) the function $e^{-it_1x_1 - it_2x_2}$ instead of f.

Then the function $\widetilde{L} \in \mathscr{B}_N(\mathbf{R}^2)$ and we have

$$\widetilde{L}(t_1, t_2) = T(I(e^{-it_1x_1 - it_2x_2})). \tag{1.24}$$

But

$$I(e^{-it_1x_1} - e^{-it_2x_2}) = \frac{e^{-it_1x} - e^{-it_2x}}{i(t_1 - t_2)}.$$

Consequently,

$$T(I(e^{-it_1x_1 - it_2x_2})) = -i\,\frac{\widetilde{T}(t_1) - \widetilde{T}(t_2)}{t_1 - t_2}.$$

By virtue of (1.24) the function $\dfrac{\widetilde{T}(t_1) - \widetilde{T}(t_2)}{t_1 - t_2}$ belongs to $\mathscr{B}_N(\mathbf{R}^2)$, and the mapping from $\mathscr{B}_{N+1}(\mathbf{R}^1)$ into $\mathscr{B}_N(\mathbf{R}^2)$ given by the relation $\varphi \to \dfrac{\delta\varphi}{\delta x}(x; x')$ is continuous and linear due to the fact that the mapping from $C_{N+1}^+(\mathbf{R})$ into $C_N^+(\mathbf{R}^2)$ defined in Theorem 1.3 is continuous and the norm in $\mathscr{B}_N(\mathbf{R}^n)$ is determined by formula (1.22). Thus we have established that the theorem that follows is valid.

Theorem 1.5. *Difference differentiation is a continuous linear mapping from* $\mathscr{B}_{N+1}(\mathbf{R}^n)$ *into* $\mathscr{B}_N(\mathbf{R}^{n+1})$.

Very often it is not easy to establish whether a function belongs to the algebra $\mathscr{B}_N(\mathbf{R}^n)$. It is well known that for $k > \frac{n}{2} + N$ the space $W_2^k(\mathbf{R}^n)$ continuously embeds in $C^{(N)}(\mathbf{R}^n)$. We have

established that $\mathscr{B}_N(\mathbf{R}^n)$ is continuously embedded in $C^{(N)}(\mathbf{R}^n)$. The lemma that follows will be valid.

Lemma 1.4. *The space $W_2^k(\mathbf{R}^n)$ for $k > \frac{n}{2} + N$ is continuously embedded in $\mathscr{B}_N(\mathbf{R}^n)$.*

Proof. Let $\varphi \in S$ and $f \in C_N(\mathbf{R}^n)$. Then the following inequality is valid:

$$\int \tilde{\varphi}(x) f(x)\, dx = \int \tilde{\varphi}(x)(1+|x|)^{N+\frac{n}{2}+\varepsilon} \frac{f(x)}{(1+|x|)^{N+n/2+\varepsilon}}\, dx \leqslant$$

$$\leqslant \|f\|_{C_N(\mathbf{R}^n)} \|\tilde{\varphi}\|_{\tilde{W}_2^{N+n/2+\varepsilon}(\mathbf{R}^n)} \cdot \sqrt{\int \frac{dx}{(1+|x|)^{n+\varepsilon}}} \leqslant$$

$$\leqslant C \|f\|_{C_N(\mathbf{R}^n)} \|\varphi\|_{W_2^{N+\frac{n}{2}+\varepsilon}(\mathbf{R}^n)}.$$

Next, S is dense in $W_2^k(\mathbf{R}^n)$; consequently for any function $\varphi \in W_2^k(\mathbf{R}^n)$ the above inequality holds and any function φ from $W_2^k(\mathbf{R}^n)$ generates a continuous linear functional from $C_N^+(\mathbf{R}^n)$ described by the formula

$$T_\varphi(f) \stackrel{\text{def}}{=} \int \tilde{\varphi}(x) f(x)\, dx$$

for any $f \in C_N(\mathbf{R}^n)$.

Show that if $\varphi \in W_2^k(\mathbf{R}^n)$ and $\varphi \neq 0$ then there exists $f \in C_N(\mathbf{R}^n)$ such that $\int \tilde{\varphi} f\, dx \neq 0$. Suppose the opposite. Then for any function $\psi \in C_0^\infty(\mathbf{R}^n)$ we have $\int \tilde{\varphi}\psi\, dx = 0$. The space $W_2^k(\mathbf{R}^n)$ is continuously embedded in $L_2(\mathbf{R}^n)$ and $C_0^\infty(\mathbf{R}^n)$ is dense in $L_2(\mathbf{R}^n)$. Hence $\varphi = 0$. Considering the inequality obtained above we see that $\tilde{W}_2^k(\mathbf{R}^n)$ is continuously embedded in $C_N^+(\mathbf{R}^n)$. Consequently $\tilde{W}_2^k(\mathbf{R}^n)$ is continuously embedded in $\mathscr{B}_N(\mathbf{R}^n)$. The Lemma is proved.

Denote by $C_N[\mathbf{R}^n, B]$ the set of all continuous functions on $\mathbf{R}^n$ with values in the Banach space B for which the norm

$$\|f\|_{C_N[\mathbf{R}^n, B]} \stackrel{\text{def}}{=} \sup \frac{\|f(x)\|_B}{(1+|x|)^N}$$

is finite.

Define the operation of the pairing of the spaces $C_N^+(\mathbf{R}^n)$ and $C_N[\mathbf{R}^n, B]$ as follows.

Let $L \in C_N^+(\mathbf{R}^n)$ and $f \in C_N[\mathbf{R}^n, B]$. Define the linear functional $\langle L, f\rangle$ on B^* by the formula

$$\langle L, f\rangle(h^*) = L(h^* f). \tag{1.25}$$

Then

$$|\langle L, f\rangle (h^*)| \leqslant \| L \|_{C_N^+(\mathbf{R}^n)} \cdot \| h^* \|_{B^*} \cdot \| f \|_{C_N[\mathbf{R}^n, B]}$$

and, consequently, $\langle L, f\rangle \in B^{**}$. The mapping $I: C_N^+(\mathbf{R}^n) \times \times C_N[\mathbf{R}^n, B] \to B^{**}$ defined by the formula (1.25) will be called the pairing of $C_N^+(\mathbf{R}^n)$ and $C_N[\mathbf{R}^n, B]$.

Lemma 1.5. *The functional $\langle L, f\rangle$ defined by (1.25) for any $f \in \in C_N[\mathbf{R}^n, B]$ and $L \in C_N^+(\mathbf{R}^n)$ belongs to B (considered as a subspace in B^{**}).*

Proof. It is known that a functional $g \in B^{**}$ belongs to B if and only if g is continuous in the weak* topology of the space B^*.

Let the sequence $(h_m^*)_{m>0} \subset B^*$ converge weakly* to zero. Show that $\langle L, f\rangle (h_m^*)$ also converges to zero, which will complete the proof of the lemma. From Lemma 1.1 it follows that for any $\varepsilon > 0$ there exists $M > 0$ such that for $n > M$ we obtain

$$| L (h^* (1 - \psi_n) f) | + \varepsilon \| h^* f \| \leqslant \varepsilon \| h^* \|_{B^*} \| f \|_{C_N[\mathbf{R}^n, B]}.$$

Any weakly* convergent sequence is bounded. Consequently for any $\varepsilon > 0$ there exists $\overline{M} > 0$ such that for all $n > \overline{M}$ the estimate is valid:

$$|\langle L, f\rangle (h_m^*) - \langle L, \psi_n f\rangle (h_m^*)| < \varepsilon. \tag{1.26}$$

The function $\psi_n f$ is continuous and of compact support. Therefore there exists a function f_n with a finite number of non-equal values such that $\| f_n - f \|_{C_N[\mathbf{R}^n, B]} < \varepsilon$. But since in the weak* topology $h_m^* \to 0$ it follows that $\langle L, f_n\rangle (h_m^*) \to 0$ for $m \to \infty$. By virtue of (1.26) we thus obtain that for any $\varepsilon > 0$ there exists $M' > 0$ such that

$$| \langle L, f\rangle (h_m^*) | < 3\varepsilon$$

for any $m > M'$. The lemma is thereby proved.

Thus the pairing of $C_N^+(\mathbf{R}^n)$ with $C_N[\mathbf{R}^n, B]$ described by (1.25) is a continuous linear mapping from $C_N^+(\mathbf{R}^n) \times C_n[\mathbf{R}^n, B]$ into B.

In complete analogy with the preceding section we may construct the algebra $\mathscr{B}_{s_1, \ldots, s_N}(\mathbf{R}^N)$, where $s_1, \ldots, s_N$ are integers. For this purpose denote by $C_{s_1, \ldots, s_N}(\mathbf{R}^N)$ the Banach space of functions continuous on $\mathbf{R}^N$ and having a finite norm

$$\| g \|_{C_{s_1, \ldots, s_N}(\mathbf{R}^N)} \overset{\text{def}}{=} \sup_{x \in \mathbf{R}^n} \frac{| g(x) |}{(1 + | x_1 |)^{s_1} \ldots (1 + | x_N |)^{s_N}}.$$

The closure of continuous functionals finite in $C_{s_1, \ldots, s_N}(\mathbf{R}^N)$ will be denoted by $C^+_{s_1, \ldots, s_N}(\mathbf{R}^N)$. Construct $\mathscr{B}_{s_1, \ldots, s_N}(\mathbf{R}^N)$

from $C^{+}_{s_1,\ldots,s_N}(\mathbf{R}^N)$ in a manner similar to the one we have used in constructing $\mathscr{B}_N(\mathbf{R}^n)$ from $C^{+}_N(\mathbf{R}^n)$. The isometric isomorphism, which is made use of, will be called the Fourier transform and denoted by $\mathscr{F}$.

Sec. 2. The Functions of Two Noncommutative Self-Adjoint Operators

Let A and B be self-adjoint operators in a separable Hilbert space H and let T be an operator of the Schmidt class $B_2(H)$.

In Sec. 9 of Chapter I the pair $(\overrightarrow{A}, \overleftarrow{B})$ of the operators commutative in the everywhere dense set defined on a Hilbert space $B_2(H)$ corresponded to the pair of operators (A, B). There exists a homomorphism

$$\mathscr{M}_{\overrightarrow{A}, \overleftarrow{B}}: C(\mathbf{R}^2) \to \mathrm{Op}(B_2(H)),$$

which translates the function $f \in C(\mathbf{R}^2)$ into the operator $f(\overrightarrow{A}, \overleftarrow{B})$, $\| f(\overrightarrow{A}, \overleftarrow{B}) \|_{B_2(H)} \leqslant \| f \|_{C(\mathbf{R}^2)}$. The homomorphism $\mathscr{M}_{\overrightarrow{A}, \overleftarrow{B}}$ induces the mapping

$$\mathscr{M}_{\overset{3}{A}, \overset{2}{T}, \overset{1}{B}}: C(\mathbf{R}^2) \to B_2(H)$$

for *a fixed* T by means of the formula

$$\mathscr{M}_{\overset{3}{A}, \overset{2}{T}, \overset{1}{B}} f \overset{\text{def}}{=} f(\overrightarrow{A}, \overleftarrow{B})\, T.$$

We shall write $\mathscr{M}$ instead of $\mathscr{M}_{\overset{3}{A}, \overset{2}{T}, \overset{1}{B}}$ and use the notation

$$\mathscr{M} f = \overset{2}{T} f\left(\overset{3}{A}, \overset{1}{B}\right).$$

Obviously, the following estimate is valid:

$$\left\| \overset{2}{T} f\left(\overset{3}{A}, \overset{1}{B}\right) \right\| \leqslant \| f \|_{C(\mathbf{R}^2)} \| T \|_2,$$

where $\| \;\|$ is the norm of an operator H. Thus $\mathscr{M}$ is a homomorphism of normed spaces. Note that the definition of the operator $\overset{2}{T} f\left(\overset{3}{A}, \overset{1}{B}\right)$ agrees with the definitions of the preceding section: if $f \in \mathscr{B}_0(\mathbf{R}^2)$ then the following formula is valid:

$$\overset{2}{T} f\left(\overset{3}{A}, \overset{1}{B}\right) h = \frac{1}{2\pi} \int_{\mathbf{R}^2} (F^{-1} f)(t_1, t_2)\, e^{-iAt_1} T e^{-iBt_2} h \, dt_1 \, dt_2$$

or any $h \in H$. Besides, the following equality is valid:

$$\overset{2}{T} f_1\left(\overset{3}{A}\right) f_2\left(\overset{1}{A}\right) = f_1(A)\, T f_2(B).$$

Lemma 2.1. *The following estimate is valid:*

$$\left\| \overset{2}{T}\varphi\left(\overset{3}{A},\ \overset{1}{B}\right)\psi\left(\overset{3}{A},\ \overset{1}{B}\right)\right\|_2 \leqslant \|\varphi\|_{C(\mathbf{R}^2)}\left\|\overset{2}{T}\psi\left(\overset{3}{A},\ \overset{1}{B}\right)\right\|_2.$$

Proof. We have

$$\left\| \overset{2}{T}\varphi\left(\overset{3}{A},\ \overset{1}{B}\right)\psi\left(\overset{3}{A},\ \overset{1}{B}\right)\right\|_2 = \|\varphi(\vec{A},\ \overleftarrow{B})\psi(\vec{A},\ \overleftarrow{B})T\|_2 \leqslant$$
$$\leqslant \|\varphi\|_{C(\mathbf{R}^2)}\|\psi(\vec{A},\ \overleftarrow{B})T\|_2 = \|\varphi\|_{C(\mathbf{R}^2)}\left\|\overset{2}{T}\psi\left(\overset{3}{A},\ \overset{1}{B}\right)\right\|_2.$$

From Lemma 2.1 it follows that the kernel of the homomorphism $\mathscr{M}$ is an ideal in $C(\mathbf{R}^2)$.

Definition. *The point $\lambda \in \mathbf{R}^2$ will be called a point of the resolvent set $\rho\left(\overset{3}{A},\overset{2}{T},\overset{1}{B}\right)$ of the pair of operators $\overset{3}{A},\overset{1}{B}$ with respect to the operator $\overset{2}{T}$ if there exists such a neighborhood U of this point that any function $\psi \in C_0^\infty(\mathbf{R}^2)$ with the support in U belongs to the kernel of the homomorphism $\mathscr{M}$.*

The complement of the set $\rho\left(\overset{3}{A},\overset{2}{T},\overset{1}{B}\right)$ in $\mathbf{R}^2$ will be called the spectrum $\sigma\left(\overset{3}{A},\overset{2}{T},\overset{1}{B}\right)$ of the pair of operators $\overset{3}{A},\overset{1}{B}$ with respect to $\overset{2}{T}$.

Theorem 2.1. (1) *Any function in $C(\mathbf{R}^2)$ with the support in $\rho\left(\overset{3}{A},\overset{2}{T},\overset{1}{B}\right)$ belongs to the kernel of the homomorphism $\mathscr{M}$.*

(2) *If φ belongs to the kernel of the homomorphism $\mathscr{M}$ then any point $\lambda \in \mathbf{R}^2$ for which $\varphi(\lambda) \neq 0$ belongs to $\rho\left(\overset{3}{A},\overset{2}{T},\overset{1}{B}\right)$.*

Proof. Let $f \in C(\mathbf{R}^2)$, $\operatorname{supp} f \subset \rho\left(\overset{3}{A},\overset{2}{T},\overset{1}{B}\right)$. Note that $\rho\left(\overset{3}{A},\overset{2}{T},\overset{1}{B}\right)$ is the maximal open set in $\mathbf{R}^2$ which satisfies the condition

$$\left[\varphi \in C_0^\infty(\mathbf{R}^2) \text{ and } \operatorname{supp}\varphi \subset \rho\left(\overset{3}{A},\ \overset{2}{T},\ \overset{1}{B}\right)\right] \Rightarrow \left[\overset{2}{T}\varphi\left(\overset{3}{A},\ \overset{1}{B}\right) = 0\right].$$

Suppose the support f is compact. Then there exists such a sequence $\{f_n\}$ of functions in C_0^∞ that $\operatorname{supp} f_n \subset \rho\left(\overset{3}{A},\overset{2}{T},\overset{1}{B}\right)$ and $\lim\limits_{n\to\infty}\|f_n(x) - f(x)\|_{C(\mathbf{R}^2)} = 0$. For this reason

$$\overset{2}{T}f\left(\overset{3}{A},\overset{1}{B}\right) = \lim_{n\to\infty}\overset{2}{T}f_n\left(\overset{3}{A},\overset{1}{B}\right) = 0.$$

Remove the assumption that $\operatorname{supp} f$ is compact. Consider the linear manifold L of vectors of the form

$$q = \psi(B)h, \quad \psi \in \bar{C}_0^\infty(\mathbf{R}),$$

which is dense in H. Obviously it suffices to prove that the restriction of the operator $\overset{2}{T}f\left(\overset{3}{A}, \overset{1}{B}\right)$ on L is equal to zero. Let $q = \psi(B)h$, $\psi \in C_0^\infty(\mathbf{R})$ and let $\{\varphi_n\}$ be such a sequence of functions in $C_0^\infty(\mathbf{R})$ that $\varphi_n(A)$ converges point-by-point to the unit operator. We have

$$\overset{2}{T}f\left(\overset{3}{A}, \overset{1}{B}\right) q = \overset{2}{T}f\left(\overset{3}{A}, \overset{1}{B}\right) \psi\left(\overset{1}{B}\right) h =$$
$$= \lim_{n\to\infty} \overset{2}{T}\varphi_n\left(\overset{3}{A}\right) f\left(\overset{3}{A}, \overset{1}{B}\right) \psi\left(\overset{1}{B}\right) h = 0,$$

since the support of the function $\varphi_n(x_1) f(x_1, x_2) \psi(x_2)$ is compact and lies in $\rho\left(\overset{3}{A}, \overset{2}{T}, \overset{1}{B}\right)$. Item (1) is proved.

Now let $\overset{2}{T}\varphi\left(\overset{3}{A}, \overset{1}{B}\right) = 0$ and $\varphi(\lambda) \neq 0$, λ being a point in $\mathbf{R}^2$. Consider the function $e_\lambda \in C_0^\infty(\mathbf{R}^2)$ which is equal to zero outside the ε-neighborhood of the point λ. For sufficiently small $\varepsilon > 0$ the function $F = e_\lambda/\varphi$ belongs to $C(\mathbf{R}^2)$. But $e_\lambda = \varphi F$. Hence the function φ belongs to the kernel of the homomorphism $\mathscr{M}$. Since the kernel of the homomorphism $\mathscr{M}$ is an ideal in $C(\mathbf{R}^2)$, $\overset{2}{T}e_\lambda\left(\overset{3}{A}, \overset{1}{B}\right) = 0$. The theorem is proved.

Theorem 2.2. *The point* $\lambda = (\lambda_1, \lambda_2)$ *belongs to* $\rho\left(\overset{3}{A}, \overset{2}{T}, \overset{1}{B}\right)$ *if and only if there exist functions* $\varphi_1, \varphi_2 \in C_0^\infty(\mathbf{R})$ *such that* $\varphi_1(\lambda_1) \neq 0$, $\varphi_2(\lambda_2) \neq 0$ *and* $\varphi_1(A)\, T\, \varphi_2(B) = 0$.

Proof. Let $\lambda = (\lambda_1, \lambda_2) \in \mathbf{R}^2$ and there exist functions $\varphi_1, \varphi_2 \in C_0^\infty(\mathbf{R})$ such that $\varphi_1(\lambda_1) \neq 0$, $\varphi_2(\lambda_2) \neq 0$ and $\varphi_1(A)\, T\varphi_2(B) = 0$. Prove that $\lambda \in \rho\left(\overset{3}{A}, \overset{2}{T}, \overset{1}{B}\right)$. The inverse statement follows directly from the definition. It is obvious that there exists such a neighborhood U of the point λ in which the function $\varphi_1(\lambda_1)\varphi_2(\lambda_2)$ does not vanish. Let ψ be an arbitrary function in $C_0^\infty(\mathbf{R}^2)$ with the support in U. Denote $\chi(x) = \dfrac{\psi(x)}{\varphi_1(x_1)\varphi_2(x_2)}$. Then $\chi \in C_0^\infty(\mathbf{R}^2)$ and $\psi(x) = \chi(x)\varphi_1(x_1)\varphi_2(x_2)$. Since the kernel of the homomorphism $\mathscr{M}$ is an ideal in $C(\mathbf{R}^2)$, $\overset{2}{T}\psi\left(\overset{3}{A}, \overset{1}{B}\right) = 0$. The theorem is proved.

Corollary. (1) $\sigma\left(\overset{3}{A}, \overset{2}{T}, \overset{1}{B}\right) \subset \sigma(A) \times \sigma(B)$;

(2) *the point* (λ_1, λ_2) *belongs to* $\sigma\left(\overset{3}{A}, \overset{2}{T}, \overset{1}{B}\right)$ *if and only if the point* (λ_2, λ_1) *belongs to* $\sigma\left(\overset{3}{B}, \overset{2}{T^*}, \overset{1}{A}\right)$.

Let $C(\rho)$ be a set of all complex functions continuous in $\mathbf{R}^2$ with supports in the open set $\rho \subset \mathbf{R}^2$. $C(\rho)$ is an ideal in $C(\mathbf{R}^2)$. Consider the Banach factor-algebra $C(\mathbf{R}^2)/\overline{C(\rho)}$. Denote by σ the complement to ρ in $\mathbf{R}^2$. Let $C(\sigma)$ be an algebra of continuous complex functions in σ with the ordinary norm

$$\| f \|_{C(\sigma)} = \sup_{x \in \sigma} | f(x) |.$$

If f_1 and f_2 are two functions in $C(\mathbf{R}^2)$ which correspond to the same element of $C(\mathbf{R}^2)/\overline{C(\rho)}$ then the shrinkage of the function f_1 coincides with the shrinkage of the function f_2 into σ. This means that the mapping of the shrinkage

$$f \mapsto f|_\sigma$$

induces the mapping $\pi_0 : C(\mathbf{R}^2)/\overline{C(\rho)} \to C(\sigma)$, which is the mapping onto all $C(\sigma)$ because any function continuous in σ may be extended to the function which is continuous in $\mathbf{R}^2$.

Lemma 2.2. *The mapping π_0 preserves the norm.* The proof is left to the reader.

From Lemma 2.2 it follows that π_0 is an isomorphism. By means of this isomorphism let us identify the algebras $C(\mathbf{R}^2)/\overline{C(\rho)}$ and $C(\sigma)$.

Theorem 2.3. *Let $\pi : f \mapsto f|_{\sigma\left(\overset{3}{A}, \overset{2}{T}, \overset{1}{B}\right)}$ be the projection of $C(\mathbf{R}^2)$ into $C\left(\sigma\left(\overset{3}{A}, \overset{2}{T}, \overset{1}{B}\right)\right)$. Then the following expansion is valid:*

$$\mathscr{M} = \mathscr{M}_\sigma \pi. \tag{2.1}$$

Here $\mathscr{M}_\sigma : C\left(\sigma\left(\overset{3}{A}, \overset{2}{T}, \overset{1}{B}\right)\right) \to B_2(H)$ is the homomorphism of Banach spaces defined uniquely by (2.1).

The estimates are valid:

$$\| \mathscr{M}_\sigma \| \leqslant \| T \|_2, \quad \| \mathscr{M}_\sigma \varphi\psi \|_2 \leqslant \| \varphi \|_{C\left(\sigma\left(\overset{3}{A}, \overset{2}{T}, \overset{1}{B}\right)\right)} \| \mathscr{M}_\sigma \psi \|_2.$$

The proof is left to the reader.

Finally, we shall consider the following theorem which is important for some applications.

Theorem 2.4. *Let φ be a real-valued continuous function in $\sigma\left(\overset{3}{A}, \overset{2}{T}, \overset{1}{B}\right)$. Then $\left\| \overset{2}{T} e^{i\varphi\left(\overset{3}{A}, \overset{1}{B}\right)} \right\|_2 = \| T \|_2$.*

This theorem follows directly from the unitariness of the operator $e^{i\varphi(\overrightarrow{A}, \overleftarrow{B})}$ (see the preceding chapter and Theorem 4.3).

Sec. 3. The Functions of Noncommutative Operators

Let the space $C_{s_1, \ldots, s_N}(\mathbf{R}^N, B)$ be defined similarly to the definition of $C_s(\mathbf{R}, B)$ in Chapter I. If $f \in C^+_{s_1, \ldots, s_N}(\mathbf{R}^N)$ and $G \in C_{s_1, \ldots, s_N}(\mathbf{R}^N, B)$ then the integral

$$\int_{\mathbf{R}^N} f(x)\, G(x)\, dx$$

will be understood in the same sense as in the case $N = 1$.

Let there exist a vector-operator (1.1). Set for any $\varphi \in \mathscr{B}_{s_1, \ldots, s_N}(\mathbf{R}^N)$

$$\overset{2}{T}{}^{(1)}\overset{4}{T}{}^{(2)} \ldots \overset{2N-2}{T}{}^{N-1} \varphi\left(\overset{1}{A_1}, \overset{3}{A_2}, \ldots, \overset{2N-1}{A_N}\right) h \overset{\text{def}}{=\!=}$$

$$= \frac{1}{(2\pi)^{N/2}} \int_{\mathbf{R}^N} \tilde{\varphi}(t)\, U(t)\, h\, dt, \tag{3.1}$$

where $\tilde{\varphi} = F^{-1}\varphi$ and $U(t)$ is a function of the operator (1.2).

Lemma 3.1. *The following estimate is valid:*

$$\left\| \overset{2}{T}{}^{(1)} \ldots \overset{2N-2}{T}{}^{(N-1)} \varphi\left(\overset{1}{A_1}, \ldots, \overset{2N-1}{A_N}\right) \right\|_{B_\tau \to B_{\tau+k_1+\ldots+k_N+l_1\ldots+l_{N-1}}} \leqslant$$

$$\leqslant c \left\| T^{(1)} \right\|_{B_{\tau+k_1} \to B_{\tau+k_1+l_1}} \times \ldots$$

$$\ldots \times \left\| \overset{2N-2}{T} \right\|_{B_{\tau+k_1+\ldots+k_{N-1}+l_1+\ldots+l_{N-2}} \to B_{\tau+k_1+\ldots+k_{N-1}+l_1+\ldots+l_{N-1}}} \times$$

$$\times \left\| \varphi \right\|_{\mathscr{B}_{s_1, \ldots, s_N}},$$

where c depends only on τ and on the operators $A_1, \ldots, A_N$.

The proof follows immediately from the inequality

$$\left\| \int_{\mathbf{R}^N} f(x)\, G(x)\, dx \right\|_B \leqslant \| f \|_{C_{s_1, \ldots, s_N}(\mathbf{R}^N)} \| G \|_{C_{s_1, \ldots, s_N}(\mathbf{R}^N, B)}.$$

Note. It is obvious that in the above constructions one may consider the degrees s_i of the generators $A_i : B_\tau \to B_{\tau+k_i}$ as functions of τ. We have not reflected this in the notation for the sake of simplicity. In every formula it is clear at what point τ one has to consider s_i. An alternative approach is to consider the Banach scales $\{B_\tau\}$ where τ varies only over a finite set of values. Then one may choose the degrees s_i independent of τ, but the operator

$$\overset{2}{T}{}^{(1)} \ldots \overset{2N-2}{T}{}^{(N-1)} \varphi\left(\overset{1}{A_1}, \ldots, \overset{2N-1}{A_n}\right) : B_\tau \to$$

$$\to B_{\tau+k_1+\ldots+k_N+l_1+\ldots+l_{N-1}}$$

will now be defined only for certain values of τ.

Example. Let

$$B_\tau = W_2^{-\tau}(\mathbf{R}), \quad D = C_0^\infty(\mathbf{R}),$$
$$A_2\varphi(x) = x\varphi(x), \quad A_1\varphi(x) = -i\varphi'(x),$$
$$T\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx}\sqrt{1+p^{2m}}\,\tilde{\varphi}(p)\,dp, \quad \varphi \in C_0^\infty(\mathbf{R}).$$

Then

$$k_1 = k_2 = 0, \quad l = m, \quad s_1 = 0, \quad s_2(\tau) = \begin{cases} -\tau \text{ for } \tau < 0, \\ 0 \text{ for } \tau \geqslant 0. \end{cases}$$

It is not difficult to verify that

$$\overset{2}{T}_\varphi\left(\overset{3}{A_2}, \overset{1}{A_1}\right)\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx}\varphi\;(x, p)\sqrt{1+p^{2m}}\,\tilde{\psi}(p)\,dp,$$

where $\tilde{\psi} = F\psi$. We shall denote

$$\overset{2}{T}_\varphi\left(\overset{3}{A_2}, \overset{1}{A_1}\right)\psi(x) = L\left(\overset{2}{x}, -i\overset{1}{\frac{\partial}{\partial x}}\right)\psi(x),$$

where $L(p, x) = \varphi(p, x)\sqrt{1+p^{2m}}$. By virtue of Lemma 3.1 we have

$$\left\| L\left(\overset{2}{x}, -i\overset{1}{\frac{\partial}{\partial x}}\right)\right\|_{W_2^r(\mathbf{R}) \to W_2^{r-m}(\mathbf{R})} \leqslant c\,\|\varphi\|_{\mathscr{B}_{r-m,\,0}(\mathbf{R}^2)},$$

where

$$\varphi(x_1\; x_2) = \frac{L(x_1, x_2)}{\sqrt{1+x_2^{2m}}}, \quad c = \text{const}.$$

Problem. Let $\varphi \in \mathscr{B}_{l,\,0}(\mathbf{R}^2)$,

$$L(x, p) = \varphi(x, p)\sqrt{1+p^{2m}}.$$

Set

$$L\left(\overset{2}{x}, \overset{1}{p}\right)U(x) \overset{\text{def}}{=}$$
$$= \frac{1}{2\pi h}\int_{-\infty}^{\infty} e^{\frac{i}{h}p\cdot x} L(x, p)\,dp \int_{-\infty}^{\infty} e^{-\frac{i}{h}p\cdot\xi}\,U(\xi)\,d\xi, \quad U \in C_0^\infty(\mathbf{R}).$$

Let S be an infinitely differential real-valued finite function. Define the operator $T \overset{\text{def}}{=} \sqrt{1+(p+S')^{2m}}$ in the following way. Consider the generator $A \overset{\text{def}}{=} p + S'$:

$$AU(x) = -ihU'(x) + S'(x)\,U(x).$$

Set $T_n(\xi)=\varphi_n(\xi)\sqrt{1+\xi^{2m}}$, where $\{\varphi_n\}$ is a sequence of functions in $C_0^\infty(\mathbf{R})$ defined in Lemma 3.13 of Chapter I.

Let $TU=\lim\limits_{n\to\infty}T_n(\bar{A})U$.

(1) Prove that T is a translator with the step m in $\{W_2^{-\tau}(\mathbf{R})\}$.

(2) Prove the equality

$$e^{-\frac{i}{h}S(x)}L\left(\overset{2}{x},\ \overset{1}{p}\right)e^{\frac{i}{h}S(x)}U(x)=L\left(\overset{2}{x},\ p+\overset{1}{S'}\right)U(x),$$

where

$$L\left(\overset{2}{x},\ p+\overset{1}{S'}\right)\overset{\text{def}}{=\!=}\sqrt{1+\overset{2}{(p+S')^m}}\,\varphi\left(\overset{3}{x},\ p+\overset{1}{S'}\right),\quad U\in W_2^{-\tau}(\mathbf{R}).$$

(3) Prove that this equality remains valid in the case of an arbitrary function $S\in C^\infty$, if it is to be considered for finite functions U.

Notation. If $\chi=\varphi\psi$ then we shall write

$$\overset{2}{T}{}^{(1)}\ldots\overset{2N-2}{T}{}^{(N-1)}\chi\left(\overset{1}{A}_1,\ \ldots,\ \overset{2N-1}{A_N}\right)=$$
$$=\overset{2}{T}{}^{(1)}\ldots\overset{2N-2}{T}{}^{(N-1)}\varphi\left(\overset{1}{A}_1,\ \ldots,\ \overset{2N-1}{A_N}\right)\times\psi\left(\overset{1}{A}_1,\ \ldots,\ \overset{2N-1}{A_N}\right).$$

Lemma 3.2. *Let* $\varphi(x)=\varphi_1(x_1)\varphi_2(x_2)\ldots\varphi_N(x_N)$, *where* $\varphi_i\in\mathscr{B}_{s_i}(\mathbf{R})$. *Then*

$$\overset{2}{T}{}^{(1)}\ldots\overset{2N-2}{T}{}^{(N-1)}\varphi\left(\overset{1}{A}_1\ \ldots,\ \overset{2N-1}{A_N}\right)=$$
$$=\varphi_N(A_N)T^{(N-1)}\varphi_{N-1}(A_{N-1})\ldots T^{(1)}\varphi_1(A_1).$$

The proof is easily obtained from (3.1).

Lemma 3.3. *The linear span of the set of functions* ψ *of the form*

$$\psi(x)=\psi_1(x_1)\ldots\psi_N(x_N),\qquad \psi_i\in\mathscr{B}_{s_i}(\mathbf{R})\tag{3.2}$$

is dense in $\mathscr{B}_{s_1,\ldots s_N}(\mathbf{R}^N)$.

Proof. It suffices to show that by means of linear combinations of functions of the form (3.2) it is possible to approximate in the space $\mathscr{B}_{s_1,\ldots,s_N}(\mathbf{R}^N)$ the Fourier transforms of functions in $C_0^\infty(\mathbf{R}^N)$ and also the exponentials

$$x\to e^{-ia_1x_1-\ldots-ia_Nx_N}.$$

For the exponentials, this is obvious because

$$e^{-ia_1x_1-\ldots-ia_Nx_N}=e^{-ia_1x_1}\ldots e^{-ia_Nx_N}.$$

Now let φ be the Fourier transform of a function in $C_0^\infty(\mathbf{R}^n)$. Then $\varphi\in W_2^l(\mathbf{R}^N)$ for any l. Any function in the Sobolev space may

be approximated in the norm of this space with any degree of accuracy by means of linear combinations of products of the form (3.2), where all ψ_i belong to $C_0^\infty(\mathbf{R})$ (prove it). To end the proof it remains to make use of embedding Theorem 2.1 of Chapter I and 1.2 of Chapter II.

Lemmas 3.1 and 3.2 signify the existence of the homomorphism of the Banach spaces:

$$\mathcal{M}\colon\ \mathcal{B}_{s_1,\ldots,s_N}(\mathbf{R}^N)\to \operatorname{Hom}(B_\tau,\ B_{\tau+k_1+\ldots+k_N+l_1+\ldots+l_{N-1}}),$$

which is defined by (3.1) and which translates the function φ into the operator

$$\overset{2}{T}{}^{(1)}\ldots\overset{2N-2}{T}{}^{(N-1)}\varphi\left(\overset{1}{A_1},\ \ldots,\ \overset{2N-1}{A_N}\right);$$

besides, it has the property

$$\overset{2}{T}{}^{(1)}\ldots\overset{2N-2}{T}{}^{(N-1)}\varphi_1\left(\overset{1}{A_1}\right)\times\varphi_2\left(\overset{3}{A_2}\right)\times\ldots\times\varphi_N\left(\overset{2N-1}{A_N}\right)=$$

$$=\varphi_N(A_N)\,T^{(N-1)}\,\varphi_{N-1}(A_{N-1})\,\ldots\,T^{(1)}\,\varphi_1(A). \tag{3.3}$$

It follows from Lemma 3.3 that this homomorphism is uniquely defined by the property (3.3).

Thus the following theorem is valid.

Theorem 3.1. *There exists a unique homomorphism*

$$\mathcal{M}:\varphi\to\overset{2}{T}{}^{(1)}\ldots\overset{2N-1}{T}{}^{(N-1)}\varphi\left(\overset{1}{A_1},\ \ldots,\ \overset{2N-1}{A_N}\right)$$

of the Banach space $\mathcal{B}_{s_1,\ldots,s_N}(\mathbf{R}^N)$ into a Banach space of everywhere defined linear bounded operators from B_τ into $B_{\tau+k_1+\ldots+k_N+l_1+\ldots+l_{N-1}}$ with the property (3.3).

Sec. 4. The Spectrum of a Vector-Operator

Let us make further agreements as to the notation. If a translator T is identical, then we shall omit it in the notation of functions of a vector-operator and number all other operators in succession. For example, instead of $\overset{2}{I}\varphi\left(\overset{1}{A},\overset{3}{A'}\right)$ we shall write $\varphi\left(\overset{1}{A},\overset{2}{A'}\right)$.

Let $O\colon D_O\to B_\tau$, where $D_O\supset D$. Then the norm of the shrinkage $O\,|_D$, which is considered as the operator from $B_{\tau'}$ into $B_{\tau''}$, will be denoted by $\|\,O\,\|_{\tau'\to\tau''}$.

The following lemma is obtained directly from the definitions.

Lemma 4.1. *Let A_i, $i=1,\ldots,N$ be generators of degree s_i with step k_i on the scale $\{B_\tau\}$ and let $T^{(i)}$, $i=1,\ldots,N-1$ be translators. Let A'_i be the operator A_i considered as a generator with*

step $k_i + \mu_i$. *Then for any* $\psi \in \mathscr{B}_{s_1, \dots, s_N}(\mathbf{R}^N)$ *and* $h \in D$ *the following equality is valid*:

$$\overset{2}{T}{}^{(1)} \dots \overset{2N-2}{T'}{}^{(N-1)} \psi\left(\overset{1}{A_1}, \dots, \overset{2N-1}{A_N}\right) h =$$
$$= \overset{2}{T}{}^{(1)} \dots \overset{2N-2}{T}{}^{(N-1)} \psi\left(\overset{1}{A'_1}, \dots, \overset{2N-1}{A'_N}\right) h.$$

Lemma 4.2. *Let*

$$\left\| \overset{2}{T}{}^{(m)} \varphi\left(\overset{1}{A_m}, \overset{3}{A_{m+1}}\right) \right\|_{\substack{\tau+k_1+\dots+k_m+l_1+\dots+l_{m-1} \to \\ \to \tau+k_1+\dots+k_m+l_1+\dots+l_m}} = M < \infty.$$

Then the following estimate is valid:

$$\left\| \overset{2}{T}{}^{(1)} \dots \overset{2N-2}{T}{}^{(N-1)} \varphi\left(\overset{2m-1}{A_m}, \overset{2m+1}{A_{m+1}}\right) \times \psi\left(\overset{1}{A_1}, \dots, \overset{2N-1}{A_N}\right) \right\| \leqslant$$
$$\leqslant cM \|\psi\|_{\mathscr{B}_{s_1, \dots, s_N}(\mathbf{R}^n)} \prod_{\substack{\alpha=1 \\ \alpha \neq m}}^{N-1} \|T^{(\alpha)}_{\tau+k_1+\dots+k_\alpha+l_1+\dots+l_{\alpha-1}}\|.$$

Proof. We shall prove this lemma for $N=2$. Namely, prove the following estimate

$$\left\| \overset{2}{T}(\varphi\psi)\left(\overset{1}{A_1}, \overset{3}{A_2}\right) \right\|_{\tau \to \tau+k_1+k_2+l} \leqslant$$
$$\leqslant c \left\| \overset{2}{T}\varphi\left(\overset{1}{A_1}, \overset{3}{A_2}\right) \right\|_{\tau+k_1 \to \tau+k_1+l} \|\psi\|_{\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)}.$$

Let $T'\colon B_{\tau+k_1} \to B_{\tau+k_1+l_1}$ be the closure of the shrinkage of the operator $\overset{2}{T}\varphi\left(\overset{1}{A_1}, \overset{3}{A_2}\right)$ on D. Then

$$\overset{2}{T}(\varphi\psi)\left(\overset{1}{A_1}, \overset{3}{A_2}\right) = \overset{2}{T'}\psi\left(\overset{1}{A_1}, \overset{3}{A_2}\right).$$

Indeed, if $h \in D$ then

$$\overset{2}{T'}\psi\left(\overset{1}{A_1}, \overset{3}{A_2}\right) h = \frac{1}{2\pi} \int_{\mathbf{R}^2} \widetilde{\psi}(t_1, t_2)\, U_{A_2}(t_2)\, T' U_{A_1}(t_1)\, h\, dt =$$
$$= \frac{1}{2\pi} \int_{\mathbf{R}^2} \widetilde{\psi}(t_1, t_2)\, U_{A_2}(t_2) \left[\overset{2}{T}\varphi\left(\overset{1}{A_1}, \overset{3}{A_2}\right)\right] U_{A_1}(t_1)\, h\, dt =$$
$$= \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} \widetilde{\psi}(t_1, t_2)\, dt \int_{\mathbf{R}^2} \widetilde{\varphi}(\tau_1, \tau_2)\, U_{A_2}(\tau_2 + t_2)\, T U_{A_1} \times$$

$$\times (\tau_1 + t_1)\, h\, d\tau = \frac{1}{2\pi} \int_{\mathbf{R}^2} (\varphi * \psi)\ (t_1,\ t_2)\ U_{A_2}(t_2)\ T U_{A_1}(t_1)\, h\, dt =$$

$$= \overset{2}{T}(\varphi\psi)\left(\overset{1}{A_1},\ \overset{3}{A_2}\right) h.$$

To complete the proof it remains to make use of Lemma 3.1.

The following theorem is proved in exactly the same way as Lemma 4.2.

Theorem 4.1. *For any integer l' the following estimate is valid:*

$$\left\| \overset{2}{T}(\varphi\psi)\left(\overset{1}{A_1},\ \overset{3}{A_2}\right) \right\|_{\tau \to \tau + k_1 + k_2 + l'} \leqslant$$
$$\leqslant c \left\| \overset{2}{T}\varphi\left(\overset{1}{A_1},\ \overset{3}{A_2}\right) \right\|_{\tau + k_1 \to \tau + k_1 + l'} \| \psi \|_{\mathscr{B}_{s_1,\, s_2}(\mathbf{R}^2)}.$$

Lemma 4.3. *There exists a set of functions $\psi \in \mathscr{B}_{s,\, s'}(\mathbf{R}^2)$ such that $\overset{2}{T}\psi\left(\overset{1}{A},\ \overset{3}{A'}\right) = 0$ is an ideal in the algebra $\mathscr{B}_{s,\, s'}(\mathbf{R}^2)$.*

Proof. Let $h \in B_\tau$. Then for any $\varphi \in \mathscr{B}_{s,\, s'}(\mathbf{R}^2)$

$$\left\| \overset{2}{T}\psi\left(\overset{1}{A},\ \overset{3}{A'}\right) \times \varphi\left(\overset{1}{A},\ \overset{3}{A'}\right) h \right\|_{\tau + k + l + k'} \leqslant$$
$$\leqslant c \| \varphi \|_{\mathscr{B}_{s_1,\, s_2}(\mathbf{R}^2)} \cdot \left\| \overset{2}{T}\psi\left(\overset{1}{A},\ \overset{3}{A'}\right) \right\|_{\tau + k \to \tau + k + l} \| h \|_\tau = 0.$$

Consequently,

$$\overset{2}{T}\psi\left(\overset{1}{A},\ \overset{3}{A'}\right) \times \varphi\left(\overset{1}{A},\ \overset{3}{A'}\right) = 0$$

for any $\varphi \in \mathscr{B}_{s,\, s'}(\mathbf{R}^2)$.

Let $\mathscr{X} = \left(\overset{1}{A_1},\ \overset{2}{T^{(1)}},\ \overset{3}{A_2},\ \ldots,\ \overset{2N-2}{T}{}^{(N-1)},\ \overset{2N-1}{A_N}\right)$ be a vector-operator. In this case we denote by X an ideal in the algebra $\mathscr{B}_{s_1,\, \ldots,\, s_N}(\mathbf{R}^N)$ containing the totality of all functions φ such that

$$\overset{2}{T^{(1)}} \ldots \overset{2N-2}{T}{}^{(N-1)} (\psi\varphi)\left(\overset{1}{A_1},\ \ldots,\ \overset{2N-1}{A_N}\right) = 0,$$
$$\forall \psi \in \mathscr{B}_{s_1,\, \ldots,\, s_N}(\mathbf{R}^N).$$

Lemma 4.3 means that in the case $N = 2$ the ideal X coincides with the kernel of the homomorphism

$$M : \varphi \to \overset{2}{T}\varphi\left(\overset{1}{A_1},\ \overset{3}{A_2}\right).$$

Definition. *A point* $\lambda \in \mathbf{R}^N$ *will be called a point of the resolvent set of the vector-operator*

$$\mathscr{X} = \left(\overset{1}{A}_1, \overset{2}{T}{}^{(1)}, \overset{3}{A}_2, \ldots, \overset{2N-3}{A}_{N-1}, \overset{2N-2}{T}{}^{(N-1)}, \overset{2N-1}{A}_N\right),$$

if there exists such a neighborhood U *of this point that any function* $\varphi \in C_0^\infty(\mathbf{R}^N)$ *with the support in* U *belongs to the ideal* X *constructed according to the homomorphism*

$$M: \psi \to \overset{2}{T}{}^{(1)} \ldots \overset{2N-2}{T}{}^{(N-1)} \psi\left(\overset{1}{A}_1, \ldots, \overset{2N-1}{A}_N\right).$$

The complement to the resolvent set will be called the spectrum σ *of the vector-operator* $\mathscr{X}$.

Lemma 4.4. *Let the function* $\varphi \in C^\infty(\mathbf{R}^N)$ *belong to the ideal* X *constructed according to the homomorphism*

$$\psi \to \overset{2}{T}{}^{(1)} \ldots \overset{2N-2}{T}{}^{(N-1)} \psi\left(\overset{1}{A}_1, \ldots, \overset{2N-1}{A}_N\right).$$

Then any point at which the function φ *is non-zero belongs to the resolvent set of the vector-operator*

$$\mathscr{X} = \left(\overset{1}{A}_1, \overset{2}{T}{}^{(1)}, \ldots, \overset{2N-1}{A}_N\right).$$

Proof. Let $\varphi(\lambda) \neq 0$ and let e_λ be a function in $C_0^\infty(\mathbf{R}^N)$ which is equal to unity in the δ_1-neighborhood of the point λ and vanishes outside the δ_2-neighborhood of this point. For sufficiently small δ_2 the function $\mathscr{F} = e_\lambda/\varphi$ belongs to $C_0^\infty(\mathbf{R}^N)$. But $e_\lambda = \varphi \cdot \mathscr{F}$. Hence the function φ is included in the ideal X; consequently, $e_\lambda \in X$. Let $\chi \in C_0^\infty(\mathbf{R}^2)$ and let the support of the function χ lie in the δ_1-neighborhood of the point λ. Then $\chi(x) = \chi(x) e_\lambda(x)$, i. e. $\chi \in X$, Q.E.D.

In the case $N = 2$ Lemmas 4.3 and 4.4 result in the criterion that follows.

Treorem 4.2. *For a point* $\lambda \in \mathbf{R}^2$ *to belong to the resolvent set of the vector-operator* $\left(\overset{1}{A}, \overset{2}{T}, \overset{3}{A'}\right)$ *it is necessary and sufficient that there exist such functions* φ_1 *and* φ_2 *in* $C_0^\infty(\mathbf{R})$ *that* $\varphi_1(\lambda_1) \neq 0$, $\varphi_2(\lambda_2) \neq 0$, *and, at the same time,* $\varphi_2(A') T \varphi_1(A) = 0$.

Let $A = (A_1, \ldots, A_k)$ be a generating set of degree s with the defining pair of spaces (B_1, B_2). Let T be a homomorphism $T: B_2 \to B_3$ and $A' = (A'_1, \ldots, A'_{k'})$ be a generating set of degree s' with the defining pair of spaces (B_3, B_4). We shall assume that there exist embeddings $B_1 \subset B_2$ and $B_3 \subset B_4$.

Let $\{U(t)\}$ be the k-parameter group generated by the set A, and let $\{V(t')\}$ be the k'-parameter group generated by the set A'.

In analogy with the above procedure we shall define a Banach algebra $\mathscr{B}_{s,\,s'}(\mathbf{R}^k \times \mathbf{R}^{k'})$ on the basis of the space $C_{s,\,s'}(\mathbf{R}^k \times \mathbf{R}^{k'})$ of functions continuous in $\mathbf{R}^k \times \mathbf{R}^{k'}$ with a finite norm

$$\|f\|_{C_{s,\,s'}(\mathbf{R}^k\times\mathbf{R}^{k'})} = \sup_{\substack{x\in\mathbf{R}^k\\ y\in\mathbf{R}^{k'}}} \frac{|f(x,\,y)|}{(1+|x|)^s(1+|y|)^{s'}}.$$

If $\varphi \in \mathscr{B}_{s,\,s'}(\mathbf{R}^k \times \mathbf{R}^{k'})$ and $h \in B_1$, then set by definition

$$\overset{2}{T}\varphi\left(\overset{1}{A},\ \overset{3}{A'}\right)h = (2\pi)^{-\frac{k+k'}{2}} \int\limits_{\mathbf{R}^{k+k'}} \widetilde{\varphi}(t,\ \tau)\, e^{-iA'\tau} T e^{-iAt} h\, dt\, d\tau,$$

where $e^{-iAt}\colon B_1 \to B_2$ is the closure of the operator $U(t)$; $e^{-iA't}\colon B_3 \to B_4$ is the closure of the operator $V(t)$.

Definition. *The point $(\lambda,\ \mu) \in \mathbf{R}^k \times \mathbf{R}^{k'}$ will be called a point of the resolvent set of the pair of generating sets $\overset{1}{A}$, $\overset{3}{A'}$ with respect to the translator $\overset{2}{T}$ if there exists such a neighborhood U of this point that any function $\varphi \in C_0^\infty(\mathbf{R}^{k+k'})$ with the support in U belongs to the following ideal X of the algebra $\mathscr{B}_{s,\,s'}(\mathbf{R}^{k+k'})$ constructed according to the homomorphism*

$$\mathscr{M}: \psi \to \overset{2}{T}\psi\left(\overset{1}{A},\ \overset{3}{A'}\right)$$

of the Banach algebra $\mathscr{B}_{s,\,s'}(\mathbf{R}^{k+k'})$ into the Banach space Hom $(B_1,\ B_2)$:

$$X = \{\varphi \mid \mathscr{M}\varphi\psi = 0,\quad \forall\psi \in \mathscr{B}_{s,\,s'}(\mathbf{R}^{k+k'})\}.$$

The complement to the resolvent set will be called the spectrum σ of the pair $\overset{1}{A}$, $\overset{3}{A'}$ with respect to the operator $\overset{2}{T}$.

Let $(\lambda,\ \mu)$ be a point of the resolvent set of the pair $\overset{1}{A}$, $\overset{3}{A'}$ with respect to $\overset{2}{T}$, and φ be a function which is infinitely differentiable everywhere except the point $(\lambda,\ \mu)$. Next, let U be a neighborhood of the point $(\lambda,\ \mu)$ mentioned in the definition, and let infinitely differentiable in U functions $\overline{\varphi}$ and $\overline{\overline{\varphi}}$ belonging to $\mathscr{B}_{s,\,s'}(\mathbf{R}^{k+k'})$ coincide with the function φ outside some closed set in U. Then $\overline{\varphi} - \overline{\overline{\varphi}} \in C_0^\infty(\mathbf{R}^{k+k'})$ and $\operatorname{supp}(\overline{\varphi} - \overline{\overline{\varphi}}) \subset U$ so that the function belongs to the ideal X. For brevity we shall denote the element $\{\overline{\varphi}\} = \{\overline{\overline{\varphi}}\}$ of the factor-algebra $\mathscr{B}_{s,\,s'}(\mathbf{R}^{k+k'})/X$ by $\{\varphi\}$ (sometimes even by φ) though φ does not necessarily belong to $\mathscr{B}_{s,\,s'}(\mathbf{R}^{k+k'})$. For the homomorphism $\mathscr{M}$, the following expansion is valid:

$$\mathscr{M} = \pi\mathscr{M}_\sigma,$$

where

$$\mathscr{B}_{s,\,s'}(\mathbf{R}^{k+k'}) \xrightarrow{\pi} \mathscr{B}_{s,\,s'}(\mathbf{R}^{k+k'})/X \xrightarrow{\mathscr{M}_\sigma} \operatorname{Hom}(B_1,\ B_4).$$

For this reason

$$\overset{2}{T}\overline{\varphi}\left(\overset{1}{A},\ \overset{3}{A'}\right) = \overset{2}{T}\overline{\overline{\varphi}}\left(\overset{1}{A},\ \overset{3}{A'}\right).$$

We shall denote this operator also by $\overset{2}{T}\varphi\left(\overset{1}{A},\ \overset{3}{A'}\right)$ and the function φ will be called its *symbol.*

Lemma 4.5. *Let the support of a function* $\psi \in \mathscr{B}_{s,\,s'}(\mathbf{R}^k \times \mathbf{R}^{k'})$ *lie in the resolvent set of the pair* $\overset{1}{A}$, $\overset{3}{A'}$ *with respect to* $\overset{2}{T}$. *Then* ψ *belongs to the ideal* X *constructed according to the homomorphism* $\mathscr{M}$.

Proof. First of all, we shall note that if the point $(\lambda,\ \mu)$ belongs to the resolvent set of the pair $\overset{1}{A}$, $\overset{3}{A'}$ with respect to $\overset{2}{T}$, then any function in $\mathscr{B}_{s,\,s'}(\mathbf{R}^k \times \mathbf{R}^{k'})$ with the support in a sufficiently small neighborhood of the point $(\lambda,\ \mu)$ belongs to X. This follows from the fact that such a function may be precisely approximated in $\mathscr{B}_{s,\,s'}(\mathbf{R}^k,\ \mathbf{R}^{k'})$ by the function in $C_0^\infty(\mathbf{R}^k \times \mathbf{R}^{k'})$ having the support in a small neighborhood of the point $(\lambda,\ \mu)$.

First of all, suppose that the function ψ is finite. Then it can be represented in the form of a sum of a finite number of functions in X and, consequently, it belongs to X itself.

Now let ψ be an arbitrary function satisfying the conditions of the lemma. There exists a sequence $\{l_n\} \subset C_0^\infty(\mathbf{R}^k \times \mathbf{R}^{k'})$ such that

$$\lim_{n\to\infty}(2\pi)^{-\frac{k+k'}{2}} \int\limits_{\mathbf{R}^{k+k'}} (F^{-1}l_n)(t)\, g(t)\, dt = g(0)$$

for any $g \in C_{s,\,s'}(\mathbf{R}^k \times \mathbf{R}^{k'})$. Set $\psi_n = \psi l_n$. We have $\operatorname{supp}\psi_n \subset \subset \operatorname{supp}\psi$. For this reason $\overset{2}{T}\psi_n\left(\overset{1}{A},\ \overset{3}{A'}\right) = 0$. Next

$$\lim_{n\to\infty}(2\pi)^{-\frac{k+k'}{2}} \int\limits_{\mathbf{R}^{k+k'}} (F^{-1}\psi_n)(t)\, g(t)\, dt =$$

$$= \lim_{n\to\infty}(2\pi)^{-\frac{k+k'}{2}} \int\limits_{\mathbf{R}^{k+k'}} [(F^{-1}\psi) * (F^{-1}l_n)](t)\, g(t)\, dt =$$

$$= \int\limits_{\mathbf{R}^{k+k'}} (F^{-1}\psi)(t)\, g(t)\, dt.$$

Hence, it follows that for any $h \in B_1$, $h^* \in B_4^*$

$$0 = \lim_{n\to\infty} h^* \left[\overset{2}{T} \psi_n \left(\overset{1}{A}, \overset{3}{A'} \right) h \right] = h^* \left[\overset{2}{T} \psi \left(\overset{1}{A}, \overset{3}{A'} \right) \right].$$

This means that $\overset{2}{T} \psi \left(\overset{1}{A}, \overset{3}{A'} \right) = 0$.

Replacing ψ by ψf in this discourse we obtain

$$\overset{2}{T} \psi \left(\overset{1}{A}, \overset{3}{A'} \right) \times f \left(\overset{1}{A}, \overset{3}{A'} \right) = 0$$

for any $f \in \mathscr{B}_{s, s'}(\mathbf{R}^{k+k'})$. This means that $\psi \in X$ and the lemma is proved.

Theorem 4.3. *Let $A = (A_1, \ldots, A_k)$ be a generating set of degree s with the defining pair of spaces (B_1, B_2), $A' = (A'_1, \ldots, A'_l)$ be a generating set of degree s' with the defining pair of spaces (B_3, B_4), let $T: B_2 \to B_3$ be a homomorphism. Let $\lambda_0 \in \mathbf{R}^k$, $\mu_0 \in \mathbf{R}^l$ and let there exist such a neighborhood U of the point (λ_0, μ_0) in $\mathbf{R}^{k+l}$ that for some $\delta > 0$ for $(\lambda, \mu) \in U$ and $\varepsilon < \varepsilon_0$ the following estimate is valid:*

$$\left\| \overset{2}{T} \left[\left(\overset{1}{A} - \lambda \right)^2 + \left(\overset{3}{A'} - \mu \right)^2 + \varepsilon^2 \right]^{-\frac{1}{2}(k+l+s+s'+\delta)} \right\|_{B_2 \to B_3} < c.$$

Then the point (λ_0, μ_0) belongs to the resolvent set of the vector-operator $\mathscr{X} = \left(\overset{1}{A}, \overset{2}{T}, \overset{3}{A'} \right)$ (c not dependent on ε)

Proof. Let $\psi \in C_0^\infty(\mathbf{R}^{k+l})$ and $\operatorname{supp} \psi \subset U$. We have to prove that $\psi(\mathscr{X}) = 0$. Without restriction of generality we set $\mu_0 = 0$, $\lambda_0 = 0$.

Let $\varphi(x, y)$, $x \in \mathbf{R}^k$, $y \in \mathbf{R}^l$ be such a function in $C_0^\infty(\mathbf{R}^{k+l})$ that

$$\sum_{i_1=-\infty}^{\infty} \cdots \sum_{i_k=-\infty}^{\infty} \sum_{j_1=-\infty}^{\infty} \cdots \sum_{j_l=-\infty}^{\infty} \varphi(x - i, y - j) = 1,$$

$$i = (i_1, \ldots, i_k), \quad j = (j_1, \ldots, j_l).$$

Then such a constant c_1 will be found that for sufficiently large n

$$\psi(x, y) = \sum_{(i, j) \in M_n} \psi(x, y)\, \varphi(nx - i, ny - j) \overset{\text{def}}{=} \sum_{(i, j) \in M_n} \psi_{ij}^{(n)}(x, y),$$

where the number of elements of the set M_n does not exceed $c_1 n^{k+l}$ and $\left(\frac{i}{n}, \frac{j}{n} \right) \in U$ for $(i, j) \in M_n$.

Denote

$$R_\varepsilon(x, y) = \left(\sqrt{x^2 + y^2 + \varepsilon^2} \right)^{k+l+s+s'+\delta}.$$

We have

$$\varphi(nx-i,\ ny-j)\,R_{1/n}\left(x-\frac{i}{n},\ y-\frac{j}{n}\right)=$$
$$=\varphi(nx-i,\ ny-j)\,R_1(nx-i,\ ny-j)\,n^{-k-l-s-s'-\delta}.$$

Hence, it follows that the estimate is valid:

$$\left\|\psi(nx-i,\ ny-j)\,R_{1/n}\left(x-\frac{i}{n},\ y-\frac{j}{n}\right)\right\|_{\mathscr{B}_{s,\ s'}(\mathrm{R}^{k+l})}\leqslant$$
$$\leqslant\|\psi R_1\|_{\mathscr{B}_{s,\ s'}(\mathrm{R}^{k+l})}\,n^{-k-l-\delta}.$$

From the identity

$$\psi_{ij}^{(n)}(x,\ y)=\left[\psi_{ij}^{(n)}(x,\ y)\,R_{1/n}\left(x-\frac{i}{n},\ y-\frac{j}{n}\right)\right]\times$$
$$\times\left[\left(x-\frac{i}{n}\right)^2+\left(y-\frac{j}{n}\right)^1+\frac{1}{n^2}\right]^{\frac{1}{2}(-k-l-s-s'-\delta)}$$

and from the estimate for the operator

$$\overset{2}{T}\left[\left(\overset{1}{A}-\frac{i}{n}\right)^2+\left(\overset{3}{A'}-\frac{j}{n}\right)^2+\frac{1}{n^2}\right]^{-\frac{1}{2}(k+l+s+s'+\delta)}$$

it follows

$$\|\psi_{ij}^{n}(\mathscr{X})\|\leqslant c_2\cdot n^{-k-l-\delta},$$

where c_2 is a constant. This means that

$$\|\psi(\mathscr{X})\|\leqslant c_2\cdot n^{-k-l-\delta}\cdot c_1 n^{k+l}=c_1c_2n^{-\delta}\to 0$$

for $n\to\infty$. The theorem is proved.

Theorem 4.4. *Let $A=(A_1,\ \ldots,\ A_k)$ be a generating set of degree s with the defining pair of spaces $(B_1,\ B_2)$, let $A'=(A'_1,\ \ldots,\ A'_l)$ be a generating set of degree s' with the defining pair of spaces $(B_3,\ B_4)$ and let $T\colon B_2\to B_3$ be a homomorphism. Suppose that for any sufficiently small real-valued $\varepsilon>0$ there exists an estimate*

$$\left\|\overset{2}{T}\left[\left(\overset{1}{A}-\lambda\right)^2+\left(\overset{3}{A'}-\mu\right)^2+\varepsilon^2\right]^{-N/2}\right\|_{B_2\to B_3}\leqslant c^N,$$

where c is a constant and N is any integer. Then the point $(\lambda,\ \mu)$ belongs to the resolvent set of the vector-operator $\mathscr{X}=\left(\overset{1}{A},\ \overset{2}{T},\ \overset{3}{A'}\right)$ (c not dependent on ε).

Proof. Suppose (without loss of generality) that $\lambda=0,\ \mu=0$. Denote $\tau_\varepsilon(x,\ y)=\sqrt{x^2+y^2+\varepsilon^2}$.

Let $\chi(x, y) \in C_0^\infty(\mathbf{R}^{k+l})$ be an arbitrary function with the support in the δ-neighborhood of zero. Then

$$\| \chi(x, y)[\tau_\delta(x, y)]^N \|_{\mathscr{B}_{s, s'}(\mathbf{R}^{k+l})} = O(\delta^N).$$

Choose $\delta < \frac{1}{c}$. Then

$$\| \chi(\mathscr{X}) \|_{B_1 \to B_2} = \left\| \overset{2}{[\![} \overset{2}{T} \left[\tau_\delta \left(\overset{1}{A}, \overset{3}{A'} \right) \right]^{-N}]\!] \chi \left(\overset{1}{A}, \overset{3}{A'} \right) \times \right.$$
$$\left. \times \left[\tau_\delta \left(\overset{1}{A}, \overset{3}{A'} \right) \right]^N \right\|_{B_1 \to B_4} \leqslant$$
$$\leqslant C \cdot c^N \| \chi(x, y)[\tau_\delta(x, y)]^N \|_{B_{s, s'}(\mathbf{R}^{k+l})} \to 0 \text{ for } N \to \infty.$$

Here the brackets $\overset{2}{[\![} \;]\!]$ mean that the numbering of operators inside the brackets does not extend to the operators outside the brackets, i. e., the expression in the brackets is considered to be the operator acting second (see Introduction). The theorem is proved.

Sec. 5. Theorem on Homomorphism

Theorem 5.1. *Let l_σ be a function in $\mathscr{B}_{s, s'}(\mathbf{R}^k \times \mathbf{R}^{k'})$. It is equal to unity in the spectrum σ of the pair of generating sets $\overset{1}{A}$, $\overset{3}{A'}$ with respect to the translator $\overset{2}{T}$. Then the following inequality is valid:*

$$\left\| \overset{2}{T}\varphi \left(\overset{1}{A}, \overset{3}{A'} \right) \times \psi \left(\overset{1}{A}, \overset{3}{A'} \right) \right\|_{B_1 \to B_4} \leqslant c \left\| \overset{2}{T}\varphi \left(\overset{1}{A}, \overset{3}{A'} \right) \right\|_{B_2 \to B_3} \times$$
$$\times \| \psi l_\sigma \|_{B_{s, s'}(\mathbf{R}^k \times \mathbf{R}^{k'})}.$$

Proof. Let $\mathscr{B}_{s, s'}(\mathbf{R}^{k+k'}/\sigma)$ be a subalgebra of the algebra $\mathscr{B}_{s, s'}(\mathbf{R}^{k+k'})$ which is the closure of the algebra of functions in $\mathscr{B}_{s, s'}(\mathbf{R}^{k+k'})$ with the support in $\mathbf{R}^{k+k'} \setminus \sigma$. For any function $f \in \mathscr{B}_{s, s'}(\mathbf{R}^{k+k'} \setminus \sigma)$

$$\overset{2}{T}f \left(\overset{1}{A}, \overset{3}{A'} \right) = 0.$$

Since $\mathscr{B}_{s, s'}(\mathbf{R}^{k+k'} \setminus \sigma)$ is an ideal in $\mathscr{B}_{s, s'}(\mathbf{R}^{k+k'})$ it follows that for $f \in \mathscr{B}_{s, s'}(\mathbf{R}^{k+k'} \setminus \sigma)$ we have

$$\overset{2}{T}(\psi \cdot \varphi) \left(\overset{1}{A}, \overset{3}{A'} \right) = \overset{2}{T}((\psi - f)\varphi) \left(\overset{1}{A}, \overset{3}{A'} \right).$$

Consequently,

$$\left\| \overset{2}{T}(\varphi\psi) \left(\overset{1}{A}, \overset{3}{A'} \right) \right\|_{B_1 \to B_4} \leqslant$$
$$\leqslant c \left\| \overset{2}{T}\varphi \left(\overset{1}{A}, \overset{3}{A'} \right) \right\|_{B_2 \to B_3} \| \psi - f \|_{B_{s, s'}(\mathbf{R}^{k+k'})}.$$

It remains to be noted that the function $l_\sigma - 1$ belongs to the algebra $\mathscr{B}_{s,\,s'}(\mathrm{R}^{k+k'} \setminus \sigma)$.

Theorem 5.2. (The Main Inequality.) *Let* σ *be a spectrum of the pair of generated sets* $\overset{1}{A}$, $\overset{3}{A'}$ *with respect to the translator* $\overset{2}{T}$. *There exists a unique homomorphism*

$$\mathscr{M} : \varphi \to \overset{2}{T}\varphi\left(\overset{1}{A},\ \overset{3}{A'}\right)$$

of the Banach space $\mathscr{B}_{s,\,s'}(\sigma)$ *into a Banach space of uniformly defined linear bounded operators from* B_1 *into* B_4 *with the property* (3.3) *and the inequality*

$$\left\| \overset{2}{T}\varphi\left(\overset{1}{A},\ \overset{3}{A'}\right) \psi\left(\overset{1}{A},\ \overset{3}{A'}\right) \right\|_{B_1 \to B_4} \leqslant$$
$$\leqslant c \left\| \overset{2}{T}\varphi\left(\overset{1}{A},\ \overset{3}{A'}\right) \right\|_{B_2 \to B_3} \| \psi \|_{\mathscr{B}_{s,\,s'}(\sigma)}$$

is true.

Corollary. *Let* σ *be the spectrum of the pair* $\overset{1}{A}$, $\overset{3}{A'}$ *with respect to* $\overset{2}{T}$. *The following inequality is valid:*

$$\left\| \sum_{i=0}^{N} \varphi_i(A')\, T\psi_i(A) \right\|_{B_1 \to B_4} \leqslant$$
$$\leqslant \left\| \sum_{i=0}^{\tau} f_i(A')\, TF_i(A) \right\|_{B_2 \to B_3} \left\| \left\{ \frac{\sum_{i=0}^{N} \varphi_i(y)\, \psi_i(x)}{\sum_{i=0}^{r} f_i(y)\, F_i(x)} \right\} \right\|_{\mathscr{B}_{s,\,s'}(\sigma)}.$$

Definition. *Let the sequence* $\{\Phi_n\}$ *converge to the function* Φ *in* $\mathscr{B}_{s,\,s'}(\mathrm{R}^k \times \mathrm{R}^{k'})$ *and let the functions* $\psi_n \colon x \to [\Phi_n(x)]^{-1}$ *also belong to* $\mathscr{B}_{s,\,s'}(\mathrm{R}^k \times \mathrm{R}^{k'})$. *At that, let the following condition be fulfilled:*

$$\left\| \overset{2}{T}\psi_n\left(\overset{1}{A},\ \overset{3}{A'}\right) \right\|_{B_2 \to B_3} \leqslant \frac{1}{\sup\limits_{\varphi} \dfrac{\left\| \overset{2}{T}\varphi\left(\overset{1}{A},\ \overset{3}{A'}\right) \right\|}{\|T\| \cdot \|\varphi\|_{\mathscr{B}_{s,\,s'}(\mathrm{R}^k \times \mathrm{R}^{k'})}}}.$$

Then the function Φ *will be called the spectral weight.*

We shall say that the spectral weight Φ *is subordinated* to the spectral weight $\mathscr{F}$ if the sequence of norms is bounded:

$$\left\| \frac{\mathscr{F}}{\Phi_n} \right\|_{\mathscr{B}_{s,\,s'}(\mathrm{R}^k \times \mathrm{R}^{k'})}.$$

Introduce in $\mathscr{B}_{s,s'}(\mathbf{R}^k \times \mathbf{R}^{k'})$ the norm

$$\|\varphi\|_\Phi \overset{\text{def}}{=\!=} \|\varphi\Phi\|_{\mathscr{B}_{s,s'}(\mathbf{R}^k\times\mathbf{R}^{k'})} \tag{5.1}$$

and the product

$$\varphi_1 \times \varphi_2 \overset{\text{def}}{=\!=} \varphi_1\varphi_2\Phi; \tag{5.2}$$

we have

$$\|\varphi_1 \times \varphi_2\|_\Phi = \|\varphi_1\varphi_2\Phi^2\|_{\mathscr{B}_{s,s'}(\mathbf{R}^k\times\mathbf{R}^{k'})} \leqslant$$
$$\leqslant \|\varphi_1\Phi\|_{\mathscr{B}_{s,s'}(\mathbf{R}^k\times\mathbf{R}^{k'})} \|\varphi_2\Phi\|_{\mathscr{B}_{s,s'}(\mathbf{R}^k\times\mathbf{R}^{k'})} = \|\varphi_1\|_\Phi \|\varphi_2\|_\Phi.$$

Thus the norm (5.1) and the product (5.2) induce the structure of the normed algebra into $\mathscr{B}_{s,s'}(\mathbf{R}^k \times \mathbf{R}^{k'})$.

The Banach algebra obtained by the completion of this normed algebra will be denoted by $\mathscr{B}_{s,s',\Phi}(\mathbf{R}^k \times \mathbf{R}^{k'})$.

Let $\mathscr{B}_{s,s',\Phi}(\mathbf{R}^{k+k'} \setminus \sigma)$ be the closure in $\mathscr{B}_{s,s',\Phi}(\mathbf{R}^k \times \mathbf{R}^{k'})$ of the subalgebra consisting of finite functions with supports in $\mathbf{R}^{k+k'} \setminus \sigma$. The factor-algebra $\mathscr{B}_{s,s',\Phi}(\mathbf{R}^k \times \mathbf{R}^{k'})/\mathscr{B}_{s,s',\Phi}(\mathbf{R}^{k+k'} \setminus \sigma)$ will be denoted by $\mathscr{B}_{s,s'.\Phi}(\sigma)$.

Theorem 5.3. *The following inequality is valid:*

$$\overset{2}{T}\psi\left(\overset{1}{A}, \overset{3}{A'}\right)\Big\| \leqslant \|\psi\|_{\mathscr{B}_{s,s',\Phi}(\sigma)}.$$

Proof. We have

$$\left\|\overset{2}{T}\psi\left(\overset{1}{A}, \overset{3}{A'}\right)\right\| = \left\|\overset{2}{T}\psi\left(\overset{1}{A}, \overset{3}{A'}\right)\Phi_n^{-1}\left(\overset{1}{A}, \overset{3}{A'}\right)\Phi_n\left(\overset{1}{A}, \overset{3}{A'}\right)\right\| \leqslant$$
$$\leqslant \left\|\overset{2}{T}\Phi_n^{-1}\left(\overset{1}{A}, \overset{3}{A'}\right)\right\|_{B_2\to B_3} \|\psi\Phi_n\|_{\mathscr{B}_{s,s'}(\mathbf{R}^k\times\mathbf{R}^{k'})} \times$$
$$\times \sup_\varphi \frac{\left\|f\varphi\left(\overset{1}{A}, \overset{3}{A'}\right)\right\|}{\|T\|\,\|\varphi\|_{\mathscr{B}_{s,s'}(\mathbf{R}^k\times\mathbf{R}^{k'})}} \leqslant \|\psi\Phi\|_{\mathscr{B}_{s,s'}(\mathbf{R}^k\times\mathbf{R}^{k'})} + \varepsilon_n,$$

where $\varepsilon_n \to 0$ for $n \to \infty$.

If the support of the function $\varphi \in \mathscr{B}_{s,s'}(\mathbf{R}^k \times \mathbf{R}^{k'})$ lies in $\mathbf{R}^{k+k'} \setminus \sigma$, then $\overset{2}{T}\psi\left(\overset{1}{A}, \overset{3}{A'}\right) = \overset{2}{T}\left[\psi\left(\overset{1}{A}, \overset{3}{A'}\right) - \varphi\left(\overset{1}{A}, \overset{3}{A'}\right)\right]$. Hence

$$\left\|\overset{2}{T}\psi\left(\overset{1}{A}, \overset{3}{A'}\right) \leqslant \right\|(\psi-\varphi)\Phi\|_{\mathscr{B}_{s,s'}(\mathbf{R}^k\times\mathbf{R}^{k'})} =$$
$$= \|\psi-\varphi\|_{\mathscr{B}_{s,s'}(\mathbf{R}^k\times\mathbf{R}^{k'})}.$$

By taking the greatest lower bound in φ in the latter inequality we obtain the proof of the theorem.

Sec. 6. Problems

(1) Let

$$A = i\frac{d}{dx} + i\,\frac{2x}{x^2+1}, \quad A' = i\,\frac{d}{dx}, \quad T = 1,$$

$$B_1 = B_2 = B_3 = B_4 = L_2(\mathbf{R}).$$

The function

$$u(x,\ t) = \frac{[(x+t)^2+1]}{x^2+1} f(x+t)$$

satisfies the equation

$$i\,\frac{\partial u}{\partial t} = i\,\frac{\partial u}{\partial x} + i\,\frac{2x}{x^2+1}\,u.$$

For this reason

$$\| e^{iAt} f \|_{L_2(\mathbf{R})} = \sqrt{\int_{-\infty}^{\infty} \left[\frac{(x+t)^2+1}{x^2+1}\right]^2 |f(x+t)|^2\,dx} \leqslant$$

$$\leqslant \max_{x\in\mathbf{R}} \left|\frac{(x+t)^2+1}{x^2+1}\right| \sqrt{\int_{-\infty}^{\infty} |f(x+t)|^2\,dx} \leqslant$$

$$\leqslant c\,(1+|t|)^2 \|f\|_{L_2(\mathbf{R})}.$$

It is obvious that

$$\| e^{iA't} f \|_{L_2(\mathbf{R})} = \| f \|_{L_2(\mathbf{R})}.$$

For any $f \in S$ we have

$$\Phi(A) f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iAt}\widetilde{\Phi}(t) f(x)\,dt =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{(x-t)^2+1}{x^2+1} f(x-t)\,\widetilde{\Phi}(t)\,dt =$$

$$= \frac{1}{x^2+1}\,\Phi\left(-i\,\frac{d}{dx}\right)(x^2+1) f(x).$$

For this reason for any Φ, $\varphi \in C_0^\infty(\mathbf{R})$ and $f \in S$

$$\Phi(A)\,\varphi(A') f(x) =$$

$$= \frac{1}{x^2+1}\,\Phi\left(-i\,\frac{d}{dx}\right)(x^2+1)\,\varphi\left(i\,\frac{d}{dx}\right) f(x) =$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{ipx}\left(1-\frac{d^2}{dp^2}\right)^{-1}\Phi(p)\left(1-\frac{d^2}{dp^2}\right)\varphi(-p)\,\widetilde{f}(p)\,dp.$$

If $\lambda \neq \mu$ and the support of the function φ lies in the small neighborhood of the point λ and the support of the function Φ lies in the small neighborhood of the point μ, then the operator

$$\Phi(p)\left(1-\frac{d^2}{dp^2}\right)\varphi(p)$$

is equal to zero. For this reason the spectrum of the pair $\left(\overset{1}{A}, \overset{2}{A'}\right)$ lies on the diagonal of the direct product $\mathbf{R} \times \mathbf{R}$.

On the other hand, the spectrum of the pair $\left(\overset{1}{A}, \overset{2}{A'}\right)$ covers the whole space $\mathbf{R}^2$, because for any $\varepsilon > 0$, $\lambda \in \mathbf{R}$ and $\mu \in \mathbf{R}$ there exist such functions Φ and $\varphi \in C_0^\infty(\mathbf{R})$ with supports in the ε-neighborhood of the point λ and in the ε-neighborhood of the point μ, respectively, that

$$\varphi(p)\left(1-\frac{d^2}{dp^2}\right)^{-1}\Phi(p) \neq 0$$

$\left(\text{the operator } \left(1-\frac{d^2}{dp^2}\right)^{-1} \text{ is non-local}\right)$.

Thus, the spectrum of the pair of non-self-adjoint operators depends on the order in which these operators are considered: in general, the spectrum of the pair $\left(\overset{1}{A}, \overset{2}{B}\right)$ is not equal to the spectrum of the pair $\left(\overset{2}{A}, \overset{1}{B}\right)$. Besides, one may say that for self-adjoint operators these spectra coincide with one another (correct to the reflection in relation to the bisectrix of the coordinate angle).

(2) Let B be the completion of $C_0^\infty(\mathbf{R})$ in the norm

$$\|f(x)\|_B^2 = \int_{-\infty}^{\infty} |e^{x^2/2} f(x)|^2\, dx$$

and B' — the completion of $C_0^\infty(\mathbf{R})$ in the norm

$$\|f(x)\|_{B'}^2 = \int_{-\infty}^{\infty} |e^{-x^2/2} f(x)|^2\, dx.$$

It is obvious that $B \subset B'$. Take the operator of embedding $1\colon B \to B'$ as T. Consider the "birth" and "death" operators

$$A = i\frac{d}{dx} + ix, \quad A' = i\frac{d}{dx} - ix.$$

The following theorem is valid.

Theorem 6.1. (a) *A is a generator of degree 0 with the defining pair (B, B) and A' is a generator of degree 0 with the defining pair (B', B');*

(b) *the spectrum of the pair* $\left(\overset{1}{A}, \overset{2}{A'}\right)$ *is the whole plane* $\mathbf{R}^2$;
(c) *the function*

$$\Phi : (x, y) \to c\,(1+(x-y)^2)^{\frac{1}{2}}\, e^{-(x-y)^2/2}$$

for sufficiently large c *is the spectral weight of the vector-operator* $\left(\overset{1}{A}, \overset{2}{A'}\right)$ *so that*

$$\left\| F\left(\overset{1}{A}, \overset{2}{A'}\right) \right\| \leqslant \| \Phi F \|_{\mathscr{B}_0(\mathbf{R}^2)}.$$

Proof. (a) The solution of the equation

$$i\frac{\partial u}{\partial t} = i\frac{\partial u}{\partial x} \pm ixu$$

takes the form $u(x, t) = e^{\mp\frac{x^2}{2}}\varphi(x+t)$. For this reason

$$e^{-iAt}f(x) = e^{-\frac{x^2}{2}}e^{-\frac{(x+t)^2}{2}}f(x+t)$$

so that

$$\|e^{-iAt}f(x)\|_B = \left\| e^{\frac{x^2}{2}}e^{-iAt}f(x)\right\|_{L_2(\mathbf{R}_x)} =$$

$$= \left\| e^{\frac{(x+t)^2}{2}} f(x+t)\right\|_{L_2(\mathbf{R}_x)} = \left\| e^{\frac{x^2}{2}}f(x)\right\|_{L_2(\mathbf{R}_x)} = \|f(x)\|_B$$

for any $f \in C_0^\infty(\mathbf{R})$. This means that $\|e^{-iAt}\| = 1$.
(b) Similarly, for any $\varphi \in C_0^\infty(\mathbf{R})$

$$\|e^{-iA't}\varphi\|_{B'} = \left[\int_{-\infty}^{\infty} e^{-x^2}e^{x^2}e^{-\frac{(x+t)^2}{2}}|\varphi(x+t)|^2\,dx\right]^{\frac{1}{2}} =$$

$$= \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}|\varphi(x)|^2\,dx\right)^{\frac{1}{2}} = \|\varphi\|_{B'}.$$

Let Φ, $f \in C_0^\infty(\mathbf{R})$.
In analogy with Example 1 we obtain

$$\Phi(A)f(x) = e^{-\frac{x^2}{2}}\Phi\left(i\frac{d}{dx}\right)e^{\frac{x^2}{2}}f(x),$$

$$\Phi(A')f(x) = e^{\frac{x^2}{2}}\Phi\left(i\frac{d}{dx}\right)e^{-\frac{x^2}{2}}f(x).$$

Let Φ, f, $\varphi \in C_0^\infty(\mathbf{R})$. Then

$$\varphi(A')\,\Phi(A)\,f(x) = e^{\frac{x^2}{2}}\,\varphi\left(i\frac{d}{dx}\right)e^{-x^2}\Phi\left(i\frac{d}{dx}\right)e^{\frac{x^2}{2}}f(x) =$$

$$= e^{\frac{x^2}{2}}\varphi\left(i\frac{d}{dx}\right)e^{-x^2}\Phi\left(i\frac{d}{dx}\right)f_1(x).$$

Let $F_{p\to x}: L_2(\mathbf{R}_p)\to L_2(\mathbf{R}_x)$ be a Fourier transform and $F_{x\to p}^{-1}$: $L_2(\mathbf{R}_x)\to L_2(\mathbf{R}_p)$ be the inverse Fourier transform. Denote $F_{x\to p}f_1(x)$ by $f_1(p)$. Then

$$\varphi(A')\,\Phi(A)\,f(x) = e^{\frac{x^2}{2}}F_{p\to x}^{-1}\varphi(p)\,e^{\frac{d^2}{dp^2}}\Phi(p)\,\widetilde{f}_1(p).$$

The expression $\varphi(p)\,e^{\frac{d^2}{dp^2}}\psi(p)$ does not diminish to zero for any finite φ and ψ. For this reason the spectrum of the vector-operator $\left(\overset{1}{A}, \overset{2}{A'}\right)$ is the whole plane.

(c) Let $\Phi(x, y) = \varphi(x - y)$. It is not difficult to verify that in this case $\Phi\left(\overset{1}{A}, \overset{2}{A'}\right)$ is a multiplication operator by some function, namely

$$\Phi\left(\overset{1}{A}, \overset{2}{A'}\right)f(x) = \frac{1}{2\pi}e^{x^2}f(x)\int_{-\infty}^{\infty}\widetilde{\varphi}(t)\,e^{-(x-t)^2}\,dt,$$

where $\widetilde{\varphi} = F^{-1}\varphi$.

Let

$$\Phi_n(x, g) = (1 + (x+y))^{\frac{1}{2}}\left[e^{-(x-y)^2/2} + \frac{1}{n}\right] = \varphi_n(x - y).$$

Then

$$\frac{1}{2\pi}e^{x^2}\int_{-\infty}^{\infty}\widetilde{\varphi}_n(t)\,e^{-(x-t)^2}\,dt =$$

$$= \frac{1}{2\pi^{3/2}}e^{x^2}\int_{\mathbf{R}^2}e^{-(x-t)^2+it\xi}\,(1+\xi^2)^{-\frac{1}{2}}\frac{e^{\xi^2/2}}{1+e^{\xi^2/2}/n}\,d\xi\,dt =$$

$$= \frac{1}{2\pi\sqrt{2}}e^{x^2}\int_{-\infty}^{\infty}e^{ix\xi}\,(1+\xi^2)^{-\frac{1}{2}}\frac{e^{\xi^2/4}}{1+e^{\xi^2/2}/n}\,d\xi.$$

It is not difficult to verify that

$$\sup_{\xi\in\mathbf{R}}(1+\xi^2)^{-\frac{1}{2}}\frac{e^{\xi^2/4}}{1+e^{\xi^2/2}/n} = 2^{-\frac{1}{2}}e^{\frac{1}{4}} + d_n,$$

where $d_n \to 0$ for $n \to \infty$. Therefore, the latter integral is bounded as a function of n. Hence the statement (c) is obtained because a multiplication operator by e^{x^2} is bounded as an operator acting on B to B'.

Problem 1. Let

$$A = -i\frac{d}{dx} + i\,\frac{x\cos x - \sin x}{x\sin x + x^2},$$

$$A' = \frac{d^2}{dx^2}, \quad T = 1,$$

$$B_1 = B_2 = B_3 = B_4 = L_2(\mathbf{R}).$$

Show that the spectrum of the vector-operator $\left(\overset{1}{A}, \overset{2}{A'}\right)$ lies in the region $\lambda_2 \leqslant 0$, $\lambda_1 - 1 \leqslant |\lambda_2|^{\frac{1}{2}} \leqslant \lambda_1 + 1$.

Solution. Note that $A = -i\frac{d}{dx} + i\frac{d}{dx}\left[\ln\left(1 + \frac{\sin x}{x}\right)\right]$. Hence, by virtue of the problem of Sec. 3 we obtain

$$\psi(A)f = e^{\ln\left(1+\frac{\sin x}{x}\right)}\psi\left(-i\frac{d}{dx}\right)e^{-\ln\left(1+\frac{\sin x}{x}\right)}f(x) =$$
$$= \left(1 + \frac{\sin x}{x}\right)\psi\left(-i\frac{d}{dx}\right)\left(1 + \frac{\sin x}{x}\right)^{-1} f(x).$$

Let $\varphi, \psi \in C_0^\infty(\mathbf{R})$. Then

$$\varphi(A')\psi(A)f =$$
$$= F^{-1}_{p\to x}\left[\varphi(-p^2)\left(1 + \frac{\sin\left(i\frac{d}{dp}\right)}{i\frac{d}{dp}}\right)\times\right.$$
$$\left.\times\,\psi(p)\left(F_{y\to p}\left(1 + \frac{\sin y}{y}\right)^{-1} f(y)\right)\right].$$

Consider in greater detail the function

$$\eta(p) \equiv \left(1 + \frac{\sin\left(i\frac{d}{dp}\right)}{i\frac{d}{dp}}\right)\chi(p) =$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} d\xi\, e^{iy(p-\xi)}\left(1 + \frac{\sin y}{y}\right)\chi(\xi),$$

$\chi \in C_0^\infty(\mathbf{R})$.

Note that

$$F_{x\to\xi}(\theta(1-|x|))=\frac{1}{\sqrt{2\pi}}\int_{-1}^{+1}e^{-i\xi x}\,dx=\sqrt{\frac{2}{\pi}}\frac{\sin\xi}{\xi},$$

or

$$F^{-1}_{y\to p}\left(1+\frac{\sin y}{y}\right)=\sqrt{\frac{\pi}{2}}\,\theta(1-|p|)+\sqrt{2\pi}\,\delta(p).$$

Hence

$$\eta(p)=\chi(p)+\frac{1}{2}\int_{p-1}^{p+1}\chi(\xi)\,d\xi.$$

If $\operatorname{supp}\chi\subset[a, b]$, then $\operatorname{supp}\eta\subset[a-1, b+1]$.

Return to operator $\varphi(A')\,\psi(A)$. From the above it is clear that if $\operatorname{supp}\psi\subset[a, b]$ and $\operatorname{supp}\varphi(-p^2)\subset\mathbf{R}\diagdown[a-1, b+1]$ then $\varphi(A')\,\psi(A)=0$. Hence

$$\sigma\left(\overset{1}{A}, \overset{2}{A'}\right)\subset\{\lambda_2\leqslant 0,\ (\lambda_1-1)\leqslant\sqrt{|\lambda_2|}\leqslant(\lambda_1+1)\}.$$

Problem 2. Let

$$A=i\frac{\partial}{\partial x_1},\quad A'=i\frac{\partial}{\partial x_2},$$

$$B_1=B_2=B_3=B_4=C_{s_1, s_2}(\mathbf{R}\times\mathbf{R}),\ T=1.$$

Show that in this instance

$$\left\|\varphi\left(\overset{1}{A}, \overset{2}{A'}\right)\times\psi\left(\overset{1}{A}, \overset{2}{A'}\right)\right\|\leqslant\left\|\varphi\left(\overset{1}{A}, \overset{2}{A'}\right)\right\|\cdot\left\|\psi\left(\overset{1}{A}, \overset{2}{A'}\right)\right\|,$$

i.e., the algebra induced in Hom $(C_{s_1, s_2}(\mathbf{R}_{x_1}, \mathbf{R}_{x_2}), C_{s_1, s_2}(\mathbf{R}_{x_1}, \mathbf{R}_{x_2}))$ by the homomorphism $\mathcal{M}$ is a normed algebra.

Solution. There exists the inequality

$$|\varphi(x+t)|\,(1+|x_1|)^{-s_1}(1+|x_2|)^{-s_2}\leqslant$$
$$\leqslant(1+|t_1|)^{s_1}(1+|t_2|)^{s_2}\,\|\varphi\|_{C_s(\mathbf{R}^2)},$$

where

$$t=(t_1, t_2),\ x=(x_1, x_2),\ s=(s_1, s_2),\quad \varphi\in C_s(\mathbf{R}^2).$$

Consequently, the set $\left(\overset{1}{A}, \overset{2}{A'}\right)$ is a generating set of degree s. Next, it $\widetilde{\varphi}\equiv F^{-1}\varphi\in C_0^\infty(\mathbf{R}^2)$ then

$$\left\|\varphi\left(\overset{1}{A}, \overset{2}{A'}\right)\chi\right\|_{C_s}=$$
$$=\sup_x(1+|x_1|)^{-s_1}(1+|x_2|)^{-s_2}\left|\int\widetilde{\varphi}(t)\,\chi(x+t)\,dt\right|\leqslant$$
$$\leqslant\int|\widetilde{\varphi}(t)|\,(1+|t_1|)^{s_1}(1+|t_2|)^{s_2}\,dt\cdot\|\chi\|_{C_s}=\|\varphi\|_{\mathcal{B}_s}\cdot\|\chi\|_{C_s}.$$

The same estimate is obtained if $\widetilde{\varphi}(t)=\delta(t-t_0)$. For this reason it follows from the definition of the space $\mathscr{B}_s(\mathbf{R}^2)$ that for all $\varphi\in\mathscr{B}_s(\mathbf{R}^2)$ the following estimate is valid:

$$\left\|\varphi\left(\overset{1}{A},\ \overset{2}{A'}\right)\right\|_{C_s\to C_s}\leqslant\|\varphi\|_{\mathscr{B}_s}.$$

On the other hand, for $\varphi\in\mathscr{B}_s$ and for any $\varepsilon>0$ such $\chi_\varepsilon\in C_s$, $\|\chi_\varepsilon\|_{C_s}=1$ will be found that

$$\left|\int_{\mathbf{R}^2}\widetilde{\varphi}(t)\,\chi_\varepsilon(t)\,dt\right|\geqslant\|\varphi\|_{\mathscr{B}_s}-\varepsilon.$$

Then

$$\left\|\varphi\left(\overset{1}{A},\ \overset{2}{A'}\right)\chi_\varepsilon\right\|_{C_s}\geqslant\left|\int\widetilde{\varphi}(t)\,\chi_\varepsilon(t)\,dt\right|\geqslant\|\varphi\|_{\mathscr{B}_s}-\varepsilon$$

so that $\left\|\varphi\left(\overset{1}{A},\ \overset{2}{A'}\right)\right\|_{C_s\to C_s}\geqslant\|\varphi\|_{\mathscr{B}_s}$. Hence, we obtain

$$\left\|\varphi\left(\overset{1}{A},\ \overset{2}{A'}\right)\right\|_{C_s\to C_s}=\|\varphi\|_{\mathscr{B}_s}.$$

It is not difficult to obtain the final inequalities now:

$$\left\|(\varphi\cdot\psi)\left(\overset{1}{A},\ \overset{2}{A'}\right)\right\|_{C_s\to C_s}=\|\varphi\cdot\psi\|_{\mathscr{B}_s}\leqslant\|\varphi\|_{\mathscr{B}_s}\cdot\|\psi\|_{\mathscr{B}_s}=$$
$$=\left\|\varphi\left(\overset{1}{A},\ \overset{2}{A'}\right)\right\|_{C_s\to C_s}\left\|\psi\left(\overset{1}{A},\ \overset{2}{A'}\right)\right\|_{C_s\to C_s}.$$

Problem 3. Let $B_1=B_2=B_3=B_4=B$, $T=1$ and let $A-A'$ be a bounded operator. Show that the function $\dfrac{K}{x_1-x_2+i}$ for a sufficiently large K is the spectral weight of the vector-operator $\left(\overset{1}{A},\ \overset{2}{A'}\right)$.

Solution. Let $e_n\in C_0^\infty(\mathbf{R})$, $e_n\to 1$ in $\mathscr{B}_l(\mathbf{R})$ for all l. Consider the sequence of functions

$$\psi_n(x_1,\ x_2)=\frac{1}{K}\left[e_n(x_1)\,e_n(x_2)\,(x_1-x_2)+i\right]\in\mathscr{B}_s(\mathbf{R}^2),$$

where $s=(s_1,\ s_2)$, s_1 is the degree of A and s_2 is the degree of A'.
We obtain

$$\frac{1}{\psi_n}=\frac{K}{e_n(x_1)\,e_n(x_2)\,(x_1-x_2)+i}=$$
$$=K\left[\frac{ie_n(x_1)\,e_n(x_2)\,(x_1-x_2)}{e_n(x_1)\,e_n(x_2)\,(x_1-x_2)+i}-i\right]\in\mathscr{B}_s(\mathbf{R}^2),$$
$$\frac{1}{\psi_n}\xrightarrow[n\to\infty]{}K\left[\frac{i(x_1-x_2)}{(x_1-x_2)+i}-i\right]=\frac{K}{x_1-x_2+i}\ \text{in}\ \mathscr{B}_s.$$

Next

$$\left\| \psi_n \left(\overset{1}{A}, \overset{2}{A'} \right) \right\| = \frac{1}{K} \left\| (A \overset{2}{-} A') e_n \left(\overset{1}{A} \right) e_n \left(\overset{3}{A'} \right) + i \right\| \leqslant$$

$$\leqslant \frac{c^2 \| A - A' \| + 1}{K},$$

where

$$\| e_n (A) \| \leqslant c, \quad \| e_n (A') \| \leqslant c.$$

Let $M = \sup\limits_{\varphi} \dfrac{\left\| \varphi \left(\overset{1}{A}, \overset{2}{A'} \right) \right\|}{\| \varphi \|_{\mathscr{B}_s}}$. Choose

$$K \geqslant M (c^2 \| A - A' \| + 1).$$

Then

$$M \cdot \left\| \psi_n \left(\overset{1}{A}, \overset{2}{A'} \right) \right\| \leqslant 1.$$

Thus the sequence ψ_n satisfies all conditions of the definition of the spectral weight, i. e., the function $\dfrac{K}{(x_1 - x_2) + i}$ is the spectral weight of $\left(\overset{1}{A}, \overset{2}{A'} \right)$.

(3) One of the most important particular cases of Theorem 5.1 is the following estimate. Let A (A') be the generating sets of degree s (s') and let $B_2 \subset B_3$. Then

$$\| f (A) - f (A') \|_{B_1 \to B_4} \leqslant$$

$$\leqslant c \| \varphi (A) - \varphi (A') \|_{B_2 \to B_3} \left\| \frac{f (x) - f (y)}{\varphi (x) - \varphi (y)} \right\|_{\mathscr{B}_{s, s'} (\mathbf{R}^{k+k'})}.$$

(4) Let $\{B_\tau\}$ be a Banach scale, T_1 and T_2 be the regular operators of degree s with the defining pair $(B_\tau, B_{\tau+\tau_1})$, where τ is an integer. Let the operator $T_1 - T_2$ be bounded as the operator acting from B_τ to $B_{\tau-\tau_2}$, where $\tau_2 > 2\tau_1$. Let $T_1 = A_1 + iA_2$, $T_2 = B_1 + iB_2$, where (A_1, A_2), (B_1, B_2) are the generating sets. Let the functions $\Phi_1 (x_1, x_2)$ and $\Phi_2 (y_1, y_2)$ be such that the function $\dfrac{\Phi_1 (x) - \Phi_2 (y)}{x_1 + ix_2 - y_1 - iy_2}$ belongs to $\mathscr{B}_{s, s'} (\sigma (T_1) \times \sigma (T_2))$. Then the operator $\Phi_1 (A_1, A_2) - \Phi_2 (B_1, B_2)$ is bounded as an operator from B_τ to $B_{\tau+2\tau_1-\tau_2}$.

(5) Let Γ be a smooth curve in $\mathbf{R}^2$. Introduce the coordinates r, t in the neighborhood of Γ, where r is the length of the normal, t is the parameter on the curve. Let

$$\varphi \in C_0^\infty (\mathbf{R}^2), \quad \varphi (x, y) = \overline{\varphi} (r, t),$$

$$\overline{\varphi}_N (r, t) = \sum_{k=0}^{N} \frac{r^k}{k!} \frac{\partial^k \overline{\varphi}}{\partial r^k} (0, t) = \varphi_N (x, y).$$

If the spectrum of the vector-operator $\left(\overset{1}{A}, T, \overset{2}{A'}\right)$ lies in Γ and the sum of the degrees of the generators A and A' does not exceed N, then

$$\overset{2}{T}\varphi\left(\overset{1}{A}, \overset{3}{A'}\right)=\overset{2}{T}\varphi_N\left(\overset{1}{A}, \overset{3}{A'}\right).$$

(6) Let $P(x, y)=\sum a_{ij}x^ix^j$ be a polynomial in two variables and A and A' be generators in some Banach space B. Set

$$P\left(\overset{1}{A}, \overset{2}{A'}\right)\overset{\text{def}}{=\!=}\sum a_{ij}A'^jA^i.$$

Problem. Let A and A' be operators in $L^2(\mathbf{R})$ considered in (1):

$$A=i\frac{d}{dx}, \quad A'=i\frac{d}{dx}+i\frac{2x}{1+x^2}.$$

Show that $\left(\overset{2}{A'}-\overset{1}{A}\right)^k=0$ if and only if $k\geqslant 3$.

Solution. It is easily seen that $\left(\overset{1}{A}-\overset{2}{A'}\right)\neq 0$ for $k=0, 1, 2$. Let $k\geqslant 3$. Obviously A is a generator of degree $s_A=0$. Next,

$$e^{-iA't}f(t)=e^{-\ln(1+x^2)+\ln(1+(x+t)^2)}f(x+t)=\frac{1+(x+t)^2}{1+x^2}f(x+t);$$

consequently A' is a generator of degree $s_{A'}=2$. For $\varphi, \psi\in C_0^\infty(\mathbf{R})$ we have

$$\psi(A')\varphi(A)f=\frac{1}{1+x^2}\psi\left(i\frac{d}{dx}\right)(1+x^2)\varphi\left(i\frac{d}{dx}\right)f(x)=$$
$$=\frac{1}{1+x^2}\left[\psi(p)\left(1-\frac{d^2}{dp^2}\right)\varphi(p)(Ff)\varphi\right].$$

For this reason if $\operatorname{supp}\varphi\cap\operatorname{supp}\psi\neq\varnothing$ then $\psi(A')\varphi(A)=0$, i.e., the spectrum $\sigma\left(\overset{1}{A}, \overset{2}{A'}\right)$ lies on the diagonal in $\mathbf{R}^2$.

In view of the preceding item the operator $F\left(\overset{1}{A}, \overset{2}{A'}\right)$ is equal to zero if on the diagonal in $\mathbf{R}^2$ the function F has zero of the order $\geqslant s_A+s_{A'}+1=3$. Consequently, for $k\geqslant 3$

$$\left(g, \left(\overset{1}{A}-\overset{2}{A'}\right)^k f\right)=\lim_{m, n\to\infty}\left(g, e_n\left(\overset{2}{A'}\right)\left(\overset{1}{A}-\overset{2}{A'}\right)e_m\left(\overset{1}{A}\right)f\right)$$

for $f\in C_0^\infty(\mathbf{R})$ and $g\in L^2(\mathbf{R})$ ($\{e_n\}$ being the sequence of functions from Lemma 3.13 of Chapter I). Hence $\left(\overset{1}{A}-\overset{2}{A'}\right)^k=0$ for $k\geqslant 3$.

(7) Let $B_1=B_2=L_2(\mathbf{R})$, $B_3=B_4=W_2^{-1}(\mathbf{R})$, and $T=i\frac{d}{dx}$ and let A and A' be multiplication operators by a smooth real-

valued function $f(x)$. It is not difficult to verify that the spectrum of the vector-operator $\left(\overset{1}{A}, \overset{2}{T}, \overset{3}{A'}\right)$ lies on the diagonal.

From the equality

$$\left(i\frac{d}{dx}\right) e^{-if(x)t} = e^{-if(x)t}\left(i\frac{d}{dx}\right) + (-it)\, e^{-if(x)t}\,(if'(x))$$

it is not difficult to obtain the formula

$$\overset{2}{T}F\left(\overset{1}{A}, \overset{3}{A'}\right)\varphi = F(f(x), f(x))\, i\frac{d\varphi(x)}{dx} +$$
$$+ F'(f(x), f(x))\, i\frac{df(x)}{dx}\varphi(x),$$

where F' is the derivative of F with respect to the first argument and $\varphi \in C_0^\infty(\mathbf{R})$.

Sec. 7. Differentiation of the Functions of an Operator Depending on a Parameter

Theorem 7.1. *Let $A_1^{(n)}$, $A_2^{(n)}$ be generators with the defining pairs of spaces (B_1, B_1) and (B_2, B_2), respectively, where $\| e^{-iA_1^{(n)}t} \| \leqslant \leqslant c_1(1 + |t|)^{s_1}$ and $\| e^{-iA_2^{(n)}t} \| \leqslant c_2(1 + |t|)^{s_2}$, c_1, s_1, c_2, s_2 being constants independent of n. Let T_n: $B_1 \to B_2$ be homomorphisms bounded in the norm by the same constant. Next, let the sequence $\{A_1^{(n)}\}$ converge point-by-point in some set dense in B_1 to the generator A_1 and let the sequence $\{A_2^{(n)}\}$ converge point-by-point in a set dense in B_2 to the generator A_2. Let the sequence $\{T_n\}$ converge point-by-point to the homomorphism T: $B_1 \to B_2$. Then, for any function $f \in \mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$, the sequence $\left\{\overset{2}{T_n}f\left(\overset{1}{A_1^{(n)}}, \overset{3}{A_2^{(n)}}\right)\right\}$ converges point-by-point to the homomorphism $\overset{2}{T}f\left(\overset{1}{A_1}, \overset{3}{A_2}\right)$: $B_1 \to B_2$.*

Proof. First of all, suppose that the function f is of the form $f(x, y) = f_1(x) f_2(y)$, where $f_1 \in \mathscr{B}_{s_1}(\mathbf{R})$, $f_2 \in \mathscr{B}_{s_2}(\mathbf{R})$. Then the following formulas are valid:

$$\overset{2}{T_n}f\left(\overset{1}{A_1^{(n)}}, \overset{3}{A_2^{(n)}}\right) = f_2(A_2^{(n)})\, T_n f_1(A_1^{(n)}),$$
$$\overset{2}{T}f\left(\overset{1}{A_1}, \overset{3}{A_2}\right) = f_2(A_2)\, T f_1(A_1).$$

For any $h \in B_1$ we have

$$\lim_{n\to\infty} f_1(A_1^{(n)})\, h = f_1(A_1)\, h.$$

Since $T_n \to T$ point-by-point and $\|T_n\| \leqslant \text{const}$ it follows that

$$T_n f_1(A_1^{(n)})\, h \xrightarrow{n\to\infty} Tf_1(A_1)\, h$$

in the norm of B_2. Since $f_2(A_2^{(n)}) \to f_2(A_2)$ point-by-point in B_2 and $\|f_2(A_2^{(n)})\| \leqslant \frac{c_2}{\sqrt{2\pi}} \|f_2\|_{\mathscr{B}_{s_2}(\mathbf{R})}$ it follows that

$$f(A_2^{(n)})\, T_n f(A_1^{(n)})\, h \xrightarrow{n\to\infty} f(A_2)\, Tf(A_1)\, h$$

in the norm of B_2. Theorem 7.1 is proved for the particular case in question.

Now let h be a fixed element of B_1. Then

$$\overset{2}{T}_n f\left(\overset{1}{A}{}_1^{(n)},\ \overset{3}{A}{}_2^{(n)}\right) h \xrightarrow{n\to\infty} \overset{2}{T} f\left(\overset{1}{A}_1,\ \overset{3}{A}_2\right) h$$

in the norm of B_2 for any f of the set P dense in $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$ of functions of the form

$$f(x, y) = \sum_{j=1}^{k} \varphi_j(x)\, \psi_j(y).$$

Denote by $A_n : \mathscr{B}_{s_1, s_2}(\mathbf{R}^2) \to B_2$ the operator acting according to the formula

$$A_n f = \overset{2}{T}_n f\left(\overset{1}{A}{}_1^{(n)},\ \overset{3}{A}{}_2^{(n)}\right) h$$

and denote by A the operator

$$Af \overset{\text{def}}{=\!=} \overset{2}{T} f\left(\overset{1}{A}_1,\ \overset{3}{A}_2\right) h.$$

Then the sequence $\{A_n\}$ converges point-by-point to A in a set P dense everywhere, and is bounded in the norm

$$\|A_n\| \leqslant \frac{c_1 c_2}{2\pi} \|T_n\|.$$

By virtue of the Banach-Steinhaus theorem $A_n \to A$ point-by-point in $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$, Q.E.D.

Note. In this theorem the generators may be replaced by the generating sets.

Let A be a generating set consisting of k commutative operators with the defining pair of spaces (B_1, B_1), let A' be a similar generating set of k' operators with the defining pair of spaces (B_2, B_2) and let T be a homomorphism $B_1 \to B_2$. Let $P(x, y)$ be a polynomial in variables $x \in \mathbf{R}^k$, $y \in \mathbf{R}^k$:

$$P(x, y) = \sum a_{ij} x^i y^j,$$

where

$$i=i_1, \ldots, i_k;\ j=j_1, \ldots, j_{k'};\ x^i=x_1^{i_1} \ldots x_k^{i_k},\ y^j=y_1^{j_1} \ldots y_{k'}^{j_{k'}}.$$

Set

$$\overset{2}{T}P\left(\overset{1}{A}, \overset{3}{A'}\right)=\sum a_{ij}A'^jTA^i.$$

Lemma 7.1. *Let* $f \in \mathscr{B}_{s, s'}(\mathbf{R}^k \times \mathbf{R}^{k'})$ *and* $fP \in \mathscr{B}_{s, s'}(\mathbf{R}^k \times \mathbf{R}^{k'})$. *If the operator* $\overset{2}{T}P\left(\overset{1}{A}, \overset{3}{A'}\right)$ *is bounded as the operator from* B_1 *to* B_2, *then*

$$\overset{2}{T}fP\left(\overset{1}{A}, \overset{3}{A'}\right)=\overset{2}{T'}f\left(\overset{1}{A}, \overset{3}{A'}\right),$$

where T' *is the extension of the operator* $\overset{2}{T}P\left(\overset{1}{A}, \overset{3}{A'}\right)$ *to the homomorphism* $B_1 \to B_2$.

Proof. Let $\{f_n\}$ be a sequence of functions of the form

$$f_n(x, y)=u_n(x)\, v_n(y),\quad x \in \mathbf{R}^k,\quad y \in \mathbf{R}^{k'}.$$

It is evident that if the operators A and A' are bounded, then the lemma is valid for functions f_n:

$$\overset{2}{T}f_nP\left(\overset{1}{A}, \overset{3}{A'}\right)=\overset{2}{T'}f_n\left(\overset{1}{A}, \overset{3}{A'}\right). \tag{7.1}$$

If $f_n \to f$ and $f_nP \to fP$ in $\mathscr{B}_{s, s'}(\mathbf{R}^{k+k'})$, then passing over to the limit for $n \to \infty$ in (7.1) we obtain the required lemma. The proof of the existence of such a sequence is left to the reader. The lemma is proved for the case of bounded operators A and A'.

Now let A and A' be arbitrary generators. Set

$$A_n=\frac{A}{1+A^2/n},\quad A'_n=\frac{A'}{1+A'^2/n}.$$

Then A_n and A'_n are bounded operators converging to the operators A and A', respectively, on sets dense everywhere. There exist constants c, c', N, N' such that

$$\|e^{-iA_nt}\| \leqslant c(1+|t|)^N,\quad \|e^{-iA'_nt}\| \leqslant c'(1+|t|)^{N'}.$$

Denote: $T_n=\varphi(A'_n)T\varphi(A_n)$. Then $T_n \to T$ point-by-point in B_1 and $\|T_n\|_{B_1 \to B_2} \leqslant$ const. Consequently, by virtue of Theorem 7.1 for any function $f \in \mathscr{B}^0(\mathbf{R}^2)$ and for any $h \in B_1$, the following formula is valid:

$$\lim_{n \to \infty} \overset{2}{T_n}f\left(\overset{1}{A_n}, \overset{3}{A'_n}\right)P\left(\overset{1}{A_n}, \overset{3}{A'_n}\right)h=\overset{2}{T}f\left(\overset{1}{A}, \overset{3}{A'}\right)P\left(\overset{1}{A}, \overset{3}{A'}\right)h.$$

On the other hand, the sequence of operators

$$T'_n=P\left(\overset{1}{A}, \overset{3}{A'}\right)\varphi\left(\overset{3}{A'_n}\right)\overset{2}{T}\varphi(A_n)=\varphi(A'_n)T'\varphi(A_n)$$

acting on B_1 to B_2 is bounded in the norm and strongly converges to T'. For this reason

$$\lim_{n\to\infty} \overset{2}{T'_n} f\left(\overset{1}{A_n}, \overset{3}{A'_n}\right) h = \overset{2}{T'} f\left(\overset{1}{A}, \overset{3}{A'}\right) h.$$

The lemma is proved.

Now consider the family $\{T(\xi)\}$ of regular operators

$$T(\xi) = A_1(\xi) + iA_2(\xi),$$

where $(A_1(\xi), A_2(\xi)) = A(\xi)$ is a generating set of degree s with the defining pair of spaces (B, B). Here the domain D of the operator $T(\xi)$ does not depend on ξ,

$$\sup_{t\in\mathbf{R}^2} \frac{\| e^{-iA(\xi)t} \|}{(1+|t|)^s} < \infty$$

and the spectrum of the operator $T(\xi)$ is contained in the closed set $\sigma \subset \mathbf{R}^2$ which is independent of ξ.

Lemma 7.2. *Let $f \in \mathscr{B}_s(\sigma)$, let the family $\{T(\xi)\}$ have the defining pair of spaces (B', B'), where $B \subset B'$, and let the spectrum of the pair of generating sets $\overset{1}{A}(\xi')$, $\overset{2}{A}(\xi'')$ with respect to the embedding of B into B' be contained in the closed set $\Sigma \subset \sigma \times \sigma \subset \mathbf{R}^4$ for any ξ', ξ'' from the domain of the function $T(\xi)$. Next, let*

$$\lim_{\xi\to\xi_0} \| T(\xi) - T(\xi_0) \|_{B\to B'} = 0.$$

Then, if the set of functions of the form

$$\frac{g(x)-g(y)}{x_1+ix_2-y_1-iy_2}, \quad g \in \mathscr{B}_s(\sigma)$$

is dense in $\mathscr{B}_s(\Sigma)$,* then

$$\lim_{\xi\to\xi_0} \| f(A(\xi)) - f(A(\xi_0)) \|_{B\to B'} = 0.$$

Proof. Let the functions $f_n(x)$, $x \in \mathbf{R}^2$ be such that

$$\Phi_n(x, y) = \frac{f_n(x)-f_n(y)}{x_1+ix_2-y_1-iy_2} \subset \mathscr{B}_s(\Sigma). \tag{7.2}$$

Let T_{ξ,ξ_0} be the extension of the operator $T(\xi) - T(\xi_0)$ to the homomorphism $B \to B'$. Then, according to the above lemma, we have

$$f_n(A(\xi)) - f_n(A(\xi_0)) = \overset{2}{T}_{\xi,\xi_0} \Phi_n\left(\overset{1}{A}(\xi), \overset{3}{A}(\xi_0)\right).$$

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* Note that the condition is carried out if $\{T(\xi)\}$ is a family of generators. In this case one may set $\sigma = \mathbf{R}$, $\Sigma = \mathbf{R}^2$.

Consequently,

$$\|f_n(A(\xi)) - f_n(A(\xi_0))\|_{B\to B'} \leqslant$$
$$\leqslant c\,\|T(\xi) - T(\xi_0)\|_{B\to B'}\,\|\Phi_n\|_{\mathscr{B}_s(\Sigma)} \to 0$$

for $\xi \to \xi_0$.

Let $f_n \to f$ in $\mathscr{B}_s(\sigma \times \sigma)$. Then

$$\|f(A(\xi)) - f(A(\xi_0))\|_{B\to B'} \leqslant \|f(A(\xi)) - f_n(A(\xi))\|_{B\to B'} +$$
$$+ \|f_n(A(\xi)) - f_n(A(\xi_0))\|_{B\to B'} + \|f_n(A(\xi_0)) -$$
$$- f(A(\xi_0))\|_{B\to B'} \leqslant c\,\{\|f - f_n\|_{\mathscr{B}_s(\sigma)} +$$
$$+ \|T(\xi) - T(\xi_0)\|_{B\to B'}\,\|\Phi_n\|_{\mathscr{B}_s(\Sigma)}\}.$$

Fix $\varepsilon > 0$ and choose such an n that the inequality

$$\|f - f_n\|_{\mathscr{B}_s(\mathbf{R}^2)} < \frac{\varepsilon}{2}$$

will be fulfilled. For a given n there exists such a $\delta > 0$ that

$$\|T(\xi) - T(\xi_0)\|_{B\to B'}\,\|\Phi_n\|_{\mathscr{B}_s(\Sigma)} < \frac{\varepsilon}{2}$$

for $|\xi - \xi_0| < \delta$. Consequently, for $|\xi - \xi_0| < \delta$ the following inequality is carried out:

$$\|f(A(\xi)) - f(A(\xi_0))\|_{B\to B'} < c\varepsilon.$$

The lemma is proved.

Problem. Formulate and prove the multidimensional analogue of Lemma 7.2.

Theorem 7.2. *Let the parameter ξ vary over an interval of the real axis, let the family of regular operators $\{T(\xi)\}$ (excluding (B, B)) have a defining pair of spaces (B', B'), where $B \subset B'$, and let for any ξ' and ξ'' of the interval the spectrum of the pair of generating sets $\overset{1}{A}(\xi'), \overset{2}{A}(\xi'')$ with respect to the embedding of B to B' be contained in the closed set $\Sigma \subset \sigma \times \sigma$. Next, let there exist a derivative $T'(\xi)$ considered in the following sense:*

$$\lim_{\varepsilon\to 0}\left\|\frac{T(\xi+\varepsilon) - T(\xi)}{\varepsilon} - T'(\xi)\right\|_{B\to B'} = 0,$$

where $\|T'(\xi)\|_{B\to B'} < \infty$. Let $f \in \mathscr{B}_s(\sigma)$. Then, if

$$\Phi(x, y) = \frac{f(x) - f(y)}{x_1 + ix_2 - y_1 - iy_2} \in \mathscr{B}_s(\Sigma), \quad x \in \mathbf{R}^2, \quad y \in \mathbf{R}^2,$$

then in the topology of the space Hom (B, B') *there exists the derivative* $\frac{d}{d\xi} f(A(\xi))$ *so that the following formula is valid*:

$$\frac{d}{d\xi} f(A(\xi)) = \overset{2}{T}{}'(\xi) \frac{f\left(\overset{1}{A}(\xi)\right) - f\left(\overset{3}{A}(\xi)\right)}{\overset{1}{T}(\xi) - \overset{3}{T}(\xi)}.$$

Corollary. *The following estimate is justified*:

$$\left\| \frac{d}{d\xi} f(A(\xi)) \right\|_{B \to B'} \leqslant c \, \| T'(\xi) \|_{B \to B'} \left\| \frac{f(x) - f(y)}{x_1 + ix_2 - y_1 - iy_2} \right\|_{\mathcal{B}_s(\Sigma)},$$

where c depends only on $\sup\limits_{t \in \mathbf{R}^2} \frac{\| e^{-iA(\xi)t} \|_{B \to B}}{(1+|t|)^s}$ *and on*

$$\sup_{t \in \mathbf{R}^2} \frac{\| e^{-iA(\xi)t} \|_{B' \to B'}}{(1+|t|)^s}.$$

Proof. From the existence and the boundedness of the operator $T'(\xi)$ it follows that $\| T(\xi + \varepsilon) - T(\xi) \|_{B \to B'} < \infty$. By virtue of Lemma 7.1 we have

$$\frac{f(A(\xi+\varepsilon)) - f(A(\xi))}{\varepsilon} =$$

$$= \frac{\overset{2}{\overline{T(\xi+\varepsilon) - T(\xi)}}}{\varepsilon} \frac{f\left(\overset{1}{A}(\xi+\varepsilon)\right) - f\left(\overset{3}{A}(\xi)\right)}{\overset{1}{T}(\xi+\varepsilon) - \overset{3}{T}(\xi)}, \tag{7.3}$$

where $\overline{T(\xi + \varepsilon) - T(\xi)}$ is the extension of the operator $T(\xi + \varepsilon) - T(\xi)$ to the homomorphism $B \to B'$. Passing in (7.3) to the limit for $\varepsilon \to 0$ and using the multidimensional analogue of Lemma 7.2 we obtain the proof of the theorem.

It is not difficult to formulate and prove the multidimensional analogue of Theorem 7.2 for the functions of a number of generating sets. The definition of such functions can be obtained almost word for word by using a similar definition from Sec. 4.

Sec. 8. Formulas of Commutation

Let A and B be generators of degrees s and s', respectively, acting in a Banach space B which contains a linear manifold E dense everywhere and invariant with respect to A, B, e^{iAt} and e^{-iBt}.

In this section we shall assume that the following condition of agreement between the operators A and B is carried out: for any function $\varphi \in C_0^\infty(\mathbf{R})$ the operators $\varphi(A)B$ and $B\varphi(A)$ are defined on E and are closed.

Lemma 8.1. *Let the operator A as well as the commutator $[A, B]$ be bounded. Then, for any $f \in \mathscr{B}_{s+1}(\mathbf{R})$ the following formula is valid:*

$$\overline{[B,\ f(A)]} = \overset{2}{\overline{[B,\ A]}}\, \frac{f\left(\overset{1}{A}\right) - f\left(\overset{3}{A}\right)}{\overset{1}{A} - \overset{3}{A}},$$

where the line above means the closure.

Proof. Let $\varphi \in C_0^\infty(\mathbf{R})$ be a real function which is contained between 0 and 1 and is equal to 1 in the neighborhood of zero. Denote $\varphi_n(x) = \varphi\left(\frac{x}{n}\right)$. Set $A_n = \varphi_n(B) A \varphi_n(B)$ and first prove the formula

$$\overline{[B,\ f(A_n)]} = \frac{f\left(\overset{1}{A_n}\right) - f\left(\overset{3}{A_n}\right)}{\overset{1}{A_n} - \overset{3}{A_n}}\, \overset{2}{\overline{[B,\ A_n]}}. \tag{8.1}$$

We have

$$\overline{[A_n,\ B]} = \varphi_n(B) A \varphi_n(B) B - B \varphi_n(B) A \varphi_n(B) = \varphi_n(B) \overline{[A,\ B]}\, \varphi_n(B).$$

This means that the right-hand member of (8.1) is equal to

$$\overset{2}{\overline{\llbracket \varphi_n(B)\,[B,\ A]\,\varphi_n(B) \rrbracket}}\, \frac{f\left(\overset{1}{A_n}\right) - f\left(\overset{3}{A_n}\right)}{\overset{1}{A_n} - \overset{3}{A_n}} =$$

$$= \overset{2}{\overline{\llbracket \varphi_n(B)\, BA\varphi_n(B) \rrbracket}}\, \frac{f\left(\overset{1}{A_n}\right) - f\left(\overset{3}{A_n}\right)}{\overset{1}{A_n} - \overset{3}{A_n}} -$$

$$- \overset{2}{\overline{\llbracket \varphi_n(B)\, AB\varphi_n(B) \rrbracket}}\, \frac{f\left(\overset{1}{A_n}\right) - f\left(\overset{3}{A_n}\right)}{\overset{1}{A_n} - \overset{3}{A_n}} =$$

$$= \overset{2}{\overline{\llbracket A_n B \rrbracket}}\, \frac{f\left(\overset{1}{A_n}\right) - f\left(\overset{3}{A_n}\right)}{\overset{1}{A_n} - \overset{3}{A_n}} + \overset{2}{\overline{\llbracket BA_n \rrbracket}}\, \frac{f\left(\overset{1}{A_n}\right) - f\left(\overset{3}{A_n}\right)}{\overset{1}{A_n} - \overset{3}{A_n}}.$$

Denote $B_m = B\varphi_m(B)$. Then for a fixed n and sufficiently large m

$$\overset{2}{\overline{\llbracket [B,\ A_n] \rrbracket}}\, \frac{f\left(\overset{1}{A_n}\right) - f\left(\overset{3}{A_n}\right)}{\overset{1}{A_n} - \overset{3}{A_n}} = - \overset{2}{\overline{\llbracket A_n B_m \rrbracket}}\, \frac{f\left(\overset{1}{A_n}\right) - f\left(\overset{3}{A_n}\right)}{\overset{1}{A_n} - \overset{3}{A_n}} +$$

$$+ \overset{2}{\overline{\llbracket B_m A_n \rrbracket}}\, \frac{f\left(\overset{1}{A_n}\right) - f\left(\overset{3}{A_n}\right)}{\overset{1}{A_n} - \overset{3}{A_n}} = - \overset{3}{A_n} \overset{2}{B_m}\, \frac{f\left(\overset{1}{A_n}\right) - f\left(\overset{3}{A_n}\right)}{\overset{1}{A_n} - \overset{3}{A_n}} +$$

$$+ \overset{2}{B_m} \overset{1}{A_n}\, \frac{f\left(\overset{1}{A_n}\right) - f\left(\overset{3}{A_n}\right)}{\overset{1}{A_n} - \overset{3}{A_n}} = \overset{2}{B_m}\left(- f\left(\overset{3}{A_n}\right) + f\left(\overset{1}{A_n}\right)\right) =$$

$$= [B_m,\ f(A_n)].$$

Thus for a fixed n and sufficiently large m the formula is valid:

$$\overset{2}{\overline{[\![[B,\ A_n]]\!]}}\,\frac{f\left(\overset{1}{A}_n\right)-f\left(\overset{3}{A}_n\right)}{\overset{1}{A}_n-\overset{3}{A}_n}=[B_m,\ f(A_n)].$$

In particular, this means that the right-hand member of this formula does not depend on m. If $h = \psi(B)\,q$, where $\psi \in C_0^\infty(\mathbf{R})$, then

$$f(A_n)\,B_m h = f(A_n)\,Bh$$

for sufficiently large m. Consequently, $B_m f(A_n)\,h$ does not depend on m for a sufficiently large m and since $\varphi_m(B) \to 1$ point-by-point, $B = B\varphi_m(B)$ and B is a closed operator, it follows that

$$B_m f(A_n)\,h = Bf(A_n)\,h$$

for a sufficiently large m. This means that (8.1) is valid in a set dense everywhere and, consequently, it is valid everywhere.

In (8.1) we shall pass over to the point-by-point limit for $n \to \infty$ with the additional assumption that the Fourier transform of the function f has a compact support. For any h

$$e^{-iA_n t}h \to e^{-iAt}h \tag{8.2}$$

uniformly in t in any interval $a \leqslant t \leqslant b$ and

$$\|e^{-iA_n t}\| \leqslant c \tag{8.3}$$

for $t \in [a,\ b]$. This follows from

$$\left\| e^{-iA_n t} - \sum_{k=0}^{N} \frac{(iA_n)^k}{k!} \right\| = O_t(\varepsilon),$$

where $O_t(\varepsilon)$ does not depend on n. Passing over to the limit in the right-hand member of (8.1) for $n \to \infty$, by virtue of (8.2) and (8.3) we obtain

$$\lim_{n\to\infty} \overset{2}{\overline{[B,\ A_n]}}\,\frac{f\left(\overset{1}{A}_n\right)-f\left(\overset{3}{A}_n\right)}{\overset{1}{A}_n-\overset{3}{A}_n}\,h=\overset{2}{\overline{[B,\ A]}}\,\frac{f\left(\overset{1}{A}\right)-f\left(\overset{3}{A}\right)}{\overset{1}{A}-\overset{3}{A}}\,h.$$

As to the limit in the left-hand member of (8.1) we consider $f(A_n) \to f(A)$ point-by-point. Hence $f(A_n)\,B \to f(A)\,B$ in E. Consequently, the sequence $\{Bf(A_n)\,h\}$ converges for any $h \in E$. By virtue of the closed nature of the operator B, we have, for $h \in E$:

$$\lim_{n\to\infty} Bf(A_n)\,h = Bf(A)\,h.$$

Thus, for any $h \in E$ the following formula is valid:

$$[B, f(A)]\, h = \overset{2}{[\overline{B, A}]}\, \frac{f\left(\overset{1}{A}\right) - f\left(\overset{3}{A}\right)}{\overset{1}{A} - \overset{3}{A}}\, h.$$

The lemma is proved for the function f which has a Fourier transform with compact support. Since the set of such functions is dense in $\mathscr{B}_{s+1}(\mathbf{R})$ the lemma is valid in the general case too.

The following theorem states that the assumption that the operator A is bounded may be rejected.

Theorem 8.1. *Let the commutator* $[A, B]$ *be bounded. Then for any* $f \in \mathscr{B}_{s+1}(\mathbf{R})$ *the following formula is valid:*

$$[B, f(A)] = \overset{2}{[\overline{B, A}]}\, \frac{f\left(\overset{1}{A}\right) - f\left(\overset{3}{A}\right)}{\overset{1}{A} - \overset{3}{A}}.$$

Proof. Let $\{\varphi_n\}$ be the sequence of functions considered in the proof of Lemma 8.1. Set $B_n = \varphi_n(A)\, B\varphi_n(A)$, $A_n = A\varphi_n(A)$. Consider the commutator

$$[B_n, A_m] = \varphi_n(A)\, B\varphi_n(A)\, A\varphi_m(A) - A\varphi_m(A)\, \varphi_n(A)\, B\varphi_n(A).$$

Let n be fixed. Then for a sufficiently large m we obtain

$$[B_n, A_m]\, h = \varphi_n(A)\, [B, A]\, \varphi_n(A)\, h, \qquad h \in E.$$

Consequently, for a sufficiently large m the operator $\overline{[B_n, A_m]}$ does not depend on m and is bounded in norm by the number $\|\, [B, A]\, \|$.

By applying Lemma 8.1, we get

$$\overline{[B_n, f(A_m)]} = \overset{2}{[\![\overline{\varphi_n(A)\, [B, A]\, \varphi_n(A)}]\!]}\, \frac{f\left(\overset{1}{A_m}\right) - f\left(\overset{3}{A_m}\right)}{\overset{1}{A_m} - \overset{3}{A_m}}.$$

For a sufficiently large m this formula may be rewritten in the form

$$\overline{[B_n, f(A_m)]} = \varphi_n\left(\overset{3}{A}\right) \overset{2}{[\overline{B, A}]}\, \varphi_n\left(\overset{1}{A}\right) \frac{f\left(\overset{1}{A}\right) - f\left(\overset{3}{A}\right)}{\overset{1}{A} - \overset{3}{A}}. \tag{8.4}$$

Let $h \in E$. Then $f(A_m)\, B_n h = f(A_m)\, \varphi_n(A)\, B\varphi_n(A)\, h = f(A)\, \varphi_n(A)\, B\varphi_n(A)\, h = f(A)\, B_n h$, if m is sufficiently large for a fixed n. In the same way, $B_n f(A_m)\, h = B_n f(A)\, h$ for a sufficiently large m. Consequently, the following formula is valid:

$$\overline{[B_n, f(A)]} = \varphi_n\left(\overset{3}{A}\right) \overset{2}{[\overline{B, A}]}\, \frac{f\left(\overset{1}{A}\right) - f\left(\overset{3}{A}\right)}{\overset{1}{A} - \overset{3}{A}}\, \varphi_n\left(\overset{1}{A}\right).$$

In this formula let us pass over to the limit for $n \to \infty$. The right-hand member converges point-by-point to

$$\overset{2}{\overline{[B,\ A]}}\,\frac{f\begin{pmatrix}1\\ A\end{pmatrix}-f\begin{pmatrix}3\\ A\end{pmatrix}}{\overset{1}{A}-\overset{3}{A}}.$$

Concerning the left-hand member we may say that for $h \in E$, $q = \psi(A)h$, $\psi \in C_0^\infty(\mathbf{R})$ we have

$$\begin{aligned}[B_n, f(A)]\,q &= \varphi_n(A)\,B\varphi_n(A)\,f(A)\,\psi(A)\,h - \\ &\quad - f(A)\,\varphi_n(A)\,B\varphi_n(A)\,\psi(A)\,h = \\ &= \varphi_n(A)\,[B, f(A)]\,\varphi_n(A)\,\psi(A)\,h.\end{aligned}$$

If n is sufficiently large, then $\varphi_n(A)\,\psi(A)\,h = \psi(A)\,h = q$ so that

$$\lim_{n\to\infty}[B_n, f(A)]\,q = \lim_{n\to\infty}\varphi_n(A)\,[B, f(A)]\,q = [B, f(A)]\,q.$$

Since the linear core of the set of elements of the form $\psi(A)\,h$, where $\psi \in C_0^\infty(\mathbf{R})$, $h \in E$, is dense everywhere, the theorem is proved.

Theorem 8.2. *Let the commutator* $[A, B]$ *be bounded. Then for any* $f(x, y) \in \mathscr{B}_{s+1,\, s'+1}(\mathbf{R}^2)$ *the following formula is valid:*

$$f\begin{pmatrix}1 & 2\\ A, & B\end{pmatrix} - f\begin{pmatrix}2 & 1\\ A, & B\end{pmatrix} = \overset{3}{\overline{[A,\ B]}}\,\frac{\delta^2 f}{\delta x\,\delta y}\begin{pmatrix}2 & 4 & 1 & 5\\ A, & A, & B, & B\end{pmatrix}. \tag{8.5}$$

Proof. It suffices to consider the case when $f(x, y) = f_1(x)\,f_2(y)$ since the set of linear combinations of such functions is dense in $\mathscr{B}_{s+1,\, s'+1}(\mathbf{R}^2)$. Then (8.5) is obtained by passing over to the closure. We have

$$[f_1(A),\ f_2(B)] = \overset{2}{\overline{[f_1(A),\ B]}}\,\frac{f_2\begin{pmatrix}1\\ B\end{pmatrix}-f_2\begin{pmatrix}3\\ B\end{pmatrix}}{\overset{1}{B}-\overset{3}{B}}.$$

Next,

$$[f_1(A),\ B] = \overset{2}{\overline{[A,\ B]}}\,\frac{f_1\begin{pmatrix}1\\ A\end{pmatrix}-f_1\begin{pmatrix}3\\ A\end{pmatrix}}{\overset{1}{A}-\overset{3}{A}}.$$

For this reason

$$[f_1(A),\ f_2(B)] = \overset{3}{\overline{[A,\ B]}}\,\frac{f_1\begin{pmatrix}2\\ A\end{pmatrix}-f_1\begin{pmatrix}4\\ A\end{pmatrix}}{\overset{2}{A}-\overset{4}{A}}\;\frac{f_2\begin{pmatrix}1\\ B\end{pmatrix}-f_2\begin{pmatrix}5\\ B\end{pmatrix}}{\overset{1}{B}-\overset{5}{B}},$$

Q.E.D.

Theorem 8.3. *Let* $f(x) \in \bigcap_s \mathscr{B}_s(\mathrm{R})$ *and* $f(x, y) \in \bigcap_{s, s'} \mathscr{B}_{s, s'}(\mathrm{R}^2)$ *and let the commutator* $[A, B]$ *be bounded. Then the following formula is valid:*

$$f\left([\![g\left(\overset{1}{A}, \overset{2}{B}\right)]\!]\right) = [\![f\left(g\left(\overset{1}{A}, \overset{2}{B}\right)\right)]\!] +$$
$$+ [\![\overset{5}{\overline{[A, B]}} \frac{\delta g}{\delta x}\left(\overset{3}{A}, \overset{7}{A}, \overset{9}{B}\right) \frac{\delta g}{\delta y}\left(\overset{3}{A}, \overset{4}{B}, \overset{6}{B}\right) \times$$
$$\times \frac{\delta^2 f}{\delta x^2}\left(g\left(\overset{1}{A}, \overset{9}{B}\right), \overset{2}{[\![} g\left(\overset{1}{A}, \overset{2}{B}\right)]\!], \overset{8}{[\![} g\left(\overset{1}{A}, \overset{2}{B}\right)]\!]\right)]\!]. \tag{8.6}$$

Proof. If the operator A is bounded, then the proof is identical with that of Introduction.

Let the operator A be unbounded. Set $A_n = \dfrac{A}{1 + \dfrac{A^2}{n}}$. Then A_n is bounded and $\| e^{-iA_n t} \| \leqslant c(1 + |t|)^s$, where c and s are constants independant of n (see Chapter I). Besides

$$\| [A_n, B] \| \leqslant \| [A, B] \| \left\| \frac{\delta^2 f}{\delta x^2} \right\|_{\mathscr{B}_{s, s'}(\mathrm{R})},$$

where $f_n(x) = \dfrac{x}{1 + \dfrac{x^2}{n}}$.

We have

$$\delta^2 f_n(x, y) = \frac{\dfrac{x}{1 + \dfrac{x^2}{n}} - \dfrac{y}{1 + \dfrac{y^2}{n}}}{x - y} = \delta^2 f_1\left(\frac{x}{\sqrt{n}}, \frac{y}{\sqrt{n}}\right).$$

Hence it follows that

$$\| \delta^2 f_n(x, y) \|_{\mathscr{B}_{s, s'}(\mathrm{R}^2)} \leqslant \| \delta^2 f_1(x, y) \|_{\mathscr{B}_{s,s'}(\mathrm{R}^2)}.$$

Therefore, the commutators $[A_n, B]$ are bounded uniformly in n. If in (8.6) A is replaced by A_n, then the obtained formula will be valid. Passing to the point-by-point limit for $n \to \infty$ in the latter formula, we obtain (8.6) for the general case, Q.E.D.

Sec. 9. Growing Symbols

Let A_1 and A_2 be generators of degrees s_1 and s_2, respectively, acting on a Banach space B. As usual we assume that A_1 and A_2 are defined in the same set E dense everywhere. The closures of the operators A_1 and A_2 in B we shall again denote by A_1 and A_2, respectively.

Definition. *Let there exist such non-negative integers k, l for the function $f(x, y)$ that the function*

$$g(x, y)=\frac{f(x, y)}{(x+i)^k (y+i)^l}$$

belongs to space $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$. Then set

$$f\left(\overset{1}{A}_1, \overset{2}{A}_2\right) \overset{\text{def}}{=} (A_2+i)^l\, g\left(\overset{1}{A}_1, \overset{2}{A}_2\right) (A_1+i)^k h \tag{9.1}$$

for any vector $h \in E$, so that $g\left(\overset{1}{A}_1, \overset{2}{A}_2\right)(A_1+i)^k h$ belongs to the domain of the operator A_2^l.

Theorem 9.1. *Let (in the notation of the definition)*

$$k' \geqslant k,\ l' \geqslant l,\ \ g'(x, y)=\frac{f(x, y)}{(x+i)^{k'} (y+i)^{l'}}$$

and h be such a vector in E that

$$g\left(\overset{1}{A}_1, \overset{2}{A}_2\right)(A_1+i)^k h \in D_{A_2^l}.$$

Then

$$(A_2+i)^l\, g\left(\overset{1}{A}_1, \overset{2}{A}_2\right)(A_1+i)^k h=(A_2+i)^{l'}\, g'\left(\overset{1}{A}_1, \overset{2}{A}_2\right)(A_2+i)^{k'} h. \tag{9.2}$$

Proof. We have

$$g(x, y)=g'(x, y)\,(x+i)^{k'-k}\,(y+i)^{l'-l}.$$

Let the sequence $\{\varphi_n\}$ of functions of the form

$$\varphi_n(x, y)=\sum_{j=1}^{m_n} \alpha_{jn}(x)\,\beta_{jn}(y),\ \ \alpha_{jn} \in \mathscr{B}_{s_1}(\mathbf{R}),\ \ \beta_{jn} \in B_{s_2}(\mathbf{R})$$

converge to $g(x, y)$ in $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$. Then the sequence of functions

$$\Psi_n(x, y)=\sum_{j=1}^{m_n} \gamma_{jn}(x)\,\delta_{jn}(y),$$

where $\gamma_{jn}(x)=\dfrac{\alpha_{jn}(x)}{(x+i)^{k'-k}}$ and $\delta_{jn}(y)=\dfrac{\beta_{jn}(y)}{(y+i)^{l'-l}}$, converges to $g'(x, y)$ in $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$. For any $q \in E$ we have

$$\varphi_n\left(\overset{1}{A}_1, \overset{2}{A}_2\right) q=\sum_{j=1}^{m_n} \beta_{jn}(A_2)\,\alpha_{jn}(A_1)\, q=$$

$$=\sum_{j=1}^{m_n} (A_2+i)^{l'-l}\,\delta_{jn}(A_2)\,\gamma_{jn}(A_1)\,(A_1+i)^{k'-k} q=$$

$$=(A_2+i)^{l'-l}\,\Psi_n\left(\overset{1}{A}_1, \overset{2}{A}_2\right)(A_1+i)^{k'-k} q.$$

Passing over to the limit for $n \to \infty$ we obtain

$$g\left(\overset{1}{A_1}, \overset{2}{A_2}\right) q = (A_2+i)^{l'-l} g'\left(\overset{1}{A_1}, \overset{2}{A_2}\right) (A_1+i)^{k'-k} q \tag{9.3}$$

(here we have used the closed nature of the operator $(A_2+i)^{l'-l}$). Putting $q = (A_1 + i)^k h$ and applying the operator $(A_2+i)^l$ to both members of (9.3) we obtain (9.2).

Theorem 9.1 shows that the element $f\left(\overset{1}{A_1}, \overset{2}{A_2}\right) h$ does not depend on the choice of powers k and l in (9.1). Note that if the right-hand member of (9.2) exists, then the left-hand member also exists. This follows directly from formula (9.3).

Theorem 9.2. *Let $f\left(\overset{1}{A_1}, \overset{2}{A_2}\right)$ be an operator defined everywhere in E and bounded in B and let T be the closure of this operator in B. Then for any function $g \in \mathcal{B}_{s_1,s_2}(\mathbf{R}^2)$ and any $h \in E$ the following formula is valid:*

$$[fg]\left(\overset{1}{A_1}, \overset{2}{A_2}\right) h = \overset{2}{T} g\left(\overset{1}{A_1}, \overset{3}{A_2}\right) h.$$

Proof. Let

$$f\left(\overset{1}{A_1}, \overset{2}{A_2}\right) h = (A_2+i)^l f_0\left(\overset{1}{A_1}, \overset{2}{A_2}\right) (A_1+i)^k h,$$

where

$$f_0(x, y) = f(x, y)/(x+i)^k (y+i)^l, \quad f_0 \in \mathcal{B}_{s_1, s_2}(\mathbf{R}^2).$$

Then for any $h \in E$ we have

$$\begin{aligned}
&\overset{2}{T} g\left(\overset{1}{A_1}, \overset{2}{A_2}\right) h = \\
&\quad = \frac{1}{2\pi} \int_{\mathbf{R}^2} \tilde{g}(t, \tau)\, e^{-iA_2\tau} (A_2+i)^l f_0\left(\overset{1}{A_1}, \overset{2}{A_2}\right) \times \\
&\quad \times (A_1+i)^k e^{-iA_1 t} h\, dt\, d\tau = (A_2+i)^l \frac{1}{2\pi} \int_{\mathbf{R}^2} \tilde{g}(t, \tau) \times \\
&\quad \times e^{-iA_2\tau} f_0\left(\overset{1}{A_1}, \overset{2}{A_2}\right) e^{-iA_2 t} (A+i)^k h\, dt\, d\tau = \\
&\quad = (A_2+i)^l [f_0 g]\left(\overset{1}{A_1}, \overset{2}{A_2}\right) (A_1+i)^k h = (fg)\left(\overset{1}{A_1}, \overset{2}{A_2}\right) h.
\end{aligned}$$

Corollary. *The following estimate is valid:*

$$\left\| (fg)\left(\overset{1}{A_1}, \overset{2}{A_2}\right) \right\|_B \leqslant c \left\| f\left(\overset{1}{A_1}, \overset{2}{A_2}\right) \right\|_B \| g \|_{\mathcal{B}_{s_1, s_2} \mathbf{R}^2}, \tag{9.4}$$

where c is the norm of the homomorphism $\mathcal{M}$: $\mathcal{B}_{s_1, s_2}(\mathbf{R}^2) \to \to \mathrm{Hom}(B, B)$ constructed on the ordered pair of operators $\overset{1}{A_1}, \overset{2}{A_2}$.

Next, we shall need the symbols $f(x_1, \ldots, x_n)$ satisfying the condition

$$\frac{f(x_1, \ldots, x_n)}{(x_1+i)^k} \in \mathscr{B}_{s_1, \ldots, s_n}(\mathbf{R}^n)$$

for some integer k. If $A_1, \ldots, A_n$ is a set of generators of degrees $s_1, \ldots, s_n$, respectively, acting on a Banach space B and defined on a linear manifold E dense everywhere, then we set, for any $h \in E$,

$$f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}\right) h \overset{\text{def}}{=} g\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}\right)(A_1+i)^k h,$$

where $g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)/(x+i)^k$. As in Theorem 9.1 this definition may be shown not to depend on the choice of k.

It is not difficult to verify that in this case the estimate analogous to (9.4) is also valid: if $f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}\right)$ is an operator bounded in B and $g \in \mathscr{B}_{s_1, \ldots, s_n}(\mathbf{R}^n)$, then

$$\left\| (fg)\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}\right) \right\|_B \leqslant c \left\| f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}\right) \right\|_B \| g \|_{\mathscr{B}_{s_1, \ldots, s_n}(\mathbf{R}^n)},$$

where c is the norm of the homomorphism $\mathscr{M}\colon \mathscr{B}_{s_1, \ldots, s_n}(\mathbf{R}^n) \to \operatorname{Hom}(B, B)$ constructed on the ordered set of operators $\overset{1}{A_1}, \ldots, \overset{n}{A_n}$.

Problem. We have defined infinitely differentiable functions of two non-commutative generators with a growth rate not exceeding the degree of the argument. Show that they form an algebra with the μ-structure (with the same proviso as in the footnote on page 210).

Finally, define growing functions of several non-commutative operators. In this case additional assumptions are necessary. Namely, *let $\{B_\tau\}$ be a Banach scale, where $B_\tau \subset B_{\tau'}$ for $\tau < \tau'$. Let $A_1, \ldots, A_n$ be generators in the scale $\{B_\tau\}$, where the operators are also translators in the same scale. Denote by $\mathscr{S}^\infty$ the space of functions $f(x_1, \ldots, x_n)$ growing with all their derivatives at a rate not exceeding $|x|^k$ (power k corresponding to the function f).*

For any function $f(x_1, \ldots, x_n)$ in $\mathscr{S}^\infty$ there exist such integers $k_1, \ldots, k_n$ that

$$g(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{(x_1+i)^{k_1} \ldots (x_n+i)^{k_n}}$$

belongs to $\mathscr{B}_{l_1, \ldots, l_n}(\mathbf{R}^n)$ for any $l_1, \ldots, l_n$. Set

$$f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}\right) \overset{\text{def}}{=} \left(\overset{2}{A_1}+1\right)^{k_1} \ldots \left(\overset{2n}{A_n}+1\right)^{k_n} g\left(\overset{1}{A_1}, \ldots, \overset{2n-1}{A_n}\right), \tag{9.5}$$

where the operators $(A_1 + i)^{k_1}, \ldots, (A_n + i)^{k_n}$ *are considered as translators.* The investigation of the correctness of the definition is left to the reader.

Problem. Verify that in the algebra of generators and translators in the scale $\{B_\tau\}$ formula (9.5) defines a μ-structure with an operation μ: $(x_1 \to A_1, \ldots, x_n \to A_n)$ translating $f(x_1, \ldots, x_n)$ into $f\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n\right)$.

Sec. 10. The Factor-Spectrum

Let $\{B_\tau\}$ be a Banach scale and A and A' be generators in the scale of degrees s and s' with steps k_1 and k_2, respectively. Let T be a translator with step l. Let S_1 and S_2 be operators defined on D and acting on B_τ for any τ. We shall call the operators S_1 and S_2 the *equivalent operators* if, for any τ and τ',

$$\| S_1 - S_2 \|_{\tau \to \tau'} < \infty.$$

Definition. *We shall say that a point* $\lambda \in \mathbf{R}^2$ *belongs to the factor-resolvent set of the vector-operator* $\left(\overset{1}{A}, \overset{2}{T}, \overset{3}{A'}\right)$ *if there exists such a neighborhood U of the point λ that for any function* $f \in \mathscr{B}_{s, s'}(\mathbf{R}^2)$ *with support in U and for any τ and τ'*

$$\left\| \overset{2}{T} f\left(\overset{1}{A}_1, \overset{3}{A'}\right) \right\|_{\tau \to \tau'} < \infty,$$

i.e., the operator $\overset{2}{T} f\left(\overset{1}{A}, \overset{3}{A'}\right)$ *is equivalent to zero.*

The complement in $\mathbf{R}^2$ *of the factor-resolvent set will be called the factor-spectrum σ of the vector-operator* $\left(\overset{1}{A}, \overset{2}{T}, \overset{3}{A'}\right)$.

Lemma 10.1. *Let f be a finite function in* $\mathscr{B}_{s, s'}(\mathbf{R}^2)$ *and the operator* $\overset{2}{T} f\left(\overset{1}{A}, \overset{3}{A'}\right)$ *be equivalent to zero. Then any point λ for which* $f(\lambda) \neq 0$ *belongs to the factor-resolvent set of the vector-operator* $\left(\overset{1}{A}, \overset{2}{T}, \overset{3}{A'}\right)$.

Proof. Let U be such a neighborhood of the point λ that $f(\lambda) \neq 0$ for $\lambda \in U$ and let φ be a function in $\mathscr{B}_{s, s'}(\mathbf{R}^2)$ with support in U. Then there exists such a function $\psi \in \mathscr{B}_{s, s'}(\mathbf{R}^2)$ that $\varphi = \psi f$. According to Theorem 4.1 we have for arbitrary μ

$$\left\| \overset{2}{T} \varphi\left(\overset{1}{A}, \overset{3}{A'}\right) \right\|_{\tau \to \tau + k_1 + k_2 + \mu} \leqslant$$
$$\leqslant c \left\| \overset{2}{T} f\left(\overset{1}{A}, \overset{3}{A'}\right) \right\|_{\tau + k_1 \to \tau + k_1 + \mu} \| \psi \|_{\mathscr{B}_{s, s'}(\mathrm{R})} < \infty,$$

Q.E.D.

Theorem 10.1. *For the point λ^0 to belong to the factor-resolvent set of the vector-operator $\left(\overset{1}{A}, \overset{2}{T}, \overset{3}{A'}\right)$ it is necessary and sufficient that for sufficiently small $\varepsilon > 0$ there exist functions φ, $\psi \in C_0^\infty(\mathbf{R})$ with support in the ε-neighborhood of the points λ_1^0 and λ_2^0, respectively, such that the following condition is carried out:*

$$\varphi(\lambda_1^0) = \psi(\lambda_2^0) = 1$$

and that the operator

$$\psi(A') T \varphi(A)$$

is equivalent to zero.

Sec. 11. The Functions of Components of a Lie Nilpotent Algebra and Their Representations

This section will be mainly devoted to the study of functions of differential operators which form an ordered representation $L_1, \ldots, L_n, \alpha$ of components of some Lie nilpotent algebra (see Sec. 9 of Introduction). We shall show that these operators are generators and translators on a suitable scale and prove the relation (9.5) of Introduction for the case when, instead of a polynomial, there exists an arbitrary function in $C_{\mathscr{L}}^\infty$. We shall also prove formula (8.10) which is necessary for the proof of the main theorem. Besides, we shall obtain formulas for reducing a function of the ordered operators $\overset{1}{L_1}, \overset{2}{L_2}, \ldots, \overset{n}{L_n}, \overset{n+1}{\alpha}$ to a pseudodifferential operator.

First point out some properties of the ordered representation.

Let X be a set of components $A_1, \ldots, A_n$ of a Lie nilpotent algebra (see Sec. 9 of Introduction). Since, by hypothesis, any commutator of length N is equal to zero, it follows that a commutator of length $N - 1$ commutes with all elements of the set X. By hypothesis, all commutators of operators are included in the set of operators $A_1, \ldots, A_n$.

Let X_1 be a subset in X of elements commuting with all elements in X. Let X_2 be such a set of elements that their commutator with any element of X belongs to X_1. In general, let $X_i \subset X$ be such a subset of elements that their commutator with any element belongs to the sum X_k for $k < i$. Obviously the number of such sets X_i does not exceed N.

Now consider the construction of an ordered representation of the set X. The method of its construction is described in Sec. 9 of Introduction. By transferring in $\overset{n+1}{A_i} q\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}\right)$ the operator A_i from position $n + 1$ to position i we obtain the expansion

of $q\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n\right)$ in a Taylor series in powers of $\overset{n}{A}_n$ at the point $\overset{n+2}{A}_n$. Hence we obtain the terms of the form

$$\frac{\left(\overset{n}{A}_n - \overset{n+2}{A}_n\right)^k \overset{n+1}{A}_i}{k!} \frac{\partial^k}{\partial x_n^k} q\left(\overset{1}{A}_1, \ldots, \overset{n-1}{A}_{n-1}, \overset{n+2}{A}_n\right).$$

Let $A_n \in X_{i(n)}$. Then $\left(\overset{1}{A}_n - \overset{3}{A}_n\right)^k \overset{2}{A}_i$ (the right-hand commutator being of the order k) belongs to such a class X_j that $j \leqslant i(n) - k$. Similarly one may conclude that, in general, the coefficient of the derivative $\dfrac{\partial^k}{\partial x_{j_1} \ldots \partial x_{j_k}}$ of the operator L_i is a constant or a linear homogeneous function of x_r such that $i(r) < \max\limits_{1 \leqslant s \leqslant k} i(j_s)$.

Note that this remark is also relevant to the construction of the representation G_i defined in Sec. 9 of Introduction.

Lemma 11.1. *Let $\hat{L} = L\left(\overset{2}{x}, -i\overset{1}{\partial/\partial x}\right)$ be an ordered representation of one of the components of the Lie nilpotent algebra. Then, for any integer k or l there exist such integers $s(k, l)$, $m(k)$, $n(l)$ that*

$$\| e^{-i\hat{L}t} \|_{C_l^{m(k)} \to C_{n(l)}^k} \leqslant \text{const}\,(1 + |t|)^{s(k, l)},$$

where C_l^k is the space of k times continuously differentiable functions with the finite norm

$$\| \varphi \|_{C_l^k} = \sup_{x \in \mathbf{R}^n} \sum_{|\gamma|=0}^{k} (1 + |x|)^l \frac{\partial^{|\gamma|}}{\partial x^\gamma} \varphi(x).$$

Proof. Denote by H_l^k the completion of the space S in the norm

$$\| \varphi \|_{H_l^k} = \left\| (1 + |x|^2)^{\frac{l}{2}} \left(1 + \left|\frac{\partial}{\partial x}\right|^2\right)^{\frac{k}{2}} \varphi \right\|_{L_2}.$$

There are embeddings

$$H_l^{k+\nu} \subset C_l^k \subset H_{l-\nu}^k, \quad \forall \nu \geqslant \frac{n}{2}.$$

Consequently, it suffices to prove the estimate

$$\| e^{-i\hat{L}t} \|_{H_l^{m(k)} \to H_{n(l)}^k} \leqslant \text{const}\,(1 + |t|)^{s(k, l)}.$$

Let $F_{x \to \xi}$ be a Fourier transform. Denote $\widetilde{L} = F_{x \to \xi} \hat{L} F_{\xi \to x}^{-1}$. Since the Fourier transform of the space H_l^k is H_k^l, it suffices to obtain

the estimate

$$\| e^{-i\widetilde{L}t} \|_{H^l_{m(k)} \to H^{n(l)}_k} \leqslant \mathrm{const}\,(1+|t|)^{s(k,l)}.$$

The operator $\widetilde{L}$ has the following form (the coordinates are numbered respectively):

$$\widetilde{L} = i \sum_{j=1}^{r_1} a_j \frac{\partial}{\partial \xi_j} - i \sum_{j=r_1+1}^{r_2} P_j(\xi_1, \ldots, \xi_{r_1}) \frac{\partial}{\partial \xi_j} - \ldots$$

$$\ldots - i \sum_{j=r_p+1}^{r_{p+1}} P_j(\xi_1, \ldots, \xi_{r_p}) \frac{\partial}{\partial \xi_j} - \ldots$$

$$\ldots - \sum_{j=r_q+1}^{n} P_j(\xi_1, \ldots, \xi_{r_q}) \frac{\partial}{\partial \xi_j} + P_0(\xi),$$

where P_j are polynomials with real coefficients and a_j are real numbers. If $\psi(\xi) \in S$, then the function $\psi(\xi, t) = e^{-i\widetilde{L}t}\psi(\xi)$ is a solution of the following Cauchy problem:

$$\begin{cases} \dfrac{\partial \psi}{\partial t} + \displaystyle\sum_{j=1}^{r_1} a_j \frac{\partial \psi}{\partial \xi_j} + \sum_{j=r_1+1}^{n} P_j \frac{\partial \psi}{\partial \xi_j} + iP_0\psi = 0, \\ \psi(\xi, 0) = \varphi(\xi). \end{cases}$$

Next, write the characteristic equation for this problem:

$$\begin{cases} \dfrac{d\xi_j}{dt} = a_j, \quad j = 1, \ldots, r_1, \\ \dfrac{d\xi_j}{dt} = P_j, \quad j = r_1+1, \ldots, n, \\ \dfrac{d\psi}{dt} = -iP_0\psi, \\ \xi(0) = \xi^0, \ \psi(0) = \varphi(\xi^0). \end{cases}$$

By solving the first n equations successively we obtain

$$\xi_j = \xi_j(\xi^0, t) = \begin{cases} \xi_j^0 + a_j t, \quad j = 1, \ldots, r_1, \\ \xi_j^0 + Q_j(\xi_1^0, \ldots, \xi_{r_p}^0, t), \ j = r_p+1, \ldots, r_{p+1}, \end{cases}$$

where Q_j are polynomials. Hence we have

$$\xi^0 = \xi^0(\xi, t) = \begin{cases} \xi_i - a_j t, \quad j = 1, \ldots, r_1, \\ \xi_j + R_j(\xi_1, \ldots, \xi_{r_p}, t), \quad j = r_p+1, \ldots, r_{p+1}, \end{cases} \tag{11.1}$$

where R_j are polynomials. Let $\overline{\psi}(\xi^0, t)$ be a function $\psi(\xi, t)$ of coordinates ξ^0 and t. From the latter equation of the characteristic system we have an explicit formula for $\overline{\psi}(\xi^0, t)$

$$\overline{\psi}(\xi^0, t) = \varphi(\xi^0) \exp\left\{-i \int_0^t P_0(\xi(\xi^0, \tau))\, d\tau\right\}. \tag{11.2}$$

Note that $\frac{D\xi}{D\xi_0} = 1$. Consequently

$$\| e^{-i\widetilde{L}t} \varphi \|^2_{L^2} = \int |\widetilde{\psi}(\xi^0, t)|^2 d\xi^0 = \| \varphi \|^2_{L_2}.$$

By putting in (11.2) $\xi^0 = \xi^0(\xi, t)$ and applying to both members an operator of the form $\xi^\beta \left(\frac{\partial}{\partial \xi}\right)^\gamma$ and using (11.1) in analogy with an estimate in L_2, we obtain the required estimate $\| e^{-i\widetilde{L}t} \|_{H^l_{m(k)} \to H^{n(l)}_k}$ for the case $l \geqslant 0$. Here, one may even set $n(l) = l$. Besides, for a given $m(k)$ one may always choose $k = k(m)$. Now make use of the fact that the spaces H^l_m and H^{-l}_{-m} are conjugated in relation to the scalar product in L_2. In other words, the norm

$$\| \varphi \|^l_m = \sup_{\|\psi\|_{H^{-l}_{-m}} = 1} |(\varphi, \psi)_{L_2}|, \quad \varphi, \psi \in S$$

is equivalent to the norm in H^l_m. The operator conjugated to the operator $e^{-i\widetilde{L}t}$ in L_2 has the form $e^{i\widetilde{L}t}$ since $\widetilde{L}$ is symmetrical in L_2. Consequently

$$\| e^{-i\widetilde{L}t} \varphi \|^{-l}_k = \sup_{\|\psi\|_{H^l_k} = 1} |(e^{-i\widetilde{L}t}\varphi, \psi)_{L_2}| = \sup_{\|\psi\|_{H^l_{-k}} = 1} |(\varphi, e^{-i\widetilde{L}t}\psi)_{L_2}|.$$

If $l > 0$, then, according to the relation proved above, there exist such numbers k' and s that for $\| \psi \|_{H^l_{-k}} = 1$

$$\| e^{-i\widetilde{L}t} \psi \|_{H^l_{k'}} \leqslant \text{const}\,(1 + |t|)^s.$$

For this reason

$$\| e^{-i\widetilde{L}t} \varphi \|^{-l}_k = \sup_{\|\psi\|_{H^l_{-k}} = 1} \| \varphi \|^{-l}_{k'} \| e^{i\widetilde{L}t} \psi \|_{H^l_{k'}} \leqslant \text{const}\, \| \varphi \|^{-l}_{k'} (1 + |t|)^s.$$

The lemma is proved.

We shall now pass over to the proof of (8.5) and (8.10) as well as of their analogues given in the item "the rule of reduction of the main problem" (see Sec. 9 of Introduction).

As it has been mentioned, this proof is based on axioms (μ_7) and (μ_8). It consists of two parts.

(1) Let $g(x) \in C^\infty$; then the operator

$$e^{iA_k t} g\left(\overset{1}{A}_1, \overset{2}{A}_2, \ldots, \overset{n}{A}\right) = e^{i\overset{n+2}{A_k} t} g\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n\right) \tag{11.3}$$

is equal to the operator $\psi\left(\overset{1}{A}_1, \overset{2}{A}_2, \ldots, \overset{n}{A}_n, t\right)$, whose symbol is expressed by the symbol $g(x)$ in the following way: $\psi(x, t) = e^{i\hat{L}_k t} g(x)$. Here $\hat{L}_k$ is an ordered representation of the operator A_k. The operator exists by virtue of the established properties $\hat{L}_k$.

Proof. Consider derivatives of the symbols $e^{ix_{n+2}t} g(x) \overset{\text{def}}{=\!=} h$ and $\psi(x, t)$ with respect to t, i.e., the symbols $ix_{n+2}h$ and $\psi_t'(x, t) = i\hat{L}_k\psi(x, t)$. It is obvious that for the former the relation (μ_8^b) of axiom (μ_8) is fulfilled. Bear in mind that the operation μ brings into correspondence x_{n+2} and $\overset{n+2}{A_k}$ and x_k and $\overset{k}{A_k}$. Since $\hat{L}_k$ is an ordered representation of A_k it follows that

$$[i\hat{L}_k\psi(x, t)]_{x_1=\overset{1}{A}_1, \ldots, x_n=\overset{n}{A}_n} = iA_k [\![\psi\left(\overset{1}{A}_1, \overset{2}{A}_2, \ldots, \overset{n}{A}_n\right)]\!].$$

Hence, it follows that the relation (μ_8^b) is also fulfilled for the symbol $\psi(x, t)$. At $t = 0$ the two operators $\hat{\psi}$ and $e^{iA_k t}\hat{g}$ are equal to $\hat{g}$. For this reason their difference satisfies the relations (μ_8^a) and (μ_8^b). This means that the operators are equal, as required.

From the second part of the axiom (μ_7) it easily follows that equality (8.5) can be extended to symbols of the class $C_{\mathscr{L}}^\infty$.

(2) Consider the operators

$$e^{i\sum_{k=1}^{n} \overset{n+1+k}{A_k} t_k} g\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n\right) \text{ and } \psi\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, t_1, \ldots, t_n\right), \tag{11.4}$$

where $\psi(x, t) = e^{i\sum_{k=1}^{n} \overset{k}{L}_k t_k} g(x)$. From the above item it follows that they are equal.

Let the pseudodifferential operator

$$f\left(i\frac{\partial}{\partial t_1}, i\frac{\partial}{\partial t_2}, \ldots, i\frac{\partial}{\partial t_n}\right) \overset{\text{def}}{=\!=} f\left(i\frac{\partial}{\partial t}\right),$$

act on the symbols of both operators (11.4), where $f \in \mathscr{S}^\infty$. By virtue of the properties of the operator $\hat{L}_k$ we have

$$f\left(i\frac{\partial}{\partial t}\right)\psi(x, t) = e^{i\sum_{k=1}^{n} \overset{k}{L}_k t_k} f\left(\overset{1}{L}_1, \ldots, \overset{n}{L}_n\right) g(x).$$

Obviously, for symbol (11.4) we have

$$f\left(i\frac{\partial}{dt}\right)e^{i\sum\limits_{k=1}^{n}x_{n+1+k}t_k}g(x_1,\ \ldots,\ x_n)=$$
$$=e^{i\sum\limits_{k=1}^{n}x_{n+1+k}t_k}f(x_{n+1},\ \ldots,\ x_{n+1+k})\,g(x_1,\ \ldots,\ x_n).$$

Because of axiom (μ_7) the corresponding operators are equal for all t ($t=0$ included). Namely, for $t=0$ we obtain the equality

$$[\![f\left(\overset{1}{A}_1,\ \ldots,\ \overset{n}{A}_n\right)]\!][\![g\left(\overset{1}{A}_1,\ \ldots,\ \overset{n}{A}_n\right)]\!]=\psi\left(\overset{1}{A}_1,\ \ldots,\ \overset{n}{A}_n\right),$$

where $\psi(x)$ is expressed by $g(x)$ in the following form:

$$\psi(x)=f\left(\overset{1}{L}_1,\ \ldots,\ \overset{n}{L}_n\right)g(x)$$

as required.

The other equalities of Sec. 8 are proved in a similar way. There exists an obvious generalization for the case when the argument is $\overset{n+1}{B}$.

We shall again use formula (8.10) for a complex-valued function of ordered operators. Let $f_i\left(\overset{1}{A}_1,\ \ldots,\overset{n}{A}_n\right)\in M,\quad i=1,\ \ldots,\ k.$ Then $f\left(\overset{1}{f_1},\ \ldots,\ \overset{k}{f_k}\right)$ can be represented in the form of the function $\psi\left(\overset{1}{A}_1,\ \ldots,\ \overset{n}{A}_n\right)$, the symbol $\psi(x)$ being defined by the formula

$$f\left(\overset{1}{[\![f_1}\left(\overset{1}{L}_1,\ \ldots,\ \overset{n}{L}_n\right)]\!],\ \ldots,\ \overset{k}{[\![f_k}\left(\overset{1}{L}_1,\ \ldots,\ \overset{n}{L}_n\right)]\!]\right)1=\psi(x). \quad (11.5)$$

Here 1 is a unity belonging to the space of symbols $\mathcal{S}^\infty$. We shall apply the above formula to describe the function of representations $\overset{1}{L}_1,\ \ldots,\ \overset{n}{L}_n$ as a complex function of the components of another nilpotent algebra, namely, that of $-i\frac{\partial}{\partial x}$, x, in other words, we shall write the function of $\overset{1}{L}_1,\ \ldots,\ \overset{n}{L}_n$ in the form of a pseudodifferential operator. Thus we shall obtain an ordered representation of operators $-i\frac{\partial}{\partial x}$, x.

It is obvious that

$$\left(i\overset{3}{\frac{\partial}{\partial x_j}}\right)f\left(-i\overset{1}{\frac{\partial}{\partial x}},\ \overset{2}{x}\right)=\psi_j\left(-i\overset{1}{\frac{\partial}{\partial x}},\ \overset{2}{x}\right),$$

where $\psi_j(y,\ x)=\left(-i\frac{\partial}{\partial x_j}+y_j\right)f(y,\ x).$

Consequently, the operators $\left(-i\overset{1}{\frac{\partial}{\partial x}}+y,\ \overset{2}{x}\right)$ constitute an ordered representation of the set $\left(i\overset{1}{\frac{\partial}{\partial x}},\ \overset{2}{x}\right)$. Then, from (11.5), it follows that

$$f\left(\overset{1}{L_1},\ \ldots,\ \overset{n}{L_n}\right)=\varphi\left(\overset{2}{x},\ -i\overset{1}{\frac{\partial}{\partial x}}\right),$$

where

$$\varphi(x,\ y)=f\left(\overset{1}{[\![}L_1\left(\overset{2}{x},\ -i\overset{1}{\frac{\partial}{\partial x}}+y\right)]\!],\ \ldots\right.$$
$$\left.\ldots,\ \overset{n}{[\![}L_n\left(\overset{2}{x},\ -i\overset{1}{\frac{\partial}{\partial x}}+y\right)]\!]\right)\,1$$

is the symbol of the pseudodifferential operator.

III. ASYMPTOTIC METHODS

Sec. 1. Canonical Transformations of Pseudodifferential Operators

We have already seen in Secs. 1 and 2 of Introduction that the commutation formula of a pseudodifferential operator with an exponential plays an essential role in Heaviside's calculus. The commutation formula of Hamiltonian with exponential featured in Ch. II in the construction of the calculus of ordered operators is also a key formula. In the theory of μ-structures (rather in its application to pseudodifferential operators) this formula was in effect accepted as the definition of functions of operators

$$-ih\frac{\partial}{\partial x}+\frac{\partial S}{\partial x}.$$

It should be noted, however, that the commutation formula of the pseudodifferential operator with exponential $e^{\frac{i}{h}S(x)}$ of the form

$$e^{-\frac{i}{h}S(x)}P\left(-ih\frac{\partial}{\partial x}\right)e^{\frac{i}{h}S(x)}=P\left(-ih\frac{\partial}{\partial x}+\frac{\partial S}{\partial x}\right)$$

may be interpreted in two ways. By taking $T=e^{\frac{i}{h}S(x)}$ as a multiplication operator by the function $e^{\frac{i}{h}S(x)}$ we obtain

$$(1)\quad T^{-1}P\left(-ih\frac{\partial}{\partial x}\right)T=P\left(-ih\frac{\partial}{\partial x}+\frac{\partial S}{\partial x}\right)$$

or

$$(2)\quad T^{*}P\left(-ih\frac{\partial}{\partial x}\right)T=P\left(-ih\frac{\partial}{\partial x}+\frac{\partial S}{\partial x}\right).$$

Up till now, for the case when $T=e^{\frac{i}{h}S\left(\overset{2}{x},\,\overset{1}{p}\right)}\varphi\left(\overset{2}{x},\,\overset{1}{p}\right)$ we generalized formula (1) and thus obtained the commutation formula of Hamiltonian with exponential. This section is devoted to the

study of the operator

$$T^*P\left(\overset{2}{x}, -ih\overset{1}{\frac{\partial}{\partial x}}\right)T, \text{ where}$$

$$T^* = e^{-\frac{i}{h}S\left(\overset{1}{x},\overset{2}{p}\right)}\bar{\varphi}\left(\overset{1}{x}, \overset{2}{p}\right).$$

A transformation of this type will be called *the canonical transformation* of the operator $P\left(\overset{2}{x}, \overset{1}{p}\right)$. It turns out that in many cases it is more convenient to use the canonical transformation but not transformation (1) of the differential (or pseudodifferential) operator $P\left(\overset{2}{x}, -ih\overset{1}{\frac{\partial}{\partial x}}\right)$.

First, we consider the composition

$$T^*T = e^{\frac{i}{h}\left[S\left(\overset{2}{x},\overset{1}{p}\right)-S\left(\overset{2}{x},\overset{3}{p}\right)\right]}\varphi\left(\overset{2}{x}, \overset{1}{p}\right)\bar{\varphi}\left(\overset{2}{x}, \overset{3}{p}\right),$$

where S is a real function.

From Theorem 1.1 (see below) it follows that

$$T^*T = \frac{\varphi\left(\tilde{x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{1}{p}\right)\bar{\varphi}\left(\tilde{x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{3}{p}\right)}{\left|J\left(\tilde{x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right); \overset{1}{p}; \overset{3}{p}\right)\right|},$$

where $\tilde{x}(x, p, p')$ is a solution of the equation

$$\tilde{x} + \frac{\delta S}{\delta p}(\tilde{x}; p, p') = x$$

and it is supposed that the Jacobian J does not vanish:

$$J(\tilde{x}; p; p') \equiv 1 + \frac{\partial}{\partial \tilde{x}}\frac{\delta S}{\delta P}(\tilde{x}; p, p') \neq 0.$$

The operator T^*T has the following symbol:

$$e^{\frac{i}{h}[S(x, p)-S(x, p')]}\varphi(x, p)\bar{\varphi}(x, p').$$

It turns out that with an accuracy of up to $O(h^\infty)$ the operator T^*T depends only on the values of its symbol on the diagonal $p = p'$. If the terms of the order $O(h^\infty)$ are omitted, then the condition on the Jacobian J may be also weakened.

Let us formulate precisely and prove the above assertions.

Lemma 1.1. *Let $\Phi \in C^{\infty}(\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n)$ and let it grow no faster than a polynomial. Then*

$$\Phi\left(\overset{3}{p}, \overset{2}{x}, \overset{1}{p}\right) f = \frac{1}{(2\pi h)^{2n}} \int e^{\frac{i}{h}(p\cdot x - p'\cdot x'')} \times$$
$$\times f(x'')\,dx''\,dp\,dp' \int e^{\frac{i}{h}(p'-p)x'} \Phi(p, x', p')\,dx' =$$
$$= F^{-1}_{p\to x} F_{x'\to p} F^{-1}_{p'\to x'} \Phi(p, x', p') F_{x''\to p'} f(x''),$$
$$\forall f \in C_0^{\infty}(\mathbf{R}^n).$$

Here $F_{x\to p}: f(x) \to \widetilde{f}(p) = \frac{1}{(2\pi h)^{n/2}} \int\limits_{\mathbf{R}^n} e^{-\frac{i}{h} p\cdot x} f(x)\,dx$

is the so-called $1/h$-Fourier transform.

The proof is obvious.

In the space $L_2(\mathbf{R}^n)$ we set the operator

$$T = \varphi\left(\overset{2}{x}, \overset{1}{p}\right) e^{\frac{i}{h} S\left(\overset{2}{x}, \overset{1}{p}\right)}, \tag{1.1}$$

where $\varphi \in C_0^{\infty}(\mathbf{R}^{2n})$ and $S \in C_0^{\infty}(\mathbf{R}^{2n})$, S being a real function.

Define the vector $\frac{\delta S}{\delta p}(x, p', p) \in \mathbf{R}^n$ whose ith component is a difference derivative

$$\frac{\delta S}{\delta p_i}(x;\ p_1', \ldots, p_{i-1}';\ p_i', p_i;\ p_{i+1}, \ldots, p_n),$$

where $p, p' \in \mathbf{R}^n$. Consider the equation

$$\widetilde{x} + \frac{\delta S}{\delta p}(\widetilde{x};\ p', p) = y \tag{1.2}$$

with respect to the vector $\widetilde{x} \in \mathbf{R}^n$. Denote by J the Jacobian of this equation

$$J(\widetilde{x}, p', p) = \det\left\| \delta_{ij} + \frac{\partial}{\partial x_j}\frac{\delta S}{\delta p_i}(\widetilde{x};\ p_1', \ldots, p_i', p_i, \ldots, p_n)\right\|.$$

Theorem 1.1. *Under condition that $J \neq 0$ the identity holds*

$$T^*T = \varphi\left(\widetilde{x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{1}{p}\right) \overline{\varphi}\left(\widetilde{x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{3}{p}\right) \times$$
$$\times \left| J^{-1}\left(\widetilde{x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{1}{p}, \overset{3}{p}\right)\right|,$$

where $\widetilde{x}(y, p', p)$ is the solution of the equation (1.2).

Proof. By virtue of Lemma 1.1

$$T^*Tf = e^{\frac{i}{h}\left[S\left(\overset{2}{x},\, \overset{1}{p}\right) - S\left(\overset{2}{x},\, \overset{3}{p}\right)\right]} \varphi\left(\overset{2}{x},\, \overset{1}{p}\right) \bar{\varphi}\left(\overset{2}{x},\, \overset{3}{p}\right) f =$$

$$= \frac{1}{(2\pi h)^{2n}} \int e^{\frac{i}{h}(p\cdot x - p'\cdot x'')} f(x'')\, dx''\, dp\, dp' \times$$

$$\times \int e^{\frac{i}{h}[(p'-p)\cdot x' + S(x',\, p') - S(x',\, p)]} \varphi(x',\, p')\bar{\varphi}(x',\, p)\, dx'.$$

In the inner integral we pass over to a new variable y according to the formula $x' = \tilde{x}(y, p', p)$, where $\tilde{x}(y, p', p)$ is the solution of equation (1.2). By virtue of the condition $J \neq 0$ the solution of (1.2) $\tilde{x}(y, p', p)$ exists and belongs to $C^\infty(\mathbf{R}^{3n})$. Taking into account the identity

$$(p' - p)\cdot y = (p' - p)\cdot\tilde{x} + S(\tilde{x},\, p') - S(\tilde{x},\, p)$$

we obtain

$$T^*Tf = \frac{1}{(2\pi h)^{2n}} \int e^{\frac{i}{h}(p\cdot x - p'\cdot x'')} f(x'')\, dx''\, dp\, dp' \times$$

$$\times \int e^{\frac{i}{h} y\cdot(p'-p)} \left| J^{-1}(\tilde{x}(y,\, p',\, p),\, p',\, p)\right| \times$$

$$\times\, \varphi(\tilde{x}(y,\, p',\, p),\, p')\, \bar{\varphi}(\tilde{x}(y,\, p',\, p),\, p)\, dy,$$

where $\varphi \in C_0^\infty(\mathbf{R}^{2n})$.

The proof of Theorem 1.1 then follows from Lemma 1.1.

Example. Let $S(x, p) = -(x - x^2)\, p$; $p,\ x \in R$, $h = 1$ and $\varphi(x, p) \equiv \varphi(x)$. Then

$$Tf = \varphi(x)\, e^{iS\left(\overset{2}{x},\, \overset{1}{p}\right)} f = \varphi(x)\, f(x^2)$$

$\left(\text{here } \overset{1}{p} = -\, i\dfrac{\overset{1}{\partial}}{\partial x} \text{ because } h = 1\right)$.

In this case equation (1.2) has the form

$$y = \tilde{x} + \frac{(-\tilde{x} + \tilde{x}^2)\, p' - (-\tilde{x} + \tilde{x}^2)\, p}{p' - p} \quad \text{or} \quad y = \tilde{x}^2.$$

The Jacobian has the form $J = 2\tilde{x}$. This is why if $\varphi(x)$ is equal to zero in the neighborhood of the point $x = 0$, then

$$T^*Tf = \frac{|\varphi(\sqrt{x})|^2 + |\varphi(-\sqrt{x})|^2}{2\sqrt{x}}\, \theta(x)\, f(x).$$

By the same method (i.e., by an analogous change of variables) we shall derive a similar formula for the operator T_1:

$$T_1 f \overset{\text{def}}{=} \frac{1}{(2\pi h)^{n/2}} \int e^{\frac{i}{h}\left[S\left(\xi,\overset{1}{p}\right)-\overset{2}{x}\xi\right]} \varphi\left(\xi, \overset{1}{p}\right) d\xi \cdot f(x)$$

whose symbol is equal to the $1/h$-Fourier transform of the symbol of the operator T. The formula which is to be obtained will help us to clarify the important role which the $1/h$-Fourier transform plays in canonical transformations of pseudodifferential operators.

Let $J_1(\tilde{\xi}, p', p)$ be the Jacobian of the equation

$$y = \frac{\delta S}{\delta p}(\tilde{\xi};\, p',\, p) + \frac{\delta S}{\delta \xi}(\tilde{\xi}+p-p',\, \tilde{\xi};\, p) \tag{1.3}$$

which is considered in regard to $\tilde{\xi}$ $\left(\text{the vector } \frac{\delta S}{\delta \xi} \text{ is defined in the same way as the vector } \frac{\delta S}{\delta p}\right)$. Thus we obtain

$$J_1(\tilde{\xi}, p', p) = \\ = \det \left\| \frac{\partial}{\partial \xi_j}\left[\frac{\delta S}{\delta p_i}(\tilde{\xi};\, p',\, p) + \frac{\delta S}{\delta \xi_i}(\tilde{\xi}+p-p',\, \tilde{\xi};\, p)\right]\right\|.$$

Theorem 1.2. *Under condition that* $J_1 \neq 0$ *the identity holds*

$$T_1^* T_1 = \varphi\left(\tilde{\xi}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{1}{p}\right) \times \\ \times \bar{\varphi}\left(\tilde{\xi}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right) + \overset{3}{p} - \overset{1}{p}, \overset{3}{p}\right) \left| J_1^{-1}\left(\tilde{\xi}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{1}{p}, \overset{3}{p}\right)\right|, \tag{1.4}$$

where $\tilde{\xi}(y, p', p)$ *is a solution of equation* (1.3).

Proof. By virtue of Lemma 1.1 we obtain

$$T_1^* T_1 f = \frac{1}{(2\pi h)^{2n}} \int e^{\frac{i}{h}(p\cdot x - p'\cdot x'')} f(x'')\, dx''\, dp\, dp' \times \\ \times \int e^{\frac{i}{h}[S(\tilde{\xi},\, p') - S(\tilde{\xi}+p-p',\, p)]}\, \bar{\varphi}(\tilde{\xi}+p-p',\, p)\, \varphi(\tilde{\xi},\, p')\, d\tilde{\xi}.$$

We make a change of variables $\tilde{\xi} \to y$ in the inner integral according to formula (1.3). By virtue of $J_1 \neq 0$ the solution $\tilde{\xi}(y, p', p)$ exists and belongs to $C^\infty(\mathbf{R}^{3n})$. By taking into account the identity

$$(p' - p)\cdot y = S(\tilde{\xi},\, p') - S(\tilde{\xi} + p - p', p)$$

we obtain the desired equality (1.4) as in Theorem 1.1, Q.E.D.

Now let us consider the general situation including the formulas of Theorems 1.1 and 1.2. For simplicity let us restrict ourselves

to a two-dimensional case: $x=(x_1, x_2)$ and $p=(p_1, p_2)$. Consider the operator

$$T_2 f \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi h}} \int e^{\frac{i}{h}\left[S\left(\xi_1, \overset{2}{x_2}, \overset{1}{p}\right) - \overset{2}{x_1}\xi_1\right]} \varphi\left(\xi_1, \overset{2}{x_2}, \overset{1}{p}\right) d\xi_1 f(x). \tag{1.5}$$

In analogy with Theorems 1.1 and 1.2 we obtain

$$T_2^* T_2 f = \frac{1}{(2\pi h)^4} \int e^{\frac{i}{h}(p\cdot x - p'\cdot x'')} f(x'')\, dx''\, dp'\, dp \times$$
$$\times \int e^{\frac{i}{h}[(p_2'-p_2)\cdot z + S(\xi_1, z, p') - S(\xi_1+p_1-p_1', z, p)]} \times$$
$$\times \overline{\varphi}(\xi_1+p_1-p_1', z, p)\, \varphi(\xi_1, z, p')\, dz\, d\xi_1.$$

By a change of variables $(\xi_1, z)\to(y_1, y_2)$ we wish to reduce the exponent in the inner integral to the form

$$\frac{i}{h}[(p_1'-p_1)\cdot y_1 + (p_2'-p_2)\cdot y_2] \equiv \frac{i}{h}(p'-p)\cdot y.$$

Therefore, we set

$$y_1 = \frac{\delta S}{\delta p_1}(\xi_1;\ z;\ p_1',\ p_1;\ p_2) + \frac{\delta S}{\delta \xi_1}(\xi_1+p_1-p_1',\ \xi_1;\ z;\ p_1,\ p_2), \tag{1.6}$$
$$y_2 = z + \frac{\delta S}{\delta p_2}(\xi_1;\ z;\ p_1';\ p_2',\ p_2).$$

Then the following identity is valid:

$$y\cdot(p'-p) = (p_2'-p_2)\cdot z + S(\xi_1, z, p_1', p_2') - S(\xi_1+p_1-p_1', z, p_1, p_2).$$

Suppose that the Jacobian J_2 of system (1.6) is not equal to zero:

$$J_2(\xi_1,\ z,\ p',\ p) \equiv \det\begin{pmatrix} \dfrac{\partial y_1}{\partial \xi_1} & \dfrac{\partial y_1}{\partial z} \\[2mm] \dfrac{\partial y_2}{\partial \xi_1} & \dfrac{\partial y_2}{dz} \end{pmatrix} \neq 0. \tag{1.7}$$

Then there exist smooth solutions $\xi_1(y, p', p)$ and $z(y, p', p)$ of system (1.6). In analogy with Theorems 1.1 and 1.2 after a change of variables $(\xi_1, z)\to(y_1, y_2)$ we obtain

$$T_2^* T_2 = \varphi\left(\xi_1\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right);\ z\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right);\ \overset{1}{p}\right) \times$$
$$\times \overline{\varphi}\left(\xi_1\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right) + \overset{3}{p_1} - \overset{1}{p_1};\ z\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right),\ \overset{3}{p}\right) \times$$
$$\times \left| J_2^{-1}\left(\xi_1\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right);\ z\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right);\ \overset{1}{p};\ \overset{3}{p}\right)\right|. \tag{1.8}$$

Thus, we have proved the theorem that follows.

Theorem 1.3. *Let the operator* T_2 *be defined by formula* (1.5) *and let condition* (1.7) *be fulfilled. Then the identity* (1.8) *holds.*

Note. In Theorems 1.1 and 1.2 it is sufficient to stipulate that the Jacobians J and J_1 should be non-zero only for $(x, p) \in \operatorname{supp} \varphi(x, p)$ and $(\xi, p) \in \operatorname{supp} \varphi(\xi, p)$, respectively. Besides, Theorems 1.1 and 1.2 can be generalized to cover the operator $\overset{3}{T}{}^*\psi\left(\overset{4}{p}-\overset{1}{p}\right)\overset{2}{T}$. Here if the Jacobian J is non-zero for $(p, p') \in \mathbf{R}^{2n}$ such that $p - p' \in \operatorname{supp} \psi$, then the correlation is valid

$$\overset{3}{T}{}^*\psi\left(\overset{4}{p}-\overset{1}{p}\right)\overset{2}{T}=\varphi\left(\tilde{x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{1}{p}\right)\times$$
$$\times\bar{\varphi}\left(\tilde{x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{3}{p}\right)\left|J^{-1}\left(\tilde{x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{1}{p}, \overset{3}{p}\right)\right|\psi\left(\overset{3}{p}-\overset{1}{p}\right),$$

where $\tilde{x}(y, p', p)$ is a solution of equation (1.1).

Under similar conditions Theorem 1.2 can be generalized:

$$\overset{3}{T}{}_1^*\psi\left(\overset{4}{p}-\overset{1}{p}\right)\overset{2}{T}_1=\varphi\left(\tilde{\xi}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{1}{p}\right)\times$$
$$\times\bar{\varphi}\left(\tilde{\xi}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right)+\overset{3}{p}-\overset{1}{p}, \overset{3}{p}\right)\left|J_1^{-1}\left(\tilde{\xi}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{1}{p}, \overset{3}{p}\right)\right|\times$$
$$\times\psi\left(\overset{3}{p}-\overset{1}{p}\right),$$

where $\tilde{\xi}(y, p', p)$ is a solution of equation (1.3) and the Jacobian $J_1 \neq 0$ for $(p', p) \in \mathbf{R}^{2n}$ such that $p - p' \in \operatorname{supp} \psi$.

An analogous assertion also takes place in the situation described in Theorem 1.3.

Lemma 1.2. *Let the function* $\psi^\delta(z) = 0$ *in a* δ*-neighborhood of point* $z = 0$, $\psi^\delta(z) = 1$ *outside a* 2δ*-neighborhood of point* $z = 0$ *and* $\psi^\delta \in C^\infty(\mathbf{R}^n)$. *Next, let* $\varphi \in C_0^\infty(\mathbf{R}^{2n})$ *and* $S(x, p) \in C^\infty(\mathbf{R}^{2n})$, S *being a real function and let the Jacobian*

$$J^0=\det\left\|\delta_{ij}+\frac{\partial^2 S(x, p)}{\partial x_i\,\partial p_j}\right\|\neq 0. \tag{1.9}$$

Then, for any N *and* $k \geqslant 0$ *we have*

$$\left\|\overset{3}{T}{}^*\psi^\delta\left(\overset{4}{p}-\overset{1}{p}\right)\overset{2}{T}\right\|_{W_2^N\to W_2^N}\leqslant c_{N,k}h^k.$$

Proof. By virtue of (1.9) the equation

$$\frac{\partial S(x', p)}{\partial x'}+p=\frac{\partial S(x', p')}{\partial x'}+p'$$

has a unique solution in regard to $p \in \mathbf{R}^n$, namely, $p = p'$. Thus, for $|p - p'| \geqslant \delta$ the vector

$$\frac{\partial}{\partial x'}[-S(x', p) + S(x', p') + (-p + p')x'] \neq 0. \tag{1.10}$$

By representing $\overset{3}{T}{}^* \psi^\delta \left(\overset{4}{p} - \overset{1}{p}\right) \overset{2}{T} f$ in an integral form it is not difficult to verify, in analogy with Theorem 1.1, that the obtained integral will differ from the integral in the proof of Theorem 1.1 by an extra factor $\psi^\delta\ (p - p')$. Since the derivative of the exponent in the inner integral does not turn to zero on the support of $\psi^\delta\ (p - p')$ (see (1.10)), the inner integral may be integrated by parts any number of times (by putting the exponential inside the differential). This immediately leads to the proof of the lemma.

Condition (1.9) can be shown to be essential here.

The meaning of Lemma 1.2 is in the fact that the operator T^*T depends largely on the values of its symbol $e^{\frac{i}{h}[S(x,\, p') - S(x,\, p)]} \times$ $\times\ \varphi(x, p')\, \overline{\varphi}(x, p)$ on the diagonal $p = p'$ (correct to $O(h^\infty)$).

Thus, if in Theorem 1.1 the condition $J \neq 0$ is weakened by replacing it by condition (1.9), then by virtue of Note of 1.1 and Lemma 1.2 we obtain the theorem that follows.

Theorem 1.4. *Let ψ^δ be the function of Lemma 1.2 and condition (1.9) be fulfilled. Then for a sufficiently small δ the following equality is valid:*

$$T^*T = \varphi\left(\tilde{x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{1}{p}\right)^{1}\varphi\left(\tilde{x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{3}{p}\right) \times$$
$$\times \left| J^{-1}\left(\tilde{x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{1}{p}, \overset{3}{p}\right)\right| \left(1 - \psi^\delta\left(\overset{3}{p} - \overset{1}{p}\right)\right) + A, \tag{1.2'}$$

where $\|A\|_{W_2^N \to W_2^N} \leqslant c_{N,k} h^k$ *for any* $N,\ k \geqslant 0$.

The latter lemma and theorem can also be formulated for the case of the operators T_1 and T_2. For the case of the operator T_1 we note that the equation

$$\frac{\partial S}{\partial \xi}(\xi, p') - \frac{\partial S}{\partial \xi}(\xi + p - p', p) = 0$$

has a unique solution $p = p'$ if the Jacobian

$$J_1^0(\xi, p) \overset{\text{def}}{=} \det \left\| \left(\frac{\partial^2 S}{\partial \xi_i\, \partial \xi_j} + \frac{\partial^2 S}{\partial \xi_i\, \partial p_j}\right)(\xi, p) \right\|$$

does not become zero. For this reason, in analogy with the preceding theorem, the theorem that follows is obtained under the same conditions on S and φ.

Theorem 1.5. *Let $J_1^0 \neq 0$. Then for sufficiently small δ the following equality is valid:*

$$T_1^* T_1 = \varphi\left(\tilde{\xi}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{1}{p}\right) \overline{\varphi}\left(\tilde{\xi}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right) + \overset{3}{p} - \overset{1}{p}, \overset{3}{p}\right) \times$$
$$\times \left| J_1^{-1}\left(\tilde{\xi}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right), \overset{1}{p}, \overset{3}{p}\right)\right| \left(1 - \psi^\delta\left(\overset{3}{p} - \overset{1}{p}\right)\right) + A,$$

where $\| A \|_{W_2^N \to W_2^N} \leqslant c_{N,k} h^k$ *for any* N, $k \geqslant 0$.

Theorem 1.3 can be reformulated in a similar way.

In the given formulas one may notice a certain duality in regard to the operators T^*T and $T_1^*T_1$. First of all we note that we may set $\overset{3}{p} = \overset{1}{p}$ correct to $O(h)$. Then, instead of the Jacobians J and J_1 we may substitute the Jacobians J^0 and J_1^0. If we denote $\varphi_1 = \dfrac{\varphi}{\sqrt{|J^0|}}$ and $\varphi_2 = \dfrac{\varphi}{\sqrt{|J_1^0|}}$ then the obtained formulas will have the form

$$\begin{aligned} T^*T &= |\varphi_1|^2\left(\tilde{x}\left(\overset{2}{x}, \overset{1}{p}\right), \overset{1}{p}\right) + O(h), \\ T_1^*T_1 &= |\varphi_2|^2\left(\tilde{\xi}\left(\overset{2}{x}, \overset{1}{p}\right), \overset{1}{p}\right) + O(h), \end{aligned} \tag{*}$$

where $\tilde{x}(x, p)$ is a solution of the equation

$$\tilde{x} + \frac{\partial S}{\partial p}(\tilde{x}, p) = x$$

and $\tilde{\xi}(x, p)$ is a solution of the equation

$$\frac{\partial S}{\partial p}(\tilde{\xi}, p) + \frac{\partial S(\tilde{\xi}, p)}{\partial \tilde{\xi}} = x.$$

Let us now dwell on the geometrical meaning of the observed duality in the case $n = 1$. Consider a one-parameter family of infinitely smooth curves in the phase plane (ξ, x):

$$\xi = \Xi(y, p), \qquad x = X(y, p), \tag{1.11}$$

where the parameter $p \in \mathbf{R}$ and y is a coordinate.

The one-parameter condition has the form

$$D = \det\begin{pmatrix} \dfrac{\partial X}{\partial y} & \dfrac{\partial X}{\partial p} \\ \dfrac{\partial \Xi}{\partial y} & \dfrac{\partial \Xi}{\partial p} \end{pmatrix} \neq 0.$$

By a suitable change of coordinate y we may always obtain $D = 1$ in new variables *. Hence without loss of generality we can set

$$D = 1. \tag{1.12}$$

* This assertion is proved in Chapter IV.

From this condition it follows that at degeneration points of projection onto the axis x, i.e., at points where $\partial \Xi/\partial y = 0$, the derivative $\partial \Xi/\partial p$ is non-zero. Similarly, at degeneration points of projection onto the axis ξ we have $\partial X/\partial p \neq 0$. At other points, either $\partial X/\partial p \neq 0$ or $\partial \Xi/\partial p \neq 0$.

This is why we can cover the family of curves (1.11) by the system of intersecting neighborhoods, in each of which whether $\partial X/\partial y$ and $\partial \Xi/\partial p$ or $\partial X/\partial p$ and $\partial \Xi/\partial y$ are non-zero. The first will be called non-singular patches and the second—singular patches. The set of patches is called a canonical atlas of the family of curves.

Along with (1.11) we shall also consider a "shifted" family of curves

$$\xi = \Xi(y, p) - p, \quad x = X(y, p).$$

Consider a finite real function $\check{\varphi}(y, p) \in C^\infty$ with support in one of patches of the canonical atlas.

We shall study two separate cases when the support $\check{\varphi}$ lies either in a non-singular or a singular patch.

Case 1. Non-singular patches. Since $\partial X/\partial y \neq 0$ in this patch, it is possible to make a change of variables $x = X(y, p)$, hence $y = y(x, p) \in C^\infty$. Denote $\tilde{\xi}(x, p) \overset{\text{def}}{=} \Xi(y(x, p), p)$. We have

$$D \equiv \left.\frac{D(X, \Xi)}{D(y, p)}\right|_{y=y(x,p)} = \frac{D(x, \tilde{\xi}(x, p))}{D(x, p)} \times$$

$$\times \left.\frac{D(X, p)}{D(y, p)}\right|_{y=y(x,p)} = \begin{vmatrix} 1 & 0 \\ \dfrac{\partial\tilde{\xi}}{\partial x} & \dfrac{\partial\tilde{\xi}}{\partial p} \end{vmatrix} \cdot \begin{vmatrix} \dfrac{\partial X}{\partial y} & \dfrac{\partial X}{\partial p} \\ 0 & 1 \end{vmatrix} = 1;$$

consequently,

$$\frac{\partial X}{\partial y}(y(x, p), p)\frac{\partial\tilde{\xi}}{\partial p}(x, p) = 1$$

or

$$\frac{\partial\tilde{\xi}}{\partial p}(x, p) = \frac{\partial y(x, p)}{\partial x}. \tag{1.13}$$

Let the patch under consideration contain the point $(y_0(p), p)$ and let $x_0(p) = X(y_0(p), p)$. Denote

$$S(x, p) = \int_{x_0(p)}^{x} [\tilde{\xi}(x', p) - p]\, dx' + S_0(p),$$

where

$$\frac{\partial S_0}{\partial p}(p) = y_0(p) - x_0(p) + (\Xi(y_0(p), p) - p)\frac{\partial x_0}{\partial p}(p). \tag{1.14}$$

Show that

$$\frac{\partial S}{\partial p}(x, p) + x = y(x, p). \tag{1.15}$$

From the definition of the function S it follows that $\frac{\partial S}{\partial x}=$ $=\tilde{\xi}(x, p)-p$. Then by virtue of (1.13)

$$\frac{\partial y(x, p)}{\partial x}=\frac{\partial\tilde{\xi}}{\partial p}(x, p)=1+\frac{\partial^2 S(x, p)}{\partial x\,\partial p}=\frac{\partial}{\partial x}\left(x+\frac{\partial S}{\partial p}\right),$$

i e., $\frac{\partial S}{\partial p}(x, p)+x-y(x, p)=f(p)$ is a function of p only. We have

$$\left.\frac{\partial S(x, p)}{\partial p}\right|_{x=x_0(p)}=\frac{\partial S_0(p)}{\partial p}-\frac{\partial S}{\partial x}(x_0(p), p)\frac{\partial x_0(p)}{\partial p}=$$

$$=\frac{\partial S_0(p)}{\partial p}-(\Xi(y_0(p), p)-p)\frac{\partial x_0(p)}{\partial p}.$$

Therefore, from (1.14) we obtain

$$f(p)=\frac{\partial S_0}{\partial p}-(\Xi(y_0(p), p)-p)\frac{\partial x_0(p)}{\partial p}+x_0(p)-y_0(p)=0,$$

from which follows (1.15).

Denote by T' an operator in the form

$$T'=e^{\frac{i}{h}S\left(\overset{2}{x},\,\overset{1}{p}\right)}\frac{\check{\varphi}\left(y\left(\overset{2}{x},\,\overset{1}{p}\right),\,\overset{1}{p}\right)}{\sqrt{\left|\frac{\partial X}{\partial y}\left(y\left(\overset{2}{x},\,\overset{1}{p}\right),\,\overset{1}{p}\right)\right|}},\quad \overset{1}{p}=-ih\frac{\overset{1}{\partial}}{\partial x}. \tag{1.16}$$

From (1.15) it follows that $1+\frac{\partial^2 S}{\partial x\,\partial p}=\frac{\partial y}{\partial x}=\left(\frac{\partial X}{\partial y}\right)^{-1}$. Therefore, it follows from (*) that

$$(T')^*T'=|\check{\varphi}|^2\left(\overset{2}{x},\,\overset{1}{p}\right)+O(h).$$

In fact, a more precise theorem exists (in its formulation in the text below we shall omit the signs $'$ and $\check{}$ to avoid any ambiguity).

Theorem 1.6. *Under the above conditions an equality is valid:*

$$T^*T=\varphi\left(\overset{2}{x},\,\overset{1}{p}\right)\varphi\left(\overset{2}{x},\,\overset{3}{p}\right)+h^2A(h),$$

where $\|A(h)\|_{W_2^N\to W_2^N}\leqslant c_N$ *for any* $N\geqslant 0$, *and* $\operatorname{Im}\varphi=0$.

Before passing over to the proof of this theorem we shall consider the case when the support $\check{\varphi}$ belongs to a singular patch.

Case 2. Singular patches. In this case $\partial\Xi/\partial y\neq 0$ and, consequently, the equation

$$\xi+p=\Xi(y, p)$$

has a solution $y(\xi, p) \in C^\infty$. Set $x(\xi, p) \overset{\text{def}}{=} X(y(\xi, p), p)$. Just as in Case 1, we have

$$1 = D\big|_{y=y(\xi,\, p)} = \frac{D(x(\xi, p), \xi+p)}{D(\xi, p)} \, \frac{D(\Xi(y, p)-p, p)}{D(y, p)}\bigg|_{y=y(\xi,\, p)} =$$

$$= \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial p} \\ 1 & 1 \end{vmatrix} \begin{vmatrix} \dfrac{\partial \Xi}{\partial y} & \dfrac{\partial \Xi}{\partial p} - 1 \\ 0 & 1 \end{vmatrix} = \frac{\partial \Xi}{\partial y}\left(\frac{\partial x}{\partial \xi} - \frac{\partial x}{\partial p}\right).$$

Hence

$$\frac{\partial y}{\partial \xi}(\xi, p) = \frac{\partial x}{\partial \xi}(\xi, p) - \frac{\partial x}{\partial p}(\xi, p). \tag{1.17}$$

Define the function

$$\widetilde{S}(\xi, p) = \int_{\xi_0(p)}^{\xi} x(\xi', p)\, d\xi' + \widetilde{S}_0(p),$$

where

$$\frac{\partial \widetilde{S}_0(p)}{\partial p} = X(y_0(p), p)\left[\frac{\partial \xi_0(p)}{\partial p} + 1\right] - y_0(p) \tag{1.18}$$

(we assume that the patch contains the point $(y_0(p), p)$; $\xi_0(p) = \Xi(y_0(p), p) - p$). Note that at the intersection of singular and non-singular patches the function $\widetilde{S}(\xi, p)$ is the Legendre transform of the function $S(x, p)$ with respect to argument x, i.e.,

$$\widetilde{S}(\xi, p) = \xi x(\xi, p) - S(x(\xi, p), p).$$

Show that

$$-\frac{\partial \widetilde{S}}{\partial p}(\xi, p) + \frac{\partial \widetilde{S}}{\partial \xi}(\xi, p) = y(\xi, p). \tag{1.19}$$

From the definition of the function $\widetilde{S}$ we have

$$\frac{\partial \widetilde{S}}{\partial \xi}(\xi, p) = x(\xi, p).$$

Hence and from (1.17) it follows that

$$\frac{\partial y}{\partial \xi} = -\frac{\partial^2 \widetilde{S}}{\partial p\, \partial \xi} + \frac{\partial^2 \widetilde{S}}{\partial \xi^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial \widetilde{S}}{\partial \xi} - \frac{\partial \widetilde{S}}{\partial p}\right);$$

therefore

$$-\frac{\partial \widetilde{S}}{\partial p} + \frac{\partial \widetilde{S}}{\partial \xi} - y(\xi, p) = f(p).$$

But at point $\xi=\xi_0(p)$ we have

$$\frac{\partial\widetilde{S}}{\partial p}(\xi,\ p)\big|_{\xi=\xi_0(p)}=\frac{\partial\widetilde{S}_0(p)}{\partial p}-\frac{\partial\widetilde{S}}{\partial\xi}(\xi_0(p),\ p)\frac{\partial\xi_0(p)}{\partial p}=$$
$$=\frac{\partial\widetilde{S}_0(p)}{\partial p}-X(y_0(p),\ p)\frac{\partial\xi_0(p)}{\partial p}.$$

For this reason and by virtue of (1.18) we obtain

$$f(p)=-\frac{\partial\widetilde{S}_0}{\partial p}+X(y_0(p),\ p)\frac{\partial\xi_0(p)}{\partial p}+X(y_0(p),\ p)-y_0(p)=0,$$

from which follows (1.19).

Equality (1.19) may be rewritten in the form

$$\frac{\partial\widetilde{S}}{\partial\xi}(\Xi(y,\ p)-p,\ p)-\frac{\partial\widetilde{S}}{\partial p}(\Xi(y,\ p)-p,\ p)=y \tag{1.19'}$$

because

$$\xi+p\equiv\Xi(y(\xi,\ p),\ p)\ \text{or}$$
$$y'\equiv y(\Xi(y',\ p)-p,\ p).$$

Denote

$$T_1'=\frac{\lambda}{\sqrt{-2\pi ih}}\int e^{\frac{i}{h}\left[\overset{2}{x}\xi-\widetilde{S}\left(\xi,\overset{1}{p}\right)\right]}\frac{\check{\varphi}\left(y\left(\xi,\ \overset{1}{p}\right),\ \overset{1}{p}\right)}{\sqrt{\left|\frac{\partial\Xi}{\partial y}\left(y\left(\xi,\ \overset{1}{p}\right),\ \overset{1}{p}\right)\right|}}d\xi, \tag{1.20}$$

where λ is a constant in modulus equal to 1 introduced for the sake of convenience. Its meaning will be made clear below.

By virtue of (1.19) we have

$$\frac{\partial\Xi}{\partial y}=\left[\frac{\partial y}{\partial\xi}\right]^{-1}=\left(\frac{\partial^2\widetilde{S}}{\partial\xi^2}-\frac{\partial^2\widetilde{S}}{\partial\xi\,\partial p}\right)^{-1}=(J_1^0)^{-1}.$$

For this reason from (*) it follows that

$$(T_1')^*\,T_1'=|\check{\varphi}|^2\left(\overset{2}{x},\ \overset{1}{p}\right)+O(h).$$

In fact, there exists a more precise relation; namely, the theorems that follow are valid (again we shall omit $'$ and $\check{}$).

Theorem 1.7. *There exists an equality*

$$T_1^*T_1=\varphi\left(\overset{2}{x},\ \overset{1}{p}\right)\varphi\left(\overset{2}{x},\ \overset{3}{p}\right)+h^2A(h),$$

where $A(h)$ *is such an operator that* $\|A(h)\|_{W_2^N\to W_2^N}\leqslant c_N$ *for any* $N\geqslant 0$, *and* $\operatorname{Im}\varphi=0$.

The proof of this theorem will be cited somewhat later, whereas now we shall formulate the result related to the canonical trans-

formation of the pseudodifferential operator $f\left(\overset{2}{x}, \overset{1}{p}\right)$ whose symbol belongs to $C^\infty(\mathbf{R}^2)$ and grows no faster than a certain degree of its arguments. In the case of a non-singular patch there exists the relation

$$T^*f\left(\overset{2}{x}, \overset{1}{p}\right)T = |\varphi|^2\left(\overset{2}{x}, \overset{1}{p}\right)f\left(X\left(\overset{2}{x}, \overset{1}{p}\right), \Xi\left(\overset{2}{x}, \overset{1}{p}\right)\right) + O(h).$$

For a singular patch there exists an identical relation

$$T_1^*f\left(\overset{2}{x}, \overset{1}{p}\right)T_1 = |\varphi|^2\left(\overset{2}{x}, \overset{1}{p}\right)f\left(X\left(\overset{2}{x}, \overset{1}{p}\right), \Xi\left(\overset{2}{x}, \overset{1}{p}\right)\right) + O(h).$$

In fact, there exists a more precise statement. Namely, the theorem that follows is valid. Its proof will be cited later.

Theorem 1.8. *Let $f \in C^\infty(\mathbf{R}^3)$, where*

$$f(p, x, p') = f(p', x, p) \text{ and } |f^{(\alpha)}| \leqslant c_k(1 + |p'|^k + |p|^k)$$

for all α and some $k > 0$, c_k do not depend on x. Let T be an operator defined by the formula (1.6). *Then, if* $\operatorname{Im}\varphi = 0$,

$$T^*\left[f\left(\overset{3}{p}, \overset{2}{x}, \overset{1}{p}\right)\right]T = \varphi\left(\overset{2}{x}, \overset{3}{p}\right)f\left(\Xi\left(\overset{2}{x}, \overset{3}{p}\right);\right.$$

$$\left.\frac{1}{2}\left[X\left(\overset{2}{x}, \overset{3}{p}\right) + X\left(\overset{2}{x}, \overset{1}{p}\right)\right]; \Xi\left(\overset{2}{x}, \overset{1}{p}\right)\right)\varphi\left(\overset{2}{x}, \overset{1}{p}\right) + h^2A_2(h), \quad (1.21)$$

where $A_2(h)$ is an operator satisfying the following relation: $\|A_2(h)\|_{W_2^N \to W_2^N} \leqslant c_N$ for all $N \geqslant 0$. Let T_1 be an operator defined by the formula (1.20). *Then, if* $\operatorname{Im}\varphi = 0$,

$$T_1^*\left[f\left(\overset{3}{p}, \overset{2}{x}, \overset{3}{p}\right)\right]T_1 = \varphi\left(\overset{2}{x}, \overset{3}{p}\right)f\left(\Xi\left(\overset{2}{x}, \overset{3}{p}\right);\right.$$

$$\left.\frac{1}{2}\left[X\left(\overset{2}{x}, \overset{1}{p}\right) + X\left(\overset{2}{x}, \overset{3}{p}\right)\right]; \Xi\left(\overset{2}{x}, \overset{1}{p}\right)\right)\varphi\left(\overset{2}{x}, \overset{1}{p}\right) + h^2A_3(h),$$

where $A_3(h)$ is an operator satisfying the following relation: $\|A_3(h)\|_{W_2^N \to W_2^N} \leqslant c_N$ for all $N \geqslant 0$.

The proof of this theorem will be cited at the end of the section.

Note. Below the following construction of the family of curves $\Xi(y, p)$, $X(y, p)$ will be used. Let $x(x^0, p^0, t)$ and $p(x^0, p^0, t)$ be a solution of the Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial p}, \quad x(0) = x^0, \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad p(0) = p^0, \quad H(x, p) \in C^\infty.$$

Then according to Liouville's theorem (see Ch. IV) we obtain

$$\begin{vmatrix} \dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial p_0} \\ \dfrac{\partial p}{\partial x_0} & \dfrac{\partial p}{\partial p_0} \end{vmatrix} = 1 \text{ at any moment of time } t.$$

The equations

$$x = x(x^0, p^0, t),$$
$$p = p(x^0, p^0, t)$$

define the family of curves (p^0 being a parameter) satisfying the above requirements at fixed t.

Proof of Theorem 1.6. According to Theorem 1.4 we have

$$T^*T = \varphi\left(y\left(\tilde{x}, \overset{1}{p}\right), \overset{1}{p}\right) \varphi\left(y\left(\tilde{x}, \overset{3}{p}\right), \overset{3}{p}\right) \times$$
$$\times \frac{\left| J^{-1}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right)\right|}{\sqrt{\left|\dfrac{\partial X}{\partial y}\left(y\left(\tilde{x}, \overset{1}{p}\right), \overset{1}{p}\right) \dfrac{\partial X}{\partial y}\left(y\left(\tilde{x}, \overset{3}{p}\right), \overset{3}{p}\right)\right|}} \times$$
$$\times\left[1 - \psi^\delta\left(\overset{1}{p} - \overset{3}{p}\right)\right] + O(h^\infty), \tag{1.22}$$

where

$$\tilde{x} = \tilde{x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right),$$
$$J^{-1}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right) = \frac{\partial \tilde{x}}{\partial x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right),$$
$$\| O(h^\infty) \|_{W_2^N \to W_2^N} \leqslant c_{k,\, N} h^k$$

for all k and $N \geqslant 0$.

Recall that $\tilde{x}(y, p, p')$ is a solution of the equation

$$\tilde{x} + \frac{S(\tilde{x}, p) - S(\tilde{x}, p')}{p - p'} = y. \tag{1.23}$$

Now observe that from (1.15) we have

$$X(y, p) + \frac{\partial S}{\partial p}(X(y, p), p) = y.$$

Therefore, it is quite natural to seek the solution of (1.23) in the form

$$\tilde{x}(y, p, p') = X(y, p) + (p' - p) X_1(y, p) +$$
$$+ O(p' - p)^2. \tag{1.24}$$

Note that by virtue of Lemma 1.2 we may consider all functions only in a small strip near the diagonal $p = p'$. Here and below we denote the function of the form $(p' - p)^2 \Phi(y, p, p')$ by the symbol $O(p' - p)^2$, where $\Phi \in C^\infty$.

By putting (1.24) in equation (1.23) and expanding all functions in p' in the neighborhood of p we shall easily find the coefficient X_1:

$$X_1(y, p) = -\frac{1}{2} I_0(y, p) \frac{\partial^2 S}{\partial p^2}(X(y, p), p), \tag{1.25}$$

where

$$I_0^{-1}(y, p) = 1 + \frac{\partial^2 S}{\partial x \, \partial p}(X(y, p), p) = \left.\frac{\partial y(x, p)}{\partial x}\right|_{x=X(y, p)}.$$

Or, in other words,

$$I_0(y, p) = \frac{\partial X}{\partial y}(y, p) \neq 0 \text{ (in a non-singular patch).}$$

Prove that

$$X_1(y, p) = \frac{1}{2} \frac{\partial X}{\partial p}(y, p), \tag{1.26}$$

i.e., expansion (1.24) has the form

$$\tilde{x}(y, p, p') = X(y, p) + \frac{1}{2}(p' - p) \frac{\partial X}{\partial p}(y, p) + O(p' - p)^2. \tag{1.27}$$

In fact, from (1.15) we obtain

$$\frac{\partial y(x, p)}{\partial p} = \frac{\partial^2 S}{\partial p^2}(x, p).$$

However,

$$\frac{\partial y}{\partial p}(x, p)\,|_{x=X(y, p)} = -\frac{1}{I_0(y, p)} \frac{\partial X}{\partial p}(y, p). \tag{1.28}$$

For this reason from (1.25) we have

$$X_1(y, p) = -\frac{1}{2} I_0(y, p) \left[-\frac{1}{I_0(y, p)} \frac{\partial X}{\partial p}(y, p) \right] = \frac{1}{2} \frac{\partial X}{\partial p},$$

i.e., (1.26) is proved.

By differentiating (1.27) with respect to y we obtain an expansion of the Jacobian $J^{-1}(y, p, p') = \frac{\partial \tilde{x}}{\partial y}(y, p, p')$ in the form

$$\begin{aligned} J^{-1}(y, p, p') &= \frac{\partial X}{\partial y} + \frac{1}{2}(p' - p) \frac{\partial^2 X}{\partial y \, \partial p} + O(p' - p)^2 = \\ &= I_0(y, p) + \frac{1}{2}(p' - p) \frac{\partial I_0}{\partial p}(y, p) + O(p' - p)^2. \end{aligned} \tag{1.29}$$

Now we shall expand the Jacobians

$$\frac{\partial X}{\partial y}(y(\tilde{x}, p), p) \text{ and } \frac{\partial X}{\partial y}(y(\tilde{x}, p'), p')$$

in degrees of $(p' - p)$. In an analogous manner after simple transformations and by taking into account (1.26) from (1.27) we obtain (here $\tilde{x} = \tilde{x}(y, p, p')$)

$$\frac{\partial X}{\partial y}(y(\tilde{x}, p), p) = I_0(y, p) + \\ + \frac{1}{2}(p' - p)\frac{\frac{\partial X}{\partial p}(y, p)}{I_0(y, p)}\frac{\partial I_0(y, p)}{\partial p} + O(p' - p)^2.$$

For the second Jacobian we have

$$\frac{\partial X}{\partial y}(y(\tilde{x}, p'), p') = I_0(y, p) + (p' - p)\left[\frac{\partial I_0(y, p)}{\partial p} - \right. \\ \left. - \frac{1}{2}\frac{1}{I_0(y, p)}\frac{\partial I_0(y, p)}{\partial y}\frac{\partial X(y, p)}{\partial p}\right] + O(p' - p)^2.$$

For these two expressions and from (1.29) we obtain

$$\frac{|J^{-1}(y, p, p')|}{\sqrt{\left|\frac{\partial X}{\partial y}(y(\tilde{x}, p), p)\frac{\partial X}{\partial y}(y(\tilde{x}, p'), p')\right|}} = 1 + O(p' - p)^2. \tag{1.30}$$

Out of all functions in the right-hand side of (1.22) only the functions $\varphi(y(\tilde{x}, p), p)$ and $\varphi(y(\tilde{x}, p'), p')$ remain to be expanded. By virtue of (1.28) their expansions in degrees of $(p' - p)$ will have the form

$$\varphi(y(\tilde{x}, p), p) = \varphi(y, p) + \frac{\partial \varphi}{\partial y}(y, p)\frac{(p' - p)}{2I_0(y, p)}\frac{\partial X}{\partial p}(y, p) + O(p' - p)^2$$

and

$$\varphi(y(\tilde{x}, p'), p') = \varphi(y, p') - \\ - \frac{\partial \varphi}{\partial y}(y, p)\frac{(p' - p)}{2I_0(y, p)}\frac{\partial X}{\partial p}(y, p) + O(p' - p)^2.$$

Note that in the second case we do not expand $\varphi(y, p')$ in the argument p'. Besides, here the remainders $O(p' - p)^2$ have the form

$$(p' - p)^2\,\Phi(y, p, p'), \quad \text{where} \quad \Phi \in C_0^\infty.$$

From these equalities and from (1.30) we obtain that the symbol of the operator in the right-hand member of (1.22) is equal to the expression

$$\varphi(x, p)\,\varphi(x, p')\,[1 - \psi^\delta(p - p')] + \\ + (p' - p)^2\,\Phi(x, p, p'), \tag{1.31}$$

where $\Phi \in C_0^\infty$. (In (1.30) we replaced the argument y in the symbol by the operator $\overset{2}{x}$; this is why in (1.31) we raplace y by x.)

Consider the operator

$$\psi^{\delta}\left(\overset{1}{p}-\overset{3}{p}\right)\varphi\left(\overset{2}{x},\ \overset{1}{p}\right)\varphi\left(\overset{2}{x},\ \overset{3}{p}\right)=$$
$$=h^{m}\theta_{m}\left(-i\,\overset{1}{\frac{\partial}{\partial x}},\ -ih\,\overset{2}{\frac{\partial}{\partial x}},\ \overset{3}{x},\ -ih\,\overset{4}{\frac{\partial}{\partial x}},\ -i\,\overset{5}{\frac{\partial}{\partial x}}\right),$$

where

$$\theta_{m}(\xi,\ p,\ x,\ p',\ \xi')=(\xi-\xi')^{m}\,\frac{\psi^{\delta}(p-p')}{(p-p')^{m}}\,\varphi(x,\ p)\,\varphi(y,\ p')\in C_{0}^{\infty},$$

m being any positive integer.

Evidently, we have the following estimate:

$$\left\|\psi^{\delta}\left(\overset{1}{p}-\overset{3}{p}\right)\varphi\left(\overset{2}{x},\ \overset{1}{p}\right)\varphi\left(\overset{2}{x},\ \overset{3}{p}\right)\right\|_{W_{2}^{N}\to W_{2}^{N}}\leqslant$$
$$\leqslant h^{m}\,\|\theta_{m}\|_{\mathscr{B}_{N}(\mathrm{R}^{5})}\leqslant c_{N,\,m}h^{m}, \tag{1.32}$$

i.e., this operator is $O(h^{\infty})$ in our notation.

In the same way,

$$\left(\overset{3}{p}-\overset{1}{p}\right)^{2}\Phi\left(\overset{2}{x},\ \overset{1}{p},\ \overset{3}{p}\right)=h^{2}A_{1},\ \text{ where }\ \|A_{1}\|_{W_{2}^{N}\to W_{2}^{N}}\leqslant c_{N}. \tag{1.33}$$

Therefore, the equality of Theorem 1.6 follows from (1.32) and (1.33). Q.E.D.

Proof of Theorem 1.7. By virtue of Theorem 1.5 we have

$$T_{1}^{*}T_{1}=\varphi\left(y\left(\tilde{\xi},\ \overset{1}{p}\right);\ \overset{1}{p}\right)\varphi\left(y\left(\tilde{\xi}+\overset{1}{p}-\overset{3}{p},\ \overset{3}{p}\right);\ \overset{3}{p}\right)\times$$
$$\times\frac{\left|J_{1}^{-1}\left(\overset{2}{x},\ \overset{1}{p},\ \overset{3}{p}\right)\right|}{\sqrt{\left|\dfrac{\partial\Xi}{\partial y}\left(y\left(\tilde{\xi},\ \overset{1}{p}\right);\ \overset{1}{p}\right)\dfrac{\partial\Xi}{\partial y}\left(y\left(\tilde{\xi}+\overset{1}{p}-\overset{3}{p};\ \overset{3}{p}\right);\ \overset{3}{p}\right)\right|}}\times$$
$$\times\left[1-\psi^{\delta}\left(\overset{1}{p}-\overset{3}{p}\right)\right]+O(h^{\infty}), \tag{1.34}$$

where

$$\tilde{\xi}=\tilde{\xi}\left(\overset{2}{x},\ \overset{1}{p},\ \overset{3}{p}\right),\quad J_{1}^{-1}\left(\overset{2}{x},\ \overset{1}{p},\ \overset{3}{p}\right)=\frac{\partial\tilde{\xi}}{\partial x}\left(\overset{2}{x},\ \overset{1}{p},\ \overset{3}{p}\right),$$
$$\|O(h^{\infty})\|_{W_{2}^{N}\to W_{2}^{N}}\leqslant c_{N,\,k}h^{k}\ \text{ for all } N,\ k\geqslant 0.$$

Here the function $\tilde{\xi}(y,\ p,\ p')$ is a solution of the equation

$$\frac{\tilde{S}(\tilde{\xi}+p-p',\ p')-\tilde{S}(\tilde{\xi},\ p')}{p-p'}-\frac{\tilde{S}(\tilde{\xi},\ p')-\tilde{S}(\tilde{\xi},\ p)}{p'-p}=y. \tag{1.35}$$

Taking into account the equality (1.19) it is natural to seek the solution in the form

$$\tilde{\xi}(y,\ p,\ p')=\Xi(y,\ p)-p+(p'-p)\,\Xi_{1}(y,\ p)+O(p'-p)^{2}.$$

Put this expansion in (1.35). As in Theorem 1.6 we obtain the coefficient Ξ_1:

$$\Xi_1(y, p) = 1 - \frac{I_1(y, p)}{2}\left(\frac{\partial^2 \tilde{S}}{\partial \xi^2}(\xi, p) - \frac{\partial^2 \tilde{S}}{\partial p^2}(\xi, p)\right)\Big|_{\xi=\Xi(y, p)-p}, \quad (1.36)$$

where

$$(I_1)^{-1} = \left(\frac{\partial^2 \tilde{S}}{\partial \xi^2} - \frac{\partial^2 \tilde{S}}{\partial \xi\, \partial p}\right)\Big|_{\xi=\Xi(y, p)-p} = \frac{\partial y}{\partial \xi}\Big|_{\xi=\Xi(y, p)-p},$$

or

$$I_1(y, p) = \frac{\partial \Xi}{\partial y}(y, p).$$

From (1.19′) we obtain (by differentiating with respect to ξ and p and summing up the obtained equalities)

$$\frac{\partial^2 \tilde{S}}{\partial \xi^2} - \frac{\partial^2 \tilde{S}}{\partial p^2} = \frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial p}.$$

From the identity $\xi + p = \Xi(y(\xi, p), p)$ we also obtain

$$2 = \frac{\partial \Xi}{\partial y}\left(\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial p}\right) + \frac{\partial \Xi}{\partial p}.$$

By virtue of (1.36) these two equalities produce

$$\Xi_1(y, p) = \frac{1}{2}\frac{\partial \Xi}{\partial p}(y, p).$$

Thus, the expansion of function $\tilde{\xi}(y, p, p')$ in degrees of $(p'-p)$ has the form

$$\tilde{\xi}(y, p, p') = \Xi(y, p) - p + \frac{1}{2}(p'-p)\frac{\partial \Xi}{\partial p}(y, p) + O(p'-p)^2. \quad (1.37)$$

By differentiating this equality with respect to y we obtain an expansion of the Jacobian

$$J_1^{-1}(y, p, p') \equiv \frac{\partial \tilde{\xi}}{\partial y}(y, p, p') = I_1(y, p) + \frac{1}{2}(p'-p)\frac{\partial I_1(y, p)}{\partial p} + O(p'-p)^2. \quad (1.38)$$

Further, in a manner analogous to the proof of Theorem 1.6, we expand the Jacobians $\frac{\partial \Xi}{\partial y}(y(\tilde{\xi}, p); p)$ and $\frac{\partial \Xi}{\partial y}(y(\tilde{\xi}+p-p', p'); p')$ as well as the amplitudes $\varphi(y(\tilde{\xi}, p); p)$ and $\varphi(y(\tilde{\xi}+p-p', p'); p')$ in degrees of $(p'-p)$.

Finally, we obtain the symbol of the operator in the right-hand member of (1.34) in the form

$$\varphi(x, p)\varphi(x, p')[1-\psi^\delta(p-p')] + (p'-p)^2\Phi(x, p, p'),$$

where $\Phi \in C_0^\infty$.

By repeating the discourse of Theorem 1.6 we can obtain that the operator with the above symbol is equal to

$$\varphi\left(\overset{2}{x}, \overset{1}{p}\right) \varphi\left(\overset{2}{x}, \overset{3}{p}\right) + h^2 A_2(h),$$

where $\|A_2\|_{W_2^N \to W_2^N} \leqslant c_N$ for all $N \geqslant 0$, Q.E.D.

Before passing over to the proof of Theorem 1.8 we shall make a comment analogous to Note 1.1.

Note. Let $f \in C^\infty(\mathbf{R}^3)$ be the function defined in Theorem 1.8. Then for the operator T defined by formula (1.16) we obtain the result in a manner analogous to the proof of Theorem 1.6:

$$\overset{4}{T}{}^* f\left(\overset{5}{p}, \overset{3}{x}, \overset{1}{p}\right) \overset{2}{T} = \varphi\left(\overset{2}{x}, \overset{3}{p}\right) f\left(\overset{3}{p}; \tilde{x}\left(\overset{2}{x}, \overset{1}{p}, \overset{3}{p}\right); \overset{1}{p}\right) \varphi\left(\overset{2}{x}, \overset{1}{p}\right) + h^2 A_2,$$

where $\tilde{x}$ is a solution of equation (1.23). Since $\tilde{x}(y, p', p) = \tilde{x}(y, p, p')$ from (1.27) we have

$$\tilde{x}(y, p', p) = \frac{1}{2}[X(y, p') + X(y, p)] + O(p' - p)^2.$$

Then in a manner analogous to the proof of Theorem 1.6 we obtain

$$\overset{4}{T}{}^* f\left(\overset{5}{p}, \overset{3}{x}, \overset{1}{p}\right) \overset{2}{T} = \varphi\left(\overset{2}{x}, \overset{3}{p}\right) \times$$
$$\times f\left(\overset{3}{p}; \frac{1}{2}\left[X\left(\overset{2}{x}, \overset{3}{p}\right) + X\left(\overset{2}{x}, \overset{1}{p}\right)\right]; \overset{1}{p}\right) \varphi\left(\overset{2}{x}, \overset{1}{p}\right) + h^2 A_2',$$

where $\|A_2'(h)\|_{W_2^N \to W_2^N} \leqslant c_N$ for all $N \geqslant 0$.

The same equality is obtained from Theorem 1.7 for the operator $\overset{4}{T}{}_1^* f\left(\overset{5}{p}, \overset{3}{x}, \overset{1}{p}\right) \overset{2}{T}_1$.

Proof of Theorem 1.8. Denote

$$\psi(x, p) \overset{\text{def}}{=\!=} \frac{\varphi(y(x, p), p)}{\sqrt{\left|\dfrac{\partial X}{\partial y}(y(x, p), p)\right|}}.$$

By using the commutation formula with an exponent we obtain

$$\overset{5}{T}{}^* f\left(\overset{4}{p}, \overset{3}{x}, \overset{2}{p}\right) \overset{1}{T} = e^{-\frac{i}{h} S\left(\overset{4}{x}, \overset{7}{p}\right)} \psi\left(\overset{6}{x}, \overset{7}{p}\right) \times$$
$$\times f\left(\left[\!\left[\overset{5}{p} + \frac{\partial S}{\partial x}\left(x, \overset{7'}{p}\right)\right]\!\right], \overset{4}{x}, \left[\!\left[\overset{3}{p} + \frac{\partial S}{\partial x}\left(x, \overset{1'}{p}\right)\right]\!\right]\right) \psi\left(\overset{2}{x}, \overset{1}{p}\right) e^{\frac{i}{h} S\left(\overset{4}{x}, \overset{1}{p}\right)}.$$

Next, let us transform the operators with numbers 3 and 5 by using

the K-formula and commutation formula, and putting the operator $p \equiv -ih\frac{\partial}{\partial x}$ in the first and seventh place, respectively. We shall get

$$\overset{5}{T}{}^*f\left(\overset{4}{p},\ \overset{3}{x},\ \overset{2}{p}\right)\overset{1}{T}=$$
$$=e^{-\frac{i}{h}S\left(\overset{2}{x},\overset{3}{p}\right)}\Phi_N\left(\overset{3}{p},\ \overset{2}{x},\ \overset{1}{p}\right)e^{\frac{i}{h}S\left(\overset{2}{x},\overset{1}{p}\right)}+h^{N+2}\mathbf{R}_N. \tag{1.39}$$

In this equality the function Φ_N has the form

$$\Phi_N(p,\ x,\ p')=\psi(x,\ p)f\psi(x,\ p')+(-ih)\psi(x,\ p)\times$$
$$\times\left[\frac{\partial f}{\partial p'}\frac{\partial\psi}{\partial x}(x,\ p')+\frac{1}{2}\frac{\partial^2 f}{\partial p'^2}\frac{\partial^2 S}{\partial x^2}(x,\ p')\psi(x,\ p')\right]+$$
$$+(ih)\left[\frac{\partial\psi}{\partial x}(x,\ p)\frac{\partial f}{\partial p}+\frac{1}{2}\psi(x,\ p)\frac{\partial^2 S}{\partial x^2}(x,\ p)\frac{\partial^2 f}{\partial p^2}\right]+$$
$$+\sum_{k=2}^{N+1}h^kF_k(p,\ x,\ p'), \tag{1.40}$$

where we omit the arguments of the function f and its derivatives

$$\left(p+\frac{\partial S}{\partial x}(x,\ p);\ x;\ p'+\frac{\partial S}{\partial x}(x,\ p')\right);$$
$$F_k\in C_0^\infty(\mathbf{R}^3),\qquad k=2,\ \ldots,\ N+1.$$

The remainder $\mathbf{R}_N$ in (1.39) has the estimate

$$\|h^N\mathbf{R}_N\|_{W_2^N\to W_2^N}\leqslant c_N \text{ for any } N\geqslant 0. \tag{1.41}$$

Since $p+\frac{\partial S}{\partial x}(x,\ p)=\Xi(y(x,\ p),\ p)$, from Theorem 1.6 and by taking into account formula (1.40) for the term of the zero order in h in equality (1.39) we obtain

$$\overset{4}{T}{}^*f\left(\Xi\left(y\left(\overset{3}{x},\ \overset{5}{p}\right),\ \overset{5}{p}\right),\ \overset{3}{x},\ \Xi\left(y\left(\overset{3}{x},\ \overset{1}{p}\right),\overset{1}{p}\right)\right)\overset{2}{T}=$$
$$=\varphi\left(\overset{2}{x},\ \overset{3}{p}\right)f\left(\Xi\left(\overset{2}{x},\ \overset{3}{p}\right),\ \frac{1}{2}\left[X\left(\overset{2}{x},\ \overset{3}{p}\right)+X\left(\overset{2}{x},\ \overset{1}{p}\right)\right],\right.$$
$$\left.\Xi\left(\overset{2}{x},\ \overset{1}{p}\right)\right)\varphi\left(\overset{2}{x},\ \overset{1}{p}\right)+h^2A_2', \tag{1.42}$$

where $\|A_2'\|_{W_2^N\to W_2^N}\leqslant c_N$.

The term of the first order in h in (1.39) can be transformed in an analogous manner. By taking into account the symmetry of the symbol f we reduce it to the form $h\left(\overset{1}{p}-\overset{3}{p}\right)Q\left(\overset{3}{p},\ \overset{2}{x},\ \overset{1}{p}\right)$,

where $Q \in C_0^\infty(\mathbf{R}^3)$. In a manner similar to the proof of Theorem 1.5 we obtain the estimate

$$\left\| h \left(\overset{1}{p} - \overset{3}{p}\right) Q\left(\overset{3}{p}, \overset{2}{x}, \overset{1}{p}\right) \right\|_{W_2^N \to W_2^N} =$$

$$= h^2 \left\| \left(\frac{\overset{1}{\partial}}{\partial x} - \frac{\overset{5}{\partial}}{\partial x} \right) Q\left(\overset{4}{p}, \overset{3}{x}, \overset{2}{p}\right) \right\|_{W_2^N \to W_2^N} \leqslant h^2 c_N. \tag{1.43}$$

The remaining terms in (1.39) (of orders $h^2, h^3, \ldots, h^{N+1}$) can also be transformed by virtue of Theorem 1.5 and the following estimate for them can be obtained:

$$\| \ldots \|_{W_2^N \to W_2^N} \leqslant h^2 c_N.$$

Consequently, from (1.39) and (1.42) and from estimates (1.41) and (1.43) the desired equality for the operator T follows.

The equality for the case of the operator T_1 is established in an analogous manner. The Theorem 1.8 is proved.

Sec. 2. The Homomorphism of Asymptotic Formulas

Let A_1 and A_2 be generators of degrees s_1 and s_2, respectively, with the same defining pair of Banach spaces (B, B). The generators are defined on the same linear manifold D. Denote again the closures of the operators by A_1 and A_2.

Theorem 2.1. *Let* $\varphi \in \mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$ *and the function* ψ:

$$\psi(x_1, x_2) = x_2^k \varphi(x_1, x_2),$$

where k *is an integer*; k *also belongs to* $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$. *Then*

$$\psi\left(\overset{1}{A_1}, \overset{2}{A_2}\right) = A_2^k \varphi\left(\overset{1}{A_1}, \overset{2}{A_2}\right).$$

Proof. Let $e \in C_0^\infty(\mathbf{R})$, $e(0) \neq 0$. Set $\varphi = \overline{\varphi} + \overline{\overline{\varphi}}$, $\psi = \overline{\psi} + \overline{\overline{\psi}}$, where $\overline{\varphi}(x_1, x_2) = \varphi(x_1, x_2)\, e(x_2)$, $\overline{\psi}(x_1, x_2) = x_2^k \overline{\varphi}(x_1, x_2)$. Let $\{\varphi_n\}$ be a sequence of functions converging to φ in $\mathscr{B}_{s_1, s_2}$ of the form

$$\varphi_n(x_1, x_2) = \sum_{j=1}^{m(n)} u_{n_j}(x_1)\, v_{n_j}(x_2),$$

where each function u_{n_j} and v_{n_j} either belongs to $C_0^\infty(\mathbf{R})$ or is the Fourier transform of the δ-function. Identically let $\{\psi_n\}$ be a sequence of functions converging to ψ in $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$ of the form

$$\psi_n(x_1, x_2) = \sum_{j=1}^{l(n)} w_{n_j}(x_1)\, z_{n_j}(x_2),$$

where the functions w_{n_j} and z_{n_j} are taken from the same class as u_{n_j} and v_{n_j}. Set

$$\varphi_{1n}(x_1, x_2) = \varphi_n(x_1, x_2)\, e(x_2), \qquad \psi_{1n}(x_1, x_2) = \varphi_{1n}(x_1, x_2)\, x_2^k,$$
$$\psi_{2n}(x_1, x_2) = \psi(x_1, x_2)(1 - e(x_2)), \qquad \varphi_{2n}(x_1, x_2) = $$
$$= \psi_{2n}(x_1, x_2)\, x_2^{-k}.$$

Then the following relations of convergence exist in $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$:

$$\lim_{n\to\infty} \varphi_{1n} = \bar{\varphi}, \quad \lim_{n\to\infty} \psi_{1n} = \bar{\psi},$$
$$\lim_{n\to\infty} \psi_{2n} = \bar{\bar{\psi}}, \quad \lim_{n\to\infty} \varphi_{2n} = \bar{\bar{\varphi}},$$

Thus,

$$\lim_{n\to\infty} (\varphi_{1n} + \varphi_{2n}) = \varphi, \quad \lim_{n\to\infty} (\psi_{1n} + \psi_{2n}) = \psi, \tag{2.1}$$

where $\psi_{1n}(x_1, x_2) + \psi_{2n}(x_1, x_2) = x_2^k[\varphi_{1n}(x_1, x_2) + \varphi_{2n}(x_1, x_2)]$. By applying the homomorphism $\mathscr{M}$ to the limiting relations (2.1) we obtain

$$\left.\begin{aligned} &\lim_{n\to\infty} \left[\varphi_{1n}\left(\overset{1}{A_1}, \overset{2}{A_2}\right) + \varphi_{2n}\left(\overset{1}{A_1}, \overset{2}{A_2}\right)\right] = \varphi\left(\overset{1}{A_1}, \overset{2}{A_2}\right), \\ &\lim_{n\to\infty} \left[\psi_{1n}\left(\overset{1}{A_1}, \overset{2}{A_2}\right) + \psi_{2n}\left(\overset{1}{A_1}, \overset{2}{A_2}\right)\right] = \psi\left(\overset{1}{A_1}, \overset{2}{A_2}\right), \end{aligned}\right\} \tag{2.2}$$

the corresponding sequences converging in the norm of operators.

Let h be an arbitrary element of the space B. From Theorem 4.1 of Chapter I it follows that

$$\left[\psi_{1n}\left(\overset{1}{A_1}, \overset{2}{A_2}\right) + \psi_{2n}\left(\overset{1}{A_1}, \overset{2}{A_2}\right)\right] h =$$
$$= A_2^k \left[\varphi_{1n}\left(\overset{1}{A_1}, \overset{2}{A_2}\right) + \varphi_{2n}\left(\overset{1}{A_1}, \overset{2}{A_2}\right)\right] h. \tag{2.3}$$

Since the operator A_2^k is closed, from (2.2) and (2.3) it follows that

$$\psi\left(\overset{1}{A_1}, \overset{2}{A_2}\right) h = A_2^k \varphi\left(\overset{1}{A_1}, \overset{2}{A_2}\right) h,$$

Q.E.D.

Theorem 2.1 shows in particular (see Note in Sec. 8) that, in the essence of the fundamental problem, we are looking for the asymptotics "in some operators": in the operator A_2 in this case, i.e., if we know the behaviour of the symbol of the "remainder" for $x_2 \to \infty$ then we can arrive at the conclusion that the "remainder" belongs to the domain of A_2^k.

Strictly speaking, Theorem 2.1 is equivalent to the following statement. Let φ and ψ belong to the space $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$ and $\varphi(x_1, x_2)$ —

$-\psi(x_1, x_2) = O(x_2^{-k})$ in the sense that the function

$$(x_1, x_2) \to x_2^k [\varphi(x_1, x_2) - \psi(x_1, x_2)]$$

belongs to $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$. Then for any $g \in B$ the vector $\varphi\left(\overset{1}{A}_1, \overset{2}{A}_2\right) g -$ $- \psi\left(\overset{1}{A}_1, \overset{2}{A}_2\right) g$ belongs to the domain of A_2^k.

As an example consider the method of a stationary phase in the case when the phase linearly depends on a parameter. Let $\varphi \in C_0^\infty(\mathbf{R})$ and $\mathscr{F}$ be an infinitely differentiable in $\mathbf{R}$ real function. Let the equation $\mathscr{F}'(\xi) = p$ have a unique solution $\xi = \Xi(p)$ on the support of the function φ for any $p \in \mathbf{R}$, where $\mathscr{F}''(\Xi(p)) \neq 0$ for any point $p \in \mathbf{R}$. Consider the integral

$$I(x, p) = \chi(p) \int_{-\infty}^{\infty} \varphi(\xi) e^{ix[\mathscr{F}(\xi) - p,\xi]} d\xi, \tag{2.4}$$

where $\chi \in C_0^\infty(\mathbf{R})$.

In the integral (2.4) proceed to a new variable of integration t according to the formula

$$x[\mathscr{F}(\xi) - p\cdot\xi - F(\Xi(p)) + p\Xi(p)] =$$
$$= |x| t^2 \operatorname{sgn}[x\mathscr{F}''(\Xi(p))].$$

Denote as in Sec. 5 of Chapter I

$$|x| = \omega, \operatorname{sgn} x\mathscr{F}''(\Xi(p)) = \sigma.$$

We obtain

$$I(x, p) = \chi(p) e^{ix[\mathscr{F}(\Xi(p)) - p(\Xi(p))]} \int_{-\infty}^{\infty} e^{i\omega\sigma t^2} \psi_0(t, p)\, dt, \tag{2.5}$$

where $\psi_0(t, p) = \varphi(\xi(t, p)) \dfrac{\partial \xi(t, p)}{\partial t}$. The function ψ_0 is infinitely differentiable in all arguments; moreover, there are such positive continuous functions $c(p)$ and $d(p)$ that $\psi_0(t, p) = 0$ for $t \notin$ $\notin [-c(p), d(p)]$. Construct a sequence of functions $\psi_j(t, p)$ according to the following recurrent formula:

$$\psi_{j+1}(t, p) = \frac{\partial}{\partial t} \frac{\psi_j(t, p) - \psi_j(0, p)}{t}.$$

It is obvious that for $t \notin [-c(p), d(p)]$

$$\psi_j(t, p) = t^{-2} P_j(t^{-1}, p),$$

where $P_j(\tau, p)$ is a polynomial in the variable τ with infinitely differentiable coefficients depending on p. In Sec. 5 of Chapter I

it is shown that the following expansion exists:

$$\sqrt{x}\, e^{-ix[\mathcal{F}(\Xi(p))-p(\Xi(p))]}\, I(x, p) =$$

$$= \sqrt{\pi}\, e^{\frac{i\pi}{4}\operatorname{sgn}\mathcal{F}''(\Xi(p))} \sum_{j=0}^{n} \left[\frac{i}{2\omega\sigma}\right]^j \psi_j(0,\ p)\,\chi(p) + \bar{r}_{n+1}(x, p), \tag{2.6}$$

where

$$\bar{r}_{n+1}(x,\ p) = \sqrt{x}\left[\frac{i\sigma}{2\omega}\right]^{n+1} \chi(p) \int_{-\infty}^{\infty} e^{i\omega\sigma t^2}\psi_{n+1}(t,\ p)\, dt, \tag{2.7}$$

the function $\sqrt{x}$ being defined in Sec. 5 of Chapter I. In the expansion (2.6) replace $\sqrt{x}$ by rad x and $\omega^{-j} = x^{-j}(-1)^j$ sgn x by $\rho_j(x)(-1)^j$ sgn x, where the function rad and ρ_j are also defined in Sec. 5 of Chapter I. Here the remainder $\bar{r}_{n+1}(x, p)$ is replaced by the function $r_{n+1}(x, p)$ which coincides with $\bar{r}_{n+1}(x, p)$ for $|x| > \alpha$, where α is an integer independent of p. Note that in (2.6) sgn $\mathcal{F}''(\Xi(p)) \stackrel{\text{def}}{=\!=} \sigma_0 = \text{const}$ because the function $\mathcal{F}''(\Xi(p))$ does not become zero anywhere. For $x \neq 0$ all terms of the expansion (2.6) are obviously infinitely differentiable in x and p with the exception of $\bar{r}_{n+1}(x, p)$. This means that the function $\bar{r}_{n+1}(x, p)$ also possesses this property. Let $e(x)$ be a function in $C_0^\infty(\mathbf{R})$ which is identically equal to 1 for $|x| < \alpha$. Set

$$r_{n+1}(x,\ p)\, e(x) = \mu(x,\ p), \quad \bar{r}_{n+1}(x,\ p)\,[1 - e(x)] = \nu(x,\ p).$$

We have

$$r_{n+1}(x,\ p) = \mu(x,\ p) + \nu(x,\ p).$$

The function μ belongs to $C_0^\infty(\mathbf{R}^2) \subset \mathcal{B}_{s_1, s_2}(\mathbf{R}^2)$. This is why the function

$$(x,\ p) \to x^{n+1}\mu(x,\ p)$$

also belongs to $C_0^\infty(\mathbf{R}^2) \subset \mathcal{B}_{s_1, s_2}(\mathbf{R}^2)$. Show that the function

$$(x,\ p) \to x^{n+1}\nu(x,\ p)$$

also belongs to $\mathcal{B}_{s_1, s_2}(\mathbf{R}^2)$ (this also means that $\nu \in \mathcal{B}_{s_1, s_2}(\mathbf{R}^2)$). For the sake of certainty set sgn $\mathcal{F}''(\Xi(p)) = 1$. Just as in Sec. 5

of Chapter I for any integer $m \geqslant n+1$ we obtain:

$$x^{n+1}\nu(x, p)=\sqrt{\pi}\, e^{\frac{i\pi}{4}} \left(\frac{i}{2}\right)^{n+1} \psi_{n+1}(0, p)\,\chi(p)\,(1-e(x))+$$
$$+\sqrt{\pi}\, e^{\frac{i\pi}{4}} \sum_{j=n+2}^{m} \left(\frac{i}{2}\right)^{j} x^{n+1-j}(1-e(x))\,\psi_j(0, p)\,\chi(p)+$$
$$+\sqrt{x}\, x^{n-m}[1-e(x)]\left(\frac{i}{2}\right)^{m+1}\times$$
$$\times\int_{-\infty}^{\infty} e^{ixt^2}\psi_{m+1}(t, p)\,\chi(p)\,dt. \tag{2.8}$$

The first term in the right-hand member of (2.8) obviously belongs to $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$. The same applies to each term of the sum $\sum_{j=n+2}^{m}$. Consider the latter term in (2.8). Set

$$f(x, p) \overset{\text{def}}{=\!=} \sqrt{x}\, x^{n-m}[1-e(x)] \int_{-\infty}^{\infty} e^{ixt^2}\psi_{m+1}(t, p)\,\chi(p)\,dt=$$
$$=\sqrt{x}\, x^{n-m}[1-e(x)]\,\chi(p)\, g_m(x, p),$$

where

$$g_m(x, p)=\int_{-\infty}^{\infty} e^{ixt^2}\psi_{m+1}(t, p)\,dt.$$

Just as in Sec. 5 of Chapter I we establish that for $x \neq 0$ the function g_m is infinitely differentiable and

$$\frac{\partial^{k_1+k_2} g_m(x, p)}{\partial x^{k_1}\,\partial p^{k_2}}=O(|x|^{-k_1}),$$

where O is an estimate, uniform in p for $p \in \operatorname{supp}\chi$. Consequently, for $m \geqslant n+2$ and for any non-negative integers k_1 and k_2

$$\int_{\mathbf{R}^2}\left|\frac{\partial^{k_1+k_2} f(x, p)}{\partial x^{k_1}\,\partial p^{k_2}}\right| dx\,dp<\infty$$

so that $f \in W_2^{s_1+1,\, s_2+1}(\mathbf{R}^2) \subset \mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$. Thus, we have shown that the functions $r_{n+1}(x, p)$ and $x^{n+1}r_{n+1}(x, p)$ belong to the space $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$. Denote

$$K(x, p)=\operatorname{rad} x e^{-ix[\mathscr{F}(\Xi(p))-p\Xi(p)]} I(x, p).$$

From the expansion

$$K(x, p)=\sqrt{\pi}\, e^{\frac{i\pi}{4\sigma_0}} \sum_{j=0}^{n} \left(\frac{i\sigma_0}{2}\right)^{j} \rho_j(x)\,\psi_j(0, p)\,\chi(p)+r_{n+1}(x, p) \tag{2.9}$$

it follows that $K \subset \mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$. By using Theorem 2.1 we obtain the following result: for any $h \in B$ the vector

$$K\left(\overset{2}{A_1}, \overset{1}{A_2}\right) h = \sqrt{\pi}\, e^{\frac{i\pi\sigma_0}{4}} \sum_{j=0}^{n} \left(\frac{i\sigma_0}{2}\right)^j \rho_j(A_1)\, \chi_j(A_2)\, h, \tag{2.10}$$

where $\chi_j(p) = \psi_j(0, p)\, \chi(p)$ belongs to the domain of the operator A_1^{n+1}.

Theorem 2.2. *Let A_1 (A_2) be generators of degree s_1 (s_2) with the same defining pair (B, B) and with the same domain $D_{A_1} = D_{A_2} = D$. Let the functions φ and $\psi \in \mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$ be correlated by the formula*

$$\psi(x_1, x_2) = P(x_1)\, \varphi(x_1, x_2),$$

where P is a polynomial. Then

$$\psi\left(\overset{1}{A_1}, \overset{2}{A_2}\right) = \overline{\varphi\left(\overset{1}{A_1}, \overset{2}{A_2}\right) P(A_1)}.$$

Proof. As in the proof of Theorem 2.1, we construct the sequences $\{\varphi_{1n}\}$, $\{\varphi_{2n}\}$, $\{\psi_{1n}\}$ and $\{\psi_{2n}\}$ of functions of the form $\sum_{j=1}^{l(n)} u_{nj}(x_1)\, v_{nj}(x_2)$ possessing the following properties:

$$\lim_{n\to\infty} (\varphi_{1n} + \varphi_{2n}) = \varphi, \quad \lim_{n\to\infty} (\psi_{1n} + \psi_{2n}) = \psi,$$

$$\psi_{1n}(x_1, x_2) = P(x_1)\, \varphi_{1n}(x_1, x_2), \quad \psi_{2n}(x_1, x_2) = P(x_1)\, \varphi_{2n}(x_1, x_2).$$

Let $h \in D$. Then from Theorem 4.1 of Chapter I it follows that

$$\left[\psi_{1n}\left(\overset{1}{A_1}, \overset{2}{A_2}\right) + \psi_{2n}\left(\overset{1}{A_1}, \overset{2}{A_2}\right)\right] h =$$

$$= \left[\varphi_{1n}\left(\overset{1}{A_1}, \overset{2}{A_2}\right) + \varphi_{2n}\left(\overset{1}{A_1}, \overset{2}{A_2}\right)\right] P(A_1)\, h.$$

Passing over to the limit for $n \to \infty$ we obtain

$$\psi\left(\overset{1}{A_1}, \overset{2}{A_2}\right) h = \varphi\left(\overset{1}{A_1}, \overset{2}{A_2}\right) P(A_1)\, h.$$

Q.E.D.

Theorem 2.2 brings out the following statement. Let φ and ψ belong to the space $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$ and $\varphi(x_1, x_2) - \psi(x_1, x_2) = O(x_1^{-k})$ in the sense that the function

$$(x_1, x_2) \to x_1^k [\varphi(x_1, x_2) - \psi(x_1, x_2)]$$

belongs to $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$. Then the operator

$$\left[\varphi\left(\overset{1}{A_1}, \overset{2}{A_2}\right) - \psi\left(\overset{1}{A_1}, \overset{2}{A_2}\right)\right] A_1^k$$

is bounded.

We can now write an analogue of the expansion (2.10) of the stationary phase method in a case when the operator A_1 acts first and A_2 second. Specifically, the operator

$$\left[K\left(\overset{1}{A}_1, \overset{2}{A}_2\right) - \sqrt{\pi}\, e^{\frac{i\pi\sigma_0}{4}} \sum_{j=0}^{n} \left(\frac{i\sigma_0}{2}\right)^j \chi_j(A_2)\,\rho_j(A_1)\right] A_1^{n+1} \tag{2.11}$$

is bounded.

Apply the obtained results of the stationary phase method to the pseudodifferential operators. The expansion (2.10) has the following form:

$$K\left(-i\overset{2}{\frac{d}{dx}}, \overset{1}{x}\right) h(x) = \sqrt{\pi}\, e^{\frac{i\pi\sigma_0}{4}} \sum_{j=0}^{n} \left(\frac{i\sigma_0}{2}\right)^j \times$$

$$\times \rho_j\left(-i\frac{d}{dx}\right) \chi_j(x)\, h(x) + q_{n+1}(x),$$

where $q_{n+1} \in W_2^{k+n+1}(\mathbf{R})$ if $h \in W_2^k(\mathbf{R})$.

The expansion (2.11) for the pseudodifferential operators has the form

$$K\left(-i\overset{1}{\frac{d}{dx}}, \overset{2}{x}\right) h(x) =$$

$$= \sqrt{\pi}\, e^{\frac{i\pi\sigma_0}{4}} \sum_{j=0}^{n} \left(\frac{i\sigma_0}{2}\right)^j \chi_j(x)\, \rho_j\left(-i\frac{d}{dx}\right) h(x) +$$

$$+ r_{n+1}\left(-i\overset{1}{\frac{d}{dx}}, \overset{2}{x}\right) h(x),$$

where the pseudodifferential operator $r_{n+1}\left(-i\overset{1}{\frac{d}{dx}}, \overset{2}{x}\right)$ possesses such a property that for any k the operator $r_{n+1}\left(-i\overset{1}{\frac{d}{dx}}, \overset{2}{x}\right) \times \left(-i\frac{d}{dx}\right)^{n+1}$ is bounded in $W_2^k(\mathbf{R})$. Consequently, for any $h \in W_2^{k-n-1}(\mathbf{R})$ the function

$$r_{n+1}\left(-i\overset{1}{\frac{d}{dx}}, \overset{2}{x}\right) h(x) =$$

$$= r_{n+1}\left(-i\overset{1}{\frac{d}{dx}}, \overset{2}{x}\right)\left(-i\frac{d}{dx}+i\right)^{n+1}\left(-i\frac{d}{dx}+i\right)^{-n-1} h(x)$$

belongs to $W_2^k(\mathbf{R})$.

Sec. 3. The Geometrical Interpretation of the Method of Stationary Phase

Consider the integral

$$I(p,\ \omega)=\frac{\operatorname{rad}\omega}{(2\pi i)^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{i\omega(-px+\mathscr{F}(x))}\varphi_1(x)\,dx, \tag{3.1}$$

where p, ω are real variables, $i^{\frac{1}{2}}=e^{i\pi/4}$ and $\varphi_1\in C_0^\infty(\mathbf{R})$; rad ω is an infinitely differentiable function defined for sufficiently large $|\omega|$ by the formula

$$\operatorname{rad}\omega=\omega^{\frac{1}{2}}\overset{\text{def}}{=\!=}e^{\frac{i\pi}{4}(1-\operatorname{sgn}\omega)}|\omega|^{\frac{1}{2}}.$$

Note that for sufficiently large $|\omega|$ the function $i^{1/2}I(p,\ \omega)$ is $\frac{1}{\omega}$-Fourier transform of the function $e^{i\omega\mathscr{F}(x)}\varphi_1(x)$ depending on the parameter ω.

Apply the method of stationary phase to the integral $I(p,\ \omega)$ using only the first term of the asymptotic expansion. Supposing that $\mathscr{F}''(x)\neq 0$ on the support of the function $\varphi_1(x)$ and that the equation $-p+\mathscr{F}'(x)=0$ has a unique solution $x=x(p)$ for $x\in\operatorname{supp}\varphi_1$, we obtain

$$e^{-i\frac{\pi}{4}(\operatorname{sgn}\mathscr{F}''(x(p))-1)\operatorname{sgn}\omega}\frac{\operatorname{rad}\omega}{(2\pi i)^{1/2}}\int_{-\infty}^{\infty}e^{i\omega(-px+\mathscr{F}(x))}\varphi_1(x)\,dx=$$

$$=\frac{\varphi_1(x(p))}{|\mathscr{F}''(x(p))|^{1/2}}e^{i\omega[-px(p)+\mathscr{F}(x(p))]}+g(p,\ \omega),$$

where the function $g(p,\ \omega)$ possesses the following property: for any $\chi\in C_0^\infty(\mathbf{R})$ and any s_1, s_2

$$\chi(p)\,e^{-i\omega[-px(p)+\mathscr{F}(x(p))]}g(p,\ \omega)\in\mathscr{B}_{s_1,\ s_2}(\mathbf{R}^2),$$
$$\omega\chi(p)\,e^{-i\omega[-px(p)+\mathscr{F}(x(p))]}g(p,\ \omega)\in\mathscr{B}_{s_1,\ s_2}(\mathbf{R}^2).$$

We shall write further $f_1(p,\ \omega)\cong f_2(p,\ \omega)$ if $f_1-f_2\in\mathscr{B}_{s_1,\ s_2}(\mathbf{R}^2)$ and $\omega[f_1(p,\ \omega)-f_2(p,\ \omega)]\in\mathscr{B}_{s_1,\ s_2}(\mathbf{R}^2)$. Thus

$$e^{-i\frac{\pi}{4}[\operatorname{sgn}\mathscr{F}''(x(p))-1]\operatorname{sgn}\omega}\chi(p)\times$$
$$\times\frac{\operatorname{rad}\omega}{(2\pi i)^{1/2}}\int_{-\infty}^{\infty}e^{i\omega[-px+\mathscr{F}(x)+px(p)-\mathscr{F}(x(p))]}\times$$
$$\times\varphi_1(x)\,dx\cong\chi(p)\frac{\varphi_1(x(p))}{|\mathscr{F}''(x(p))|^{1/2}}. \tag{3.2}$$

In the space of points (x, p) which is usually called the *phase space* in physics the equation

$$\mathcal{F}'(x) = p \tag{3.3}$$

describes some line. Introduce the parameter α on the curve (3.3):

$$x = x(\alpha), \quad p = p(\alpha).$$

The condition $\mathcal{F}''(x) \neq 0$ means that equation $p = p(\alpha)$ with respect to α has a unique solution $\alpha = \alpha(p)$, where the function $\alpha(p)$ is infinitely differentiable. In this case the curve (3.3) is said to be diffeomorphically projected onto the axis p. Thus any function on the curve (3.3) may be regarded as a function of x and p or α. Allowing for inaccuracy in the notation we shall sometimes write an equality of the type

$$\varphi(x) = \varphi(p) = \varphi(\alpha),$$

as well as the expressions $\alpha = \alpha(x)$ for the solution of the equation $x = x(\alpha)$, and $\alpha = \alpha(p)$ for the solution of the equation $p = p(\alpha)$ (of course, this applies to cases where there could be no misunderstanding). We have

$$\mathcal{F}(x) = \mathcal{F}(x(\alpha_0)) + \int_{x(\alpha_0)}^{x} \mathcal{F}'(x)\,dx = \mathcal{F}(x(\alpha_0)) + \int_{\alpha_0}^{\alpha} p\,dx, \tag{3.4}$$

$$-px(p) + \mathcal{F}(x(p)) = -px(p) + \int_{\alpha_0}^{\alpha} p\,dx + \mathcal{F}(x(\alpha_0)) =$$

$$= -\int_{\alpha_0}^{\alpha} x\,dp - p(\alpha_0)\,x(\alpha_0) + \mathcal{F}(x(\alpha_0)),$$

where $\int_{\alpha_0}^{\alpha} x\,dp$ and $\int_{\alpha_0}^{\alpha} p\,dx$ are curvilinear integrals along the curve (3.3). The integral $\int_{\alpha_0}^{\alpha} p\,dx$ along the curve in the phase space will be called the *action* (as accepted in physics).

By analogy with (3.2), set

$$\varphi_1(x) = \left.\frac{\varphi(\alpha)}{\left|\frac{dx}{d\alpha}\right|^{1/2}}\right|_{\alpha=\alpha(x)}.$$

By virtue of (3.4) and of equalities

$$\mathcal{F}''(x) = \frac{dp}{dx}, \quad \mathcal{F}''\frac{dx}{d\alpha} = \frac{dp}{d\alpha}$$

we obtain

$$\frac{1}{\omega+i}\chi(p)\frac{\operatorname{rad}\omega}{\sqrt{2\pi i}}\int_{-\infty}^{\infty} e^{-i\omega px}\left(e^{i\omega\int_{\alpha_0}^{\alpha} p\,dx}\frac{\varphi(\alpha)}{\left|\frac{dx}{d\alpha}\right|^{1/2}}\right)\Bigg|_{\alpha=\alpha(x)} dx \cong$$

$$\cong e^{-i\frac{\pi}{2}\sigma\operatorname{sgn}\omega}\left(e^{i\omega\left[-\int_{\alpha_0}^{\alpha} x\,dp-p(\alpha_0)x(\alpha_0)\right]}\frac{\varphi(\alpha)}{\left|\frac{dp}{d\alpha}\right|^{1/2}}\right)\Bigg|_{\alpha=\alpha(p)}\times$$

$$\times\chi(p)\frac{1}{\omega+i}, \tag{3.5}$$

where

$$\sigma=\frac{1}{2}\left(1-\operatorname{sgn}\frac{dp}{dx}\right).$$

Sec. 4. The Canonical Operator on an Unclosed Curve

Let $M^1: x = x(\alpha),\ p = p(\alpha)$ be an unclosed infinitely smooth curve without self-intersection with coordinates (x, p) in the phase space and let $\varphi(\alpha)$ be an infinitely differentiable function with a compact support. Define the real functions on M^1

$$S(\alpha)=\int_{\alpha_0}^{\alpha} p\,dx,\quad \tilde{S}(\alpha)=\int_{\alpha_0}^{\alpha} x\,dp+x(\alpha_0)\,p(\alpha_0)=$$
$$=-S(\alpha)+p(\alpha)\,x(\alpha).$$

Let an open arc U of the curve M^1 containing the support of the function φ be diffeomorphically projected onto the axes x and p, i.e., $\frac{dx}{d\alpha}\neq 0$, $\frac{dp}{d\alpha}\neq 0$ on U. Then, instead of α, we may choose either x or p as a parameter on U:

$$\alpha=\alpha(x),\quad \alpha=\alpha(p).$$

The relations are valid:

$$p=\frac{dS}{dx},\quad x=\frac{d\tilde{S}}{dp},\quad S+\tilde{S}=px,$$

i.e., the transition from x and $S(x)$ to p and $\tilde{S}(p)$ is a Legendre transformation.

Exchanging x and p in (3.5) and passing over to the complex-adjoint values we obtain, for any infinitely differentiable function $\rho(\omega)$ which is equal to zero in a neighborhood of zero and is equal

to unity in a neighborhood of infinity:

$$e^{-i\omega S(\alpha(x))}\chi(x)\,\rho(\omega)\sqrt{\frac{\omega}{-2\pi i}}\times$$
$$\times\int_{-\infty}^{\infty} e^{i\omega px}e^{-i\omega\widetilde{S}(\alpha)}\left|\frac{dp}{d\alpha}\right|^{-\frac{1}{2}}\varphi(\alpha)\Big|_{\alpha=\alpha(p)}dp\cong$$
$$\cong\chi(x)\,\rho(\omega)\,(i\operatorname{sgn}\omega)^{\sigma}\left|\frac{dx}{d\alpha}\right|^{-\frac{1}{2}}\varphi(\alpha)\Big|_{\alpha=\alpha(x)}$$

under the condition that the support of the function χ is contained in the projection of U onto the axis x. But if the support of an infinitely differentiable function $\chi(x)$ does not intersect with the support of the function $\varphi(\alpha(x))$ and $\chi(x)=1$ in a neighborhood of infinity, then the function

$$g(x,\ \omega)=\chi(x)\,\rho(\omega)\sqrt{\frac{\omega}{-2\pi i}}\times$$
$$\times\int_{-\infty}^{\infty} e^{i\omega px-i\omega\widetilde{S}(\alpha)}\left|\frac{dp}{d\alpha}\right|^{-\frac{1}{2}}\varphi(\alpha)\Big|_{\alpha=\alpha(p)}dp$$

belongs to the space $W_2^s(\mathbf{R}^2)$ for any s. The function $\omega^k g(x,\omega)$, where k is an arbitrary integer, has the same property. Consequently, the equality is valid:

$$\chi(x)\,\rho(\omega)\sqrt{\frac{\omega}{-2\pi i}}\times$$
$$\times\int_{-\infty}^{\infty} e^{i\omega(px-\widetilde{S}(\alpha))}\left|\frac{dp}{d\alpha}\right|^{-\frac{1}{2}}\varphi(\alpha)\Big|_{\alpha=\alpha(p)}dp=$$
$$=\chi(x)\,\rho(\omega)\,(i\operatorname{sgn}\omega)^{\sigma}e^{i\omega S(\alpha)}\left|\frac{dx}{d\alpha}\right|^{-\frac{1}{2}}\varphi(\alpha)\Big|_{\alpha=\alpha(x)}+$$
$$+e^{i\omega S(\alpha(x))}g_1(x,\ \omega)+g_2(x,\ \omega),\tag{4.1}$$

where g_1 and g_2 are infinitely differentiable functions $g_1(x,\omega)=0$ for $x\in\operatorname{supp}\varphi(\alpha(x))$, the functions $\omega g_1(x,\omega)$ and $\omega g_2(x,\omega)$ belonging to the space $\mathscr{B}_{s_1,\,s_2}(\mathbf{R}^2)$ for any non-negative integers s_1 and s_2.

In the right-hand member of (4.1) the function

$$e^{i\omega S(\alpha)}\left|\frac{dx}{d\alpha}\right|^{-\frac{1}{2}}$$

is infinitely differentiable in U, but outside the arc U it may no longer be smooth in points, where $\frac{dx}{dp}=0$. These points will be

called focal points. Besides, the left-hand member of (4.1) is proportional to the ω-Fourier transform of the function

$$e^{-i\omega\widetilde{S}(\alpha)}\left|\frac{dp}{d\alpha}\right|^{-\frac{1}{2}}\cdot\varphi(\alpha)\Big|_{\alpha=\alpha(p)},$$

where the factor $e^{-i\omega\widetilde{S}(\alpha)}\left|\frac{dp}{d\alpha}\right|^{-\frac{1}{2}}$ is an infinitely differentiable function in the neighborhood of a focal point. This consideration suggests the idea of defining the following operator K which is *a canonical operator* translating an infinitely differentiable finite function in M^1 into a function of variables ω and x.

Let the family $\{U_j\}_{j\in I}$ of open arcs cover M^1:

$$M_1=\bigcup_{j\in I} U_j$$

(in M^1 the ends are not included), every arc U_j intersecting only a finite number of arcs U_j. Next, let on every arc U_j

$$\frac{dx}{d\alpha}\neq 0 \quad \text{or} \quad \frac{dp}{d\alpha}\neq 0.$$

Choose on every arc U_j either x or p as a parameter (as a local coordinate). If x is chosen as a local coodrinate on U_j, then we call U_j *a non-singular patch* of the curve M^1; otherwise, U_j will be called *a singular patch*. The function $\alpha=\alpha_j(x)$ or $\alpha=\alpha_j(p)$ corresponds to every patch U_j. The set of singular or non-singular patches on M^1 will be called the *canonical atlas of the curve* M^1.

Let the function $\varphi(\alpha)$ have a compact support $\operatorname{supp}\varphi\in M^1$. Consider the partition of unity corresponding to the canonical atlas, i.e., infinitely differentiable functions $\{e_j(\alpha)\}$ which satisfy the condition

$$1=\sum_j e_j(\alpha) \text{ for } \alpha\in\operatorname{supp}\varphi,\ \operatorname{supp} e_j\subset U_j.$$

Then

$$\varphi(\alpha)=\sum e_j(\alpha)\,\varphi(\alpha)=\sum\varphi_j(\alpha),\ \operatorname{supp}\varphi_j\subset U_j.$$

For the sake of certainty let U_{j_1} be a non-singular patch which contains a point α_0. For the function $\psi(\alpha)$ with the support in U_{j_1} we set

$$[K_{j_1}\psi](x)=e^{i\omega S(\alpha)}\left|\frac{dx}{d\alpha}\right|^{-\frac{1}{2}}\psi(\alpha)\Big|_{\alpha=\alpha_j(x)}. \tag{4.2}$$

Let U_{j_2} be a singular patch crossing with U_{j_1}. Define *a canonical operator* K_{j_2} acting on the function with support in U_{j_2} by means

of the formula

$$[K_{j_2}\psi](x) = c\sqrt{\frac{\omega}{-2\pi i}}\times$$

$$\times\int_{-\infty}^{\infty} e^{i\omega px - i\omega\tilde{S}(\alpha)}\left|\frac{dp}{d\alpha}\right|^{-\frac{1}{2}}\psi(\alpha)\Big|_{\alpha=\alpha_{j_2}(p)}dp,$$

where the constant c will be defined below. Require that for any infinitely differentiable finite function $\psi(\alpha)$ with the support contained in $U_{j_1}\cap U_{j_2}$ the following relationship should be valid:

$$e^{-i\omega S(\alpha_1(x))}\chi(x)\rho(\omega)[K_{j_1}\psi - K_{j_2}\psi](x,\omega)\cong 0$$

regardless of the function $\chi\in C_0^\infty(\mathbf{R})$ with the support contained in the projection of $U_{j_1}\cap U_{j_2}$ onto the axis x and regardless of the function $\rho\in C^\infty(\mathbf{R})$, which is equal to zero in a neighborhood of zero and equal to 1 in a neighborhood of infinity.

By virtue of (4.1) we have

$$e^{-i\omega S(\alpha_{j_1}(x))}\chi(x)\rho(\omega)[K_{j_2}\psi](x,\omega)\cong$$

$$\cong c_2(i\operatorname{sgn}\omega)^{\sigma_{U_{j_1}\cap U_{j_2}}}e^{-i\omega S(\alpha_1(x))}\chi(x)\rho(\omega)[K_{j_1}\psi](x,\omega),$$

where

$$\sigma_{U_1\cap U_2} = \frac{1}{2}\left(1-\operatorname{sgn}\frac{dp}{dx}\right)\Big|_{\alpha\in U_{j_1}\cap U_{j_2}}.$$

Consequently, we may set

$$c = (i\operatorname{sgn}\omega)^{-\sigma_{U_{j_1}\cap U_{j_2}}}.$$

Identically for non-singular patch U_{j_3} which intersects with U_{j_2} we set

$$[K_{j_3}\psi](x) = (i\operatorname{sgn}\omega)^{-\sigma_{U_{j_1}\cap U_{j_2}}+\sigma_{U_{j_2}\cap U_{j_3}}}\times$$

$$\times e^{i\omega S(\alpha)}\left|\frac{dx}{d\alpha}\right|^{-\frac{1}{2}}\psi(\alpha)\Big|_{\alpha=\alpha_{j_3}(x)};$$

here $\operatorname{supp}\psi\in U_3$.

In general let $U_1, U_2, \ldots, U_n$ be a finite sequence of patches of a given canonical atlas, where singular and non-singular patches of the sequence follow in succession and two adjacent patches of the sequence intersect.

Then set

$$[K_{j_n}\psi](x) = (i\operatorname{sgn}\omega)^{-\sigma_{U_{j_1}\cap U_{j_2}}+\sigma_{U_{j_2}\cap U_{j_3}}-\cdots+\sigma_{U_{j_{n-1}}\cap U_{j_n}}}\times$$

$$\times\left|\frac{dx}{d\alpha}\right|^{-\frac{1}{2}}\psi(\alpha)\Big|_{\alpha=\alpha_{j_n}(x)}$$

for odd n and

$$K_{j_n}\psi(x) = (i\,\mathrm{sgn}\,\omega)^{-\sigma_{U_{j_1}\cap U_{j_2}}+\dots-\sigma_{U_{j_{n-1}}\cap U_{j_n}}}\times$$

$$\times\sqrt{\frac{\omega}{-2\pi i}}\int_{-\infty}^{\infty} e^{i\omega(px-\tilde{S}(\alpha))}\left|\frac{dp}{d\alpha}\right|^{-\frac{1}{2}}\psi(\alpha)\Big|_{\alpha=\alpha_{j_n}(p)}dp.$$

For even n define the index and the canonical operator. The number

$$\mathrm{ind}\,(j_1, \dots, j_n) \overset{\mathrm{def}}{=\!=} \sum_{k=1}^{n-1}(-1)^k\sigma_{U_{j_k}\cap U_{j_{k+1}}}$$

will be called *the index of the chain of patches* $U_{j_1}, \dots, U_{j_n}$. For the given definition of the canonical operator K_j in a patch to be uncontradictory it is necessary and sufficient that the index $\mathrm{ind}\,(j_1, \dots, j_n)$ should depend only on j_1 and j_n or identically that the index of the closed chain of patches should be equal to zero. This statement holds for an unclosed curve M^1. In fact, let U be an arc of the curve M^1 which is diffeomorphically projected onto the axis p and α' and α'' be two points on the arc U which are non-focal $\left(\text{i.e., } \frac{dx}{dp}\neq 0 \text{ at these points}\right)$. Set

$$\gamma(\alpha', \alpha'') = \frac{1}{2}\left[\mathrm{sgn}\frac{dp}{dx}(\alpha') - \mathrm{sgn}\frac{dp}{dx}(\alpha'')\right].$$

If the points α', α'' are on the arc which is diffeomorphically projected on the axis x, then we set

$$\gamma(\alpha', \alpha'') = 0.$$

It is easy to see that both definitions yield the same result in cases when they are applied simultaneously. Now let l be a path on the curve M^1 with ends β', β'', i.e., a continuous mapping

$$t \to \alpha(t)$$

of the segment $[0, 1]$ in M^1 satisfying the conditions

$$\alpha(0) = \beta', \quad \alpha(1) = \beta''.$$

Let the points β' and β'' be non-focal. We set

$$\gamma[l] = \sum_{k=1}^{n-1}\gamma(\alpha(t_k), \quad \alpha(t_{k+1})), \tag{4.3}$$

where $\beta' = \alpha(0)$, $\beta'' = \alpha(1)$ and $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = 1$, where the numbers $t_1, \dots, t_{n-1}$ are chosen in such a way that on the right-hand side of (4.3) every segment of the path with ends $\alpha(t_k)$, $\alpha(t_{k+1})$ either has no focal points or entirely lies on the

arc diffeomorphically projected onto the axis p (this choice is always possible). Call $\gamma\,[l]$ the index of the path l. This term is justified because $\alpha\,(t)$ does not depend on the choice of intermediate points $\alpha\,(t_1), \ldots, \alpha\,(t_{n-1})$ on the path l. In order to verify this statement it is sufficient to note that $\gamma\,[l]$ does not change if the point t' is placed between t_k and t_{k+1}. In this approach a summand is to be added to $\gamma\,[l]$:

$$\gamma\,(\alpha\,(t_k),\ \alpha\,(t')) + \gamma\,(\alpha\,(t'),\ \alpha\,(t_{k+1})) - \gamma\,(\alpha\,(t_k),\ \alpha\,(t_{k+1})).$$

The summand is always zero as follows directly from the definition.

Let two paths l: $\alpha = f\,(t)$, l': $\alpha = g\,(t)$ be homotopic on M^1, i.e., there exists a continuous function $\mathcal{F}(t, s)$, $0 \leqslant t \leqslant 1$, $0 \leqslant s \leqslant 1$ with such values in M^1 that $\mathcal{F}(t, 0) = f\,(t)$ and $\mathcal{F}(t, 1) = g\,(t)$. If for any $s \in [0, 1]$ the points $\mathcal{F}(0, s)$ and $\mathcal{F}(1, s)$ are not focal then the indices of the two paths l and l' coincide. The proof of this statement is left to the reader. Now, it is not difficult to verify that the index of the closed chain of patches on M^1 is equal to zero. In fact for any chain of patches $U_{j_1}, \ldots, U_{j_n}$ the number ind $(j_1, \ldots, j_n)$ is equal to the index of some path l with the initial point in U_{j_1} and the final point at the intersection $U_{j_{n-1}} \cap U_{j_n}$. Let $j_n = j_1$. Then we may assume that the initial and final points of the path l coincide, say with a point α_0, i.e., the path l is closed. But any closed path of the unclosed curve may be continuously transformed into the path l': $\alpha = \text{const}$ (into the point). It is obvious that $\gamma\,[l'] = 0$. Consequently,

$$\gamma\,[l] = \operatorname{ind}\,(j_1, \ldots, j_1) = 0.$$

The concept of the path index on M^1 becomes especially clear if in every compact subset M^1 there exists only a finite number of focal points and at every point the derivative $\frac{dx}{dp}$ changes its sign (if these conditions are fulfilled, then the curve M^1 is said to be in the general position with respect to the projection onto the axis x). Then we can consider the index $\gamma\,(\alpha)$ of the point α as an integer-valued function on M^1 in the following way: $\gamma\,(\alpha_0) = 0$; by the transition through a focal point in the direction of decreasing dx/dp the index $\gamma\,(\alpha)$ increases stepwise by one; by the transition in the direction of increasing dx/dp the index $\gamma\,(\alpha)$ decreases by one; at focal points the function $\gamma\,(\alpha)$ is not defined and at other points it is continuous (hence locally constant). We shall stress that the initial point α_0 should be non-focal. For any path l on M' which begins at the point α_0 and ends at the point α we have

$$\gamma\,[l] = \gamma\,(\alpha). \tag{4.4}$$

In the case when the curve M^1 is not in a general position we shall consider (4.4) as the definition of the function $\gamma\,(\alpha)$. Finally, define

the *canonical operator* by formula

$$[K\varphi](x,\omega)=\sum_{j\in I_1}(i\operatorname{sgn}\omega)^{\gamma_j}e^{i\omega S(\alpha)}\left|\frac{dx}{d\alpha}\right|^{-\frac{1}{2}}\times$$
$$\times e_j(\alpha)\varphi(\alpha)\big|_{\alpha=\alpha_j(x)}+$$
$$+\sum_{j\in I_2}(i\operatorname{sgn}\omega)^{\gamma_j}\sqrt{\frac{\omega}{-2\pi i}}\int_{-\infty}^{\infty}e^{i\omega px-i\omega\tilde{S}(\alpha)}\times$$
$$\times\left|\frac{dp}{d\alpha}\right|^{-\frac{1}{2}}e_j(\alpha)\varphi(\alpha)\Big|_{\alpha=\alpha_j(p)}dp,$$

where I_1 is a set of numbers of non-singular patches and I_2 is a set of numbers of singular patches and

$$\gamma_j=\gamma(\alpha),\qquad \alpha\in U_j$$

if U_j is a non-singular patch. If U_j is a singular patch, then $\gamma(\alpha)$ may accept different values at different points of the arc U_j. Let

$$\gamma_j=\min_{\alpha\in U_j}\gamma(\alpha).$$

An operator $\mathcal{K}$ will be called a smooth canonical operator if it satisfies the following conditions:

(a) $[\mathcal{K}\varphi](x,\omega)=[K\varphi](x,\omega)$ *if* $|\omega|$ *is larger than some constant independent of* φ;

(b) $[\mathcal{K}\varphi](x,\omega)\in C^{\infty}(\mathbf{R}^2)$.

Let A_1 and A_2 be generators acting on a Banach space B. Since $[\mathcal{K}\varphi](x,\omega)$ is a smooth function of x and ω we may consider operators $[\mathcal{K}\varphi]\left(\overset{1}{A}_1,\overset{2}{A}_2\right)$ and $[\mathcal{K}\varphi]\left(\overset{2}{A}_1,\overset{1}{A}_2\right)$. Introduce the following notation:

$$\mathcal{K}\left(\overset{1}{A}_1,\overset{2}{A}_2\right)\varphi(\alpha)h\overset{\text{def}}{=\!=}[\mathcal{K}\varphi]\left(\overset{1}{A}_1,\overset{2}{A}_2\right)h,$$
$$\mathcal{K}\left(\overset{2}{A}_1,\overset{1}{A}_2\right)\varphi(\alpha)h\overset{\text{def}}{=\!=}[\mathcal{K}\varphi]\left(\overset{2}{A}_1,\overset{1}{A}_2\right)h.$$

Definition. *We shall say that the pair of generators* $\overset{1}{A}_1$ *and* $\overset{2}{A}_2$ $\left(\overset{2}{A}_1\right.$ *and* $\left.\overset{1}{A}_2\right)$ *acting on* B *is in agreement with the curve* M^1 *in the phase plane* (x,p) *if for any non-singular patch* U *of the curve* M^1 *and for any infinitely differentiable function* $\chi(x)$ *with the support contained in the projection of the arc* U *onto the axis* x *the operator* $\chi\left(\overset{1}{A}_1\right)e^{i\overset{2}{A}_2\varphi\left(\overset{1}{A}_1\right)}$ $\left(\textit{respectively},\ \chi\left(\overset{2}{A}_1\right)e^{i\overset{1}{A}_2\varphi\left(\overset{2}{A}_1\right)}\right)$, *where* $\varphi(x)=S(\alpha(x))$ *is bonded in* B.

Theorem. *Let $\mathcal{K}$ and $\mathcal{K}'$ be two canonical operators on M^1 corresponding to different canonical atlases $\{U_j\}_{j\in I}$ and $\{U'_j\}_{j\in I'}$ to partitions of unity $\{e_j\}$ and $\{e'_j\}$ and to different techniques of "smoothing" in ω, but to the same initial point α_0.*
Then

(a) *if the pair $\overset{1}{A}_1, \overset{2}{A}_2$ is in agreement with M^1, then the operator*

$$A_2\left[\mathcal{K}\left(\overset{1}{A}_1, \overset{2}{A}_2\right)\varphi(\alpha)-\mathcal{K}'\left(\overset{1}{A}_1, \overset{2}{A}_2\right)\varphi(\alpha)\right]$$

is bounded in B;

(b) *if the pair $\overset{2}{A}_1, \overset{1}{A}_2$ is in agreement with M^1, then the operator*

$$\left[\mathcal{K}\left(\overset{2}{A}_1, \overset{1}{A}_2\right)\varphi(\alpha)-\mathcal{K}'\left(\overset{2}{A}_1, \overset{1}{A}_2\right)\varphi(\alpha)\right]A_2$$

is bounded in B.

Proof. Consider the sum $\{V_j\}_{j\in I''}$ of the canonical atlases $\{U_j\}_{j\in I}$ and $\{U'_j\}_{j\in I'}$, i.e., the set I'' is the sum (disjunctive) of the sets I and I', $V_j = U_j$ for $j \in I$ and $V_j = U'_j$ for $j \in I'$. Continue the partitions of unity $\{e_j\}$ and $\{e'_j\}$ in the following way:

$$e_j(\alpha) = 0 \quad \text{for } j \in I', \; e'_j(\alpha) = 0 \quad \text{for } j \in I.$$

Here I and I' are regarded as unintersecting subsets of the set I''. Then $\mathcal{K}$ and $\mathcal{K}'$ may be regarded as two canonical operators corresponding to one and the same canonical atlas $\{V_j\}$, but to different partitions of unity $\{e_j\}$ and $\{e'_j\}$. For this reason and without any loss of generality we may consider that from the start the canonical atlases $\{U_j\}$ and $\{U'_j\}$ have been identical.

For sufficiently large modulus ω we have

$$[(\mathcal{K}-\mathcal{K}')\varphi](x, \omega) =$$

$$= \sum_{j\in I_1} (i \operatorname{sgn}\omega)^{\nu_j} e^{i\omega S(\alpha)} \left|\frac{dx}{d\alpha}\right|^{-\frac{1}{2}} [e_j(\alpha)-e'_j(\alpha)] \times$$

$$\times \varphi(\alpha)\big|_{\alpha=\alpha_j(x)} + \sum_{j\in I_2} (i \operatorname{sgn}\omega)^{\nu_j} \sqrt{\frac{\omega}{-2\pi i}} \int_{-\infty}^{\infty} e^{i\omega px - i\omega\widetilde{S}(\alpha)} \times$$

$$\times \left|\frac{dp}{d\alpha}\right|^{-\frac{1}{2}} [e_j(\alpha)-e'_j(\alpha)]\varphi(\alpha)\bigg|_{\alpha=\alpha_j(p)} dp.$$

Without any loss of generality one may suppose that any two different non-singular patches of the atlas $\{U_j\}$ do not intersect and any two different singular patches of this atlas do not intersect either. In fact, let U_{j_0} be a non-singular (singular, respectively)

patch of a given atlas and let I_0 be a set of the numbers of all non-singular (singular) patches intersecting with U_{j_0}. Then the arc $\bigcup_{j\in I_0} U_j$ with x (or p) as a local coordinate may be chosen as a patch of some canonical atlas in M^1. In general, call two elements j' and j'' of the set I equivalent if the patches $U_{j'}$ and $U_{j''}$ are simultaneously non-singular or singular and intersecting.

Let $\bar{I}$ be the quotient set of the set I according to a given equivalency relation. For every equivalence class $\bar{j} \in \bar{I}$ we set

$$U_{\bar{j}} = \bigcup_{j\in \bar{I}} U_j, \quad e_{\bar{j}} = \sum_{j\in \bar{I}} e_j,$$

where we choose x (or p) as a local coordinate on $U_{\bar{j}}$ if U_j is a non-singular (singular) patch for $j \in \bar{j}$. Then $\{U_{\bar{j}}\}_{\bar{j}\in \bar{I}}$ is the canonical atlas on M^1 and $\{e_{\bar{j}}\}_{\bar{j}\in\bar{I}}$ is partition of unity corresponding to this atlas. Here any two different non-singular (or singular) patches of the atlas $\{U_{\bar{j}}\}$ do not intersect and the canonical operator corresponding to the atlas $\{U_j\}$ and to partition of unity $\{e_j\}$ obviously coincides with the canonical operator corresponding to the atlas $\{U_{\bar{j}}\}$ and to partition of unity $\{e_{\bar{j}}\}$.

Thus, suppose that the atlas $\{U_j\}$ coincides with the atlas $\{U_{\bar{j}}\}$. Then $[e_j(\alpha) - e_j'(\alpha)]\,\varphi(\alpha) = 0$, if the point α is not included in any of the non-singular (singular) patches of the atlas, because in this case $e_j(\alpha)\,\varphi(\alpha) = e_j'(\alpha)\,\varphi(\alpha) = \varphi(\alpha)$. Let U_j be a singular patch of the atlas $\{U_j\}$ and let $U_{k_1}, \ldots, U_{k_m}$ be non-singular patches intersecting with U_j. Then the support of the function $[e_j(\alpha) - e_j'(\alpha)]\,\varphi(\alpha)$ is contained in $\bigcup_{\mu=1}^{m} \overline{(U_j \cap U_{k_\mu})}$, where at any point $\alpha \in U_{k_\mu} \cap U_j$ the following equalities are valid:

$$e_j(\alpha)\,\varphi(\alpha) + e_{k_\mu}(\alpha)\,\varphi(\alpha) = 1,$$

$$e_j'(\alpha)\,\varphi(\alpha) + e_{k_\mu}'(\alpha)\,\varphi(\alpha) = 1.$$

Hence at the intersection $U_j \cap U_{k_\mu}$ the following equality is valid:

$$\varphi(\alpha)\,[\varphi_j(\alpha) - e_j'(\alpha) + e_{k_\mu}(\alpha) - e_{k_\mu}'(\alpha)] = 0.$$

Here we note that since the functions e_{k_μ} and e_{k_μ}' are equal to zero in a neighborhood of each end of the arc it follows that the support of the function $[e_j(\alpha) - e_j'(\alpha)]\,\varphi(\alpha)$ is contained in some open arc which is diffeomorphically projected onto the axis x. By applying (4.1) we obtain that the difference $[(K - K')\,\varphi](x, \omega)$ is the sum of a finite number of terms of the form

$$e^{i\omega S(\alpha_j(x))} g_1(x, \omega) + g_2(x, \omega),$$

where g_1 and g_2 are infinitely differentiable functions $g_1(x, \omega) = 0$ outside some closed set which is contained in the projection of the non-singular patch U_j onto the axis x and the functions $\omega g_1(x, \omega)$ and $\omega g_2(x, \omega)$ belong to $\mathscr{B}_{s_1, s_2}(\mathbf{R}^2)$ for any non-negative integers s_1 and s_2. The operators $A_2 g_1\left(\overset{1}{A}_1, \overset{2}{A}_2\right)$, $g_1\left(\overset{2}{A}_1, \overset{1}{A}_2\right)A_2$, $A_2 g_2\left(\overset{1}{A}_1, \overset{2}{A}_2\right)$ and $g_2\left(\overset{2}{A}_1, \overset{1}{A}_2\right)A_2$ are bounded in B. The proof of the theorem now follows directly from Theorem 9.2 of Chapter II.

Sec. 5. The Method of Stationary Phase

Let $h \in \mathbf{R}$, $y \in \mathbf{R}^n$, Φ be an infinitely differentiable real function and φ be a complex function in $C_0^\infty(\mathbf{R}^{n+2})$ whose support is contained in $(a, b) \times Y \times (-c, c)$, where $a < b$, $c > 0$ are real numbers and Y is a ball in $\mathbf{R}^n$. Consider the integral

$$I(y, h) = \int_{-\infty}^{\infty} e^{\frac{i}{h}\Phi(p, y, h)} \varphi(p, y, h)\, dp. \tag{5.1}$$

Suppose that the equation

$$\frac{\partial \Phi}{\partial p}(p, y, h) = 0, \quad p \in (a, b), \quad y \in Y, \quad |h| < c$$

has a unique solution $p = p(y, h)$, where

$$\frac{\partial^2 \Phi}{\partial p^2}(p(y, h), y, h) \neq 0.$$

We shall now make in (5.1) a change of variables

$$\Phi(p, y, h) - \Phi(p(y, h), y, h) = t^2 \sigma, \tag{5.2}$$

where $\sigma = \operatorname{sgn} \frac{\partial^2 \Phi}{\partial p^2}(p(y, h), y, h)$. We obtain

$$I(y, h) = e^{\frac{i}{h}\Phi(p(y, h), y, h)} \int_{-\infty}^{\infty} e^{\frac{i}{h}\sigma t^2} \psi(t, y, h)\, dt, \tag{5.3}$$

where $p(t, y, h)$ is the solution of the equation (5.2),

$$\psi(t, y, h) = \varphi(p(t, y, h), y, h) \frac{\partial p(t, y, h)}{\partial t}.$$

The function ψ belongs to $C_0^\infty(\mathbf{R}^{n+2})$ with support contained in $(d, e) \times Y \times (-c, c)$, d, e being real numbers. By applying to (5.3) the results of Sec. 5 of Chapter I we obtain the following expan-

sion

$$\frac{1}{\sqrt{|h|}}e^{-i\frac{\pi}{4}\sigma\operatorname{sgn}h}e^{-\frac{i}{h}\Phi(p(y,h),y,h)}I(y,h)=$$
$$=\sqrt{\pi}\sum_{j=0}^{N}\left(\frac{i\sigma h}{2}\right)^{j}\psi_j(0,y,h)+r_{N+1}(y,h), \tag{5.4}$$

where

$$r_{N+1}(y,h)=\frac{1}{\sqrt{|h|}}e^{-i\frac{\pi}{4}\sigma\operatorname{sgn}h}\left(\frac{i\sigma h}{2}\right)^{N+1}\times$$
$$\times\int_{-\infty}^{\infty}e^{\frac{i}{h}\sigma t^2}\psi_{N+1}(t)\,dt, \tag{5.5}$$

$$\psi_1(t,y,h)=\psi(t,y,h),$$
$$\psi_{j+1}(t,y,h)=\frac{\partial}{\partial t}\frac{\psi_j(t,y,h)-\psi_j(0,y,h)}{t}. \tag{5.6}$$

Note that for sufficiently large in modulus t every function $\psi_j(t,y,h)$ has the form $\frac{1}{t^2}P_{y,h}\left(\frac{1}{t}\right)$, where $P_{y,h}$ is a polynomial whose coefficients are infinitely differentiable functions of y and h supported in $Y\times(-c,c)$. Besides, the functions

$$\chi_j(y,h)\overset{\text{def}}{=\!=}\psi_j(0,y,h)$$

belong to $C_0^\infty(\mathbf{R}^{n+1})$ and are supported in $Y\times(-c,c)$.

Estimate the remainder. Set $h=\frac{1}{\omega}$ and consider the function $R_{N+1}(y,\omega)\overset{\text{def}}{=\!=}r_{N+1}\left(y,\frac{1}{\omega}\right)$. This function is non-zero only for $|\omega|>1/c$. Show that the function $\omega^{N+1}R_{N+1}(y,\omega)$ belongs to $\mathscr{B}_{s_1,s_2}(\mathbf{R}^n\times\mathbf{R})$ for any s_1, s_2. In fact, for any integer $m>N+1$ we have

$$\omega^{N+1}R_{N+1}(y,\omega)=\sqrt{\pi}\left(\frac{i\sigma}{2}\right)^{N+1}\chi_{N+1}\left(y,\frac{1}{\omega}\right)+$$
$$+\sqrt{\pi}\sum_{j=N+2}^{m}\left(\frac{i\sigma}{2}\right)^{j}\omega^{N+1-j}\chi_j\left(y,\frac{1}{\omega}\right)+$$
$$+\sqrt{|\omega|}\,e^{-i\frac{\pi}{4}\sigma\operatorname{sgn}\omega}\left(\frac{i\sigma}{2}\right)^{N+1}\omega^{N-m}\times$$
$$\times\int_{-\infty}^{\infty}e^{i\omega\sigma t^2}\psi_{m+1}\left(t,y,\frac{1}{\omega}\right)dt. \tag{5.7}$$

The first term in the right-hand side of (5.7) belongs to

$$\mathscr{B}_{s_1, s_2}(\mathbf{R}^n \times \mathbf{R})$$

since

$$\chi_{N+1}\left(y, \frac{1}{\omega}\right) = \chi_{N+1}(y, 0) + \frac{1}{\omega}\bar{\chi}\left(y, \frac{1}{\omega}\right),$$

where $\bar{\chi}(y, h)$ is an infinitely differentiable function. Hence the function $\frac{1}{\omega}\bar{\chi}\left(y, \frac{1}{\omega}\right)$ belongs to any Sobolev space. Next all terms in the sum $\sum\limits_{j=N+2}^{m}$ in the right-hand side of (5.7) are square-integrable and all their derivatives possess the same property. Finally, consider the function

$$f(y, \omega) = \sqrt{|\omega|}\, e^{-i\frac{\pi}{4}\sigma \operatorname{sgn} \omega} \omega^{N-m} \times$$

$$\times \int_{-\infty}^{\infty} e^{i\omega\sigma t^2} \psi_{m+1}\left(t, y, \frac{1}{\omega}\right) dt \overset{\text{def}}{=\!=} \sqrt{|\omega|} \times$$

$$\times e^{-i\frac{\pi}{4}\sigma \operatorname{sgn} \omega} \omega^{N-m} g(y, \omega).$$

The function $g(y, \omega)$ is non-zero only for $y \in Y$ and is bounded. We shall estimate its derivatives. For any multi-index $k = (k_1, \ldots, k_n)$ and any number l we have

$$\frac{\partial^{|k|+l} g(y, \omega)}{\partial y^k \, \partial \omega^l} = \sum_{j=0}^{l} C_l^j \left(-\frac{1}{2\omega}\right)^j \times$$

$$\times \int_{-\infty}^{\infty} e^{i\omega\sigma t^2} \left(1 + t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial \omega}\right)^{l-j} \left(\frac{\partial}{\partial y}\right)^k \psi_{m+1}\left(t, y, \frac{1}{\omega}\right) dt. \tag{5.8}$$

From (5.8) it follows that the function $g(y, \omega)$ is infinitely differentiable and all its derivatives are bounded. Consequently, for a sufficiently large m the function $f(y, \omega)$ belongs to any Sobolev space given beforehand. This proves that the remainder r_{N+1} in the expansion (5.4) satisfies the condition

$$\omega^{N+1} r_{N+1}\left(y, \frac{1}{\omega}\right) \in \mathscr{B}_{s_1, s_2}(\mathbf{R}^n \times \mathbf{R}).$$

In conclusion, we shall consider the case when the critical point of the phase $\Phi(p, y, h)$ is absent. Consider the integral (5.1) and let the functions Φ and φ satisfy all the above conditions excluding the solvability of the equation $\frac{\partial \Phi}{\partial p} = 0$.

Instead of this, we suppose that

$$\frac{\partial \Phi}{\partial p}(p, y, h) \neq 0$$

for $p \in (a, b)$, $y \in Y$ and $|h| < c$. So we can make the following change of variables in the integral (5.1):

$$\Phi(p, y, h) = t, \quad p = p(t, y, h).$$

We obtain

$$I(y, h) = \int_{-\infty}^{\infty} e^{\frac{i}{h}t} \psi(t, y, h)\, dt,$$

where

$$\psi(t, y, h) = \varphi(p(t, y, h), y, h) \frac{\partial p(t, y, h)}{\partial t}.$$

As before, the function ψ belongs to $C_0^\infty(\mathbf{R}^{n+2})$, where its support is contained in $(d, e) \times Y \times (-c, c)$, d, e being integers. By integrating by parts N times we obtain

$$I(y, h) = (ih)^N \int_{-\infty}^{\infty} e^{\frac{it}{h}} \frac{\partial^N \psi(t, y, h)}{\partial t^N}\, dt. \tag{5.9}$$

Set $h = \frac{1}{\omega}$. From (5.9) it follows that if in (5.1) $\frac{\partial \Phi}{\partial p} \neq 0$ on the support of the functions φ, then for any integer m the function $\omega^m I(y, 1/\omega)$ belongs to any Sobolev space which means that it also belongs to any space $\mathscr{B}_{s_1, s_2}(\mathbf{R}^n \times \mathbf{R})$.

Sec. 6. The Canonical Operator on the Unclosed Curve Depending on Parameters Defined Correct to $O\left(\frac{1}{\omega}\right)$

1. Consider the family of unclosed curves $M^1(\beta, \omega)$ in the phase plane (x, p). Let $\beta \in \mathbf{R}^n$, ω belong to an extended numerical axis and the equations of curves $M^1(\beta, \omega)$ are

$$x = x(\alpha, \beta, \omega), \tag{6.1}$$

$$p = p(\alpha, \beta, \omega), \tag{6.2}$$

where α is the (same) parameter on the curves of the family in question and $x(\alpha, \beta, \omega)$ and $p(\alpha, \beta, \omega)$ are infinitely differentiable functions (also for $\omega = \infty$).

Introduce the following functions $S(\alpha, \beta, \omega)$ and $\widetilde{S}(\alpha, \beta, \omega)$:

$$dS = p(\alpha, \beta, \omega)\, dx(\alpha, \beta, \omega),$$

$$S(0, \beta, \omega) = 0,$$

$$\widetilde{S}(\alpha, \beta, \omega) = -S(\alpha, \beta, \omega) + x(\alpha, \beta, \omega)\, p(\alpha, \beta, \omega). \tag{6.3}$$

Let $\{U(\beta, \omega)\}_{\beta\in B,\, \omega\in\Omega}$ be a family of the arcs of the curves $M^1(\beta, \omega)$ corresponding to the interval (a, b) containing the parameter α, and let Ω be an open domain without zero on an extended numerical axis and B be a domain in $\mathbf{R}^n$. Suppose that every arc $U(\beta, \omega)$ is diffeomorphically projected onto the axis x and also onto the axis p so that the equalities (6.1) and (6.2) have on $(a, b) \times B \times \Omega$ a unique solution with respect to α.

$$\alpha = \bar{\alpha}(x, \beta, \omega), \qquad \bar{\bar{\alpha}} = \alpha(p, \beta, \omega).$$

Let $\varphi(\alpha, \beta, \omega)$ be an infinitely differentiable function with a compact support which is contained in $(a, b) \times B \times \Omega$. Consider the integral

$$I(x, \beta, \omega) = \sqrt{\frac{\omega}{-2\pi i}} \int_{-\infty}^{\infty} e^{i\omega p\cdot x - i\omega\tilde{S}(\alpha, \beta, \omega)} \left| \frac{\partial p}{\partial \alpha} \right|^{-\frac{1}{2}} \times$$

$$\times \varphi(\alpha, \beta, \omega)\Big|_{\alpha=\bar{\bar{\alpha}}(p, \beta, \omega)} dp.$$

For the calculation of this integral we can apply the method of stationary phase. If $\chi(x)$ is an arbitrary function in $C_0^\infty(\mathbf{R})$ then

$$\chi(x) I(x, \beta, \omega) =$$

$$= \chi(x)(i \operatorname{sgn} \omega)^\sigma e^{i\omega S(\alpha, \beta, \omega)} \left| \frac{\partial x}{\partial \alpha} \right|^{-\frac{1}{2}} \varphi(\alpha, \beta, \omega)\Big|_{\alpha=\bar{\alpha}(x, \beta, \omega)} +$$

$$+ e^{i\omega S(\bar{\alpha}(x, \beta, \omega), \beta, \omega)} g_1(x, \beta, \omega) + g_2(x, \beta, \omega).$$

Here the functions $g_1(x, \beta, \omega)$ and $g_2(x, \beta, \omega)$ are non-zero only for $\omega \in \Omega$, the functions $\omega g_1(x, \beta, \omega)$ and $\omega g_2(x, \beta, \omega)$ belonging to the space $\mathscr{B}_{s_1, s_2, s_3}(\mathbf{R} \times \mathbf{R}^n \times \mathbf{R})$ for any s_1, s_2, s_3;

$$\sigma = \frac{1}{2}\left(1 - \operatorname{sgn} \frac{\partial p/\partial\alpha}{\partial x/\partial\alpha}\right)\Big|_{\alpha=\bar{\alpha}(x, \beta, \omega)}.$$

Now let $\chi(x)$ be an infinitely differentiable function equal to zero on the combined projection of the arcs $U(\beta, \omega)$ onto the axis x and equal to 1 in a neighborhood of infinity. Then we can use the results of the preceding section which deal with the absence of critical points of the phase if we set

$$y = \left(\beta, \frac{1}{x}\right).$$

We obtain that for any N and any s_1, s_2 and s_3

$$\omega^N \chi(x) I(x, \beta, \omega) \in \mathscr{B}_{s_1, s_2, s_3}(\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}).$$

Putting unity in the form $1 = \chi(x) + (1 - \chi(x))$, where $\chi(x) \in C_0^\infty(\mathbf{R})$, we finally obtain the following relation:

$$I(x, \beta, \omega) = (i \operatorname{sgn} \omega)^\sigma e^{i\omega S(\alpha, \beta, \omega)} \left| \frac{\partial x}{\partial \alpha} \right|^{-\frac{1}{2}} \times$$
$$\times \varphi(\alpha, \beta, \omega)\big|_{\alpha=\bar{\alpha}(x, \beta, \omega)} + e^{i\omega S(\bar{\alpha}(x, \beta, \omega), \beta, \omega)} g_1(x, \beta, \omega) +$$
$$+ g_2(x, \beta, \omega), \tag{6.4}$$

where the functions g_1 and g_2 have the same properties as above.

2. The definition of the index. The formula (6.4) which is a generalization of (4.1) permits us to define the canonical operator in the family of curves $M^1(\beta, \omega)$. Let C be a compact set in the space of parameters α, β and ω containing no points at which $\omega = 0$. Next, let $\{U_j\}$, $j = 1, \ldots, N$ be a family of open connected sets in the parameter space which covers C and let $\{e_j\}$, $j = 1, \ldots, N$ be partition of unity corresponding to the covering $\{U_j\}$

$$e_j(\alpha, \beta, \omega) \in C_0^\infty; \sum_{j=1}^N e_j(\alpha, \beta, \omega) = 1$$

for $(\alpha, \beta, \omega) \in C$.

Obviously, the covering $\{U_j\}$ of the compact set C may be chosen so that for any j one of the following conditions is fulfilled:

(a) $\frac{\partial x}{\partial \alpha}(\alpha, \beta, \omega) \neq 0$ for $(\alpha, \beta, \omega) \in U_j$,

(b) $\frac{\partial p}{\partial \alpha}(\alpha, \beta, \omega) \neq 0$ for $(\alpha, \beta, \omega) \in U_j$.

Divide the set of numbers $1, 2, \ldots, N$ into two classes—I_1 and I_2—in such a way that for $j \in I_1$ the condition (a) will hold and for $j \in I_2$ the condition (b) will hold. The family $\{U_j\}$ with the pair $\{I_1, I_2\}$ will be called a canonical atlas of a neighborhood of the set C in the space of parameters. The set U_j for $j \in I_1$ will be called a non-singular patch of this atlas and for $j \in I_2$—a singular patch. The function $\alpha_j(x, \beta, \omega)$, which is the solution of the following equation with respect to α

$$x = x(\alpha, \beta, \omega),$$
$$(\alpha, \beta, \omega) \in U_j,$$

corresponds to every non-singular patch U_j and the function $\alpha_{j'}(p, \beta, \omega)$, which is the solution of the following equation

$$p = p(\alpha, \beta, \omega),$$
$$(\alpha, \beta, \omega) \in U_{j'}$$

corresponds to a singular patch $U_{j'}$.

The canonical atlas $\{U_j\}$ and $\{I_1, I_2\}$ of a neighborhood of the compact set C may always be chosen in the space of parameters in such a way that for any $j_1 \in I_1$ and $j_2 \in I_2$ the intersection $U_{j_1} \cap \cap U_{j_2}$ is connected. We shall suppose that this condition is fulfilled. The number

$$\sigma_{U_{j_1} \cap U_{j_2}} = -\frac{1}{2}\left(1 - \operatorname{sgn}\frac{\frac{\partial p}{\partial \alpha}(\alpha, \beta, \omega)}{\frac{\partial x}{\partial \alpha}(\alpha, \beta, \omega)}\right)\Bigg|_{(\alpha, \beta, \omega) \in U_{j_1} \cap U_{j_2}}$$

will be called an index of intersection of the non-singular patch U_{j_1} and singular patch U_{j_2} which have a non-empty intersection.

Set by definition

$$\sigma_{U_{j_1} \cap U_{j_2}} = -\sigma_{U_{j_2} \cap U_{j_1}}.$$

If both patches U_{j_1} and U_{j_2} are non-singular, or singular, then set

$$\sigma_{U_{j_1} \cap U_{j_2}} = 0.$$

Let $U_{j_1}, \ldots, U_{j_m}$ be a finite sequence of patches of the canonical atlas such that any two successive patches of this sequence intersect. Such a finite sequence will be called a chain of patches. The number

$$\operatorname{ind}(j_1, \ldots, j_m) = \sigma_{U_{j_1} \cap U_{j_2}} + \ldots + \sigma_{U_{j_{m-1}} \cap U_{j_m}}$$

will be called an index of this chain of patches.

By means of the index of the chain of patches we introduce the index of an arbitrary path l in the space of parameters (α, β, ω) which begins and ends at points belonging to some non-singular patches. Let $U_{j_1}, \ldots, U_{j_m}$ be such a chain of patches begun and ended by non-singular patches that the path l is equivalent to the sum of the segments

$$l = l_1 + \ldots + l_m$$

satisfying the condition $l_\mu \subset U_{j_\mu}$. Then the number

$$\gamma[l] = \operatorname{ind}(j_1, \ldots, j_m)$$

will be called an index of the path l. This definition of the index of the path l is correct, i.e., it is independent of the choice of the chain of patches $U_{j_1}, \ldots, U_{j_m}$ and of the partition of the path l into segments $l_1, \ldots, l_m$.

In order to verify this statement, first suppose that by the same partition of the path l into segments we use another chain of patches $U_{j'_1}, \ldots, U_{j'_m}$ for the definition of the index of the path.

Then

$$\operatorname{ind}(j_1, \ldots, j_m) = \operatorname{ind}(j'_1, \ldots, j'_m). \tag{6.5}$$

In fact, let M_μ be the end of a segment l_μ. Set

$$\sigma_\mu \overset{\text{def}}{=} \frac{1}{2}\left(1 - \operatorname{sgn}\frac{\dfrac{\partial p}{\partial \alpha}(M_\mu)}{\dfrac{\partial x}{\partial \alpha}(M_\mu)}\right)$$

if $\frac{\partial x}{\partial \alpha}(M_\mu) \neq 0$, $\frac{\partial p}{\partial \alpha}(M_{\mu'}) \neq 0$ and $\sigma_\mu \overset{\text{def}}{=} 0$ in the opposite case. Regard the path l as a road which in every part l_μ has a highway (a non-singular patch) or a canal (a singular patch) (or the former and the latter). At the points there exist stations where one can change from a car to a motor-ship (if the highway and the canal intersect at this point). The passenger begins and ends his trip in a car. At every station he has the right to change the type of transport. Leaving the car he receives the sum σ_μ for the loss of the speed and, leaving the motor-ship, he pays the sum σ_μ. The sum does not change regardless of the part of the road where the highway is alongside the canal. It is clear that the sum total for the whole trip does not depend on the choice of the scheme for changing transports and this means that (6.5) is valid. Since one may go over singular or non-singular patches several times it should be noted that the index of the road does not change by partition of the segment l_μ into parts.

From the definition of the index of the path it follows that $\gamma[l]$ does not change by continuous deformation of l if the ends of the path remain unmoved. Since, in our case, the parameter α changes in some part of the real line it follows that any closed path l may be transformed into a point by continuous deformation in the space of parameters α, β, ω. This means that $\gamma[l] = 0$ for any closed path l.

Hence it follows that the index of the path depends only on its initial and final points.

Therefore, the index of the chain of patches beginning and ending with non-singular patches depends only on the first and last patches on the chain.

This fact is identically proved for an arbitrary chain of patches.

3. The definition of the canonical operator. Now we can define the canonical operator K on the family of curves $\{M^1(\beta, \omega)\}$. Fix a point P_0 in the space of parameters and let this point belong to some non-singular patch U_{j_0}. For any function $\varphi(\alpha, \beta, \omega)$ sup-

ported in the set C by definition we set

$$[K\varphi](x, \beta, \omega) = \sum_{j\in I_1} (i \operatorname{sgn} \omega)^{\gamma_j} e^{i\omega S(\alpha, \beta, \omega)} \times$$

$$\times \left| \frac{\partial x}{\partial \alpha} \right|^{-\frac{1}{2}} e_j(\alpha, \beta, \omega)\, \varphi(\alpha, \beta, \omega)\,|_{\alpha=\alpha_j(x, \beta, \omega)} +$$

$$+ \sum_{j\in I_2} (i \operatorname{sgn} \omega)^{\gamma_j} \sqrt{\frac{\omega}{-2\pi i}} \int_{-\infty}^{\infty} e^{i\omega(px - \widetilde{S}(\alpha, \beta, \omega))} \times$$

$$\times \left| \frac{\partial p}{\partial \alpha} \right|^{-\frac{1}{2}} e_j(\alpha, \beta, \omega)\, \varphi(\alpha, \beta, \omega)\,|_{\alpha=\alpha_j(p, \beta, \omega)}\, dp, \qquad (6.6)$$

where γ_j is the index of the chain of patches starting with the patch U_{j_0} and ending with the patch U_j.

The canonical operator K defined by (6.6) depends on the choice of the canonical atlas $\{U_j\}$ and on the partition of unity $\{e_j\}$ (as well as on the choice of the initial point P_0 and on the arbitrary choice of the definition of the action S associated with the possibility of the reparametrization of the family of curves).

Let $A_1, \ldots, A_{n+2}$ be generators acting on a Banach space B. Consider the operator

$$[K\varphi]\left(\overset{k}{A_1}, \overset{2}{A_2}, \ldots, \overset{k-1}{A_{k-1}}, \overset{k+1}{A_k}, \ldots, \overset{n+2}{A_{n+1}}, \overset{1}{A_{n+2}}\right).$$

Definition. We shall say that the ordered set of generators $A = (\overset{k}{A_1}, \overset{2}{A_2}, \ldots, \overset{k-1}{A_{k-1}}, \overset{k+1}{A_k}, \ldots, \overset{n+2}{A}_{n+1}, \overset{1}{A_{n+2}})$ acting on B is in agreement with the family of curves $M^1(\beta, \omega)$ in the phase plane (x, p) if for any non-singular patch U in the space of parameters α, β, ω (for family of curves $M^1(\beta, \omega)$) and for any infinitely differentiable function $\chi(\alpha, \beta, \omega)$ supported in U, the operator $\Phi(A)$ corresponding to the symbol

$$\Phi(x, \beta, \omega) = \chi(\alpha(x, \beta, \omega), \beta, \omega)\, e^{i\omega S(\alpha(x, \beta, \omega), \beta, \omega)}$$

is bounded in B.

Theorem 6.1. Let K and K' be two canonical operators on the family $\{M^1(\beta, \omega)\}$ corresponding to different canonical atlases $\{U_j\}_{j\in I}$, I_1, I_2 and $\{U'_j\}_{j\in I'}$, I'_1, I'_2 (of the neighborhood of compact set C in the space of parameters) and to the partitions of unity $\{e_j\}_{j\in I}$ and $\{e'_j\}_{j\in I'}$ respectively. Next, let the set of generators

$$A = \left(\overset{k}{A_1}, \overset{2}{A_2}, \ldots, \overset{k-1}{A_{k-1}}, \overset{k+1}{A_k}, \ldots, \overset{n+2}{A_{n+1}}, \overset{1}{A_{n+2}}\right)$$

acting on a Banach space B be such that the operators $A_1, \ldots, A_{k-1}$ commute with each other and with the operator A_{n+2} and the

operators $A_k, \ldots, A_{n+1}$ commute with each other and with the operator A_1; and let this set be in agreement with the family $\{M^1(\beta, \omega)\}$. Then for any infinitely differentiable function $\varphi(\alpha, \beta, \omega)$ with support in C the operator

$$\{[K\varphi](A) - [K'\varphi](A)\} A_{n+2}$$

is bounded in B.

The proof of this theorem is based on (6.4) and is analogous to the proof of the theorem of Sec. 4.

Note. Below we shall also need the dependence on the translator. This case is analogous to the preceding case. For the sake of simplicity let $n = 1$. In Theorem 6.1 one can replace the vector-operator A by the vector-operator

$$\left(\overset{1}{A_3}, \overset{2}{A_2}, \overset{3}{T}, \overset{4}{A_1}\right),$$

where A_2 commutes with A_3 and $T: B \to B$ is a translator, or by the vector-operator

$$\left(\overset{1}{A_3}, \overset{2}{T}, \overset{3}{A_2}, \overset{4}{A_1}\right),$$

where A_2 commute with A_1.

Sec. 7. V-Objects on the Curve

1. The definition of the spaces L_A and L_Σ. Let M^1 be an infinitely smooth curve on the plane $\mathbf{R}^2$ and let the Cartesian coordinates x and y be fixed on this plane. If an open arc U of the curve M^1 is diffeomorphically projected onto the axis x, then one can choose x as a parameter on this arc. If such a choice is made, then U will be called a non-singular patch of the curve M^1. In the same way, the arc of the curve M^1 on which y is chosen as a parameter will be called a singular patch of the curve M^1.

A set of all singular and non-singular patches of this curve will be called a complete canonical atlas $\mathcal{A}_\infty$ of the curve M^1. Any finite family $\{U_j\}_{j=1, \ldots, N}$ of patches of the complete canonical atlas $\mathcal{A}_\infty$ covering M^1 will be called a finite canonical atlas of the curve M^1. We shall suppose that the finite atlases of the curve M^1 exist (for example, M^1 is a compact curve).

Let $\mathcal{A}_H$ be a set of a non-singular patches of the atlas $\mathcal{A}_\infty$ and $\mathcal{A}_0$ be a set of singular patches of this atlas. Every patch $U \in \mathcal{A}_H$ is defined by the equation

$$y = y_U(x)$$

and every patch $U \in \mathcal{A}_0$ is defined by the equation

$$x = x_U(y).$$

It is obvious that in the case when the two families of functions

$$\{y_U\}_{U \in A_H}, \quad \{x_U\}_{U \in A_0}$$

are given they are equivalent to the curve M^1.

For any of the two intersecting patches $U \in A_H$ and $U' \in \mathcal{A}_0$ there exists a linear operator $V_{U,\ U'}$ acting on infinitely differentiable functions of y with support in the projection onto the axis y of the intersection $U \cap U'$ and translating them into infinitely differentiable functions of x with support in the projection onto the axis x of the intersection $U \cap U'$. It is required that the following two axioms be fulfilled:

V_1 (the locality of the operators $V_{U,\ U'}$). The value of the function $V_{U,\ U'}\varphi$ at the point x depends only on the germ of the function φ at the point $y_U(x)$ (i.e. only on the restriction of the function φ by any small neighborhood of the point $y_U(x)$).

V_2 (the conditions of concordance)

$$[V_{U_1,\ U_1'}\varphi]\,(x_0) = [V_{U_2,\ U_2'}\varphi]\,(x_0)$$

for any function φ with support in the projection onto the axis y of the intersection $U_1' \cap U_2'$ and for any point x_0 of the projection onto the axis x of the intersection $U_1 \cap U_2$.

Let $\mathcal{A} = \{U_j\}$ be the finite atlas of the curve M^1, I_1 be the set of numbers of non-singular patches and I_2 be the set of numbers of singular patches of this atlas. Consider the triple

$$\varphi = (\mathcal{A},\ \{\varphi_j\,(x)\}_{j \in I_1},\ \{\varphi_j\,(y)\}_{j \in I_2}), \tag{7.1}$$

where $\varphi_j\,(x)$ is a function in $C_0^\infty\,(\mathbf{R})$ with support in the projection U_j onto the axis x and $\varphi_j\,(y)$ is a function in $C_0^\infty\,(\mathbf{R})$ with support in the projection U_j onto the axis y. The set of all possible elements φ of the form (7.1) becomes a vector space for a fixed $\mathcal{A}$ if linear operations will be introduced by the formulas

$$\lambda\varphi = (\mathcal{A},\ \{\lambda\varphi_j\,(x)\}_{j \in I_1},\ \{\lambda\varphi_j\,(y)\}_{j \in I_2}),$$

$$(\mathcal{A},\ \{\varphi_j\,(x)\},\ \{\varphi_j\,(y)\}) + (\mathcal{A},\ \{\psi_j\,(x)\},\ \{\psi_j\,(y)\}) =$$
$$= (\mathcal{A},\ \{\varphi_j\,(x) + \psi_j\,(x)\},\ \{\varphi_j\,(y) + \psi_j\,(y)\}).$$

The obtained vector space will be denoted by $L_{\mathcal{A}}$. Denote by L_Σ the join of all possible sets $L_{\mathcal{A}}$.

2. The definition of V-objects. Let

$$\varphi = (\mathcal{A},\ \{\varphi_j\,(x)\}_{j \in I_1},\ \{\varphi_j\,(y)\}_{j \in I_2}), \quad \mathcal{A} = \{U_j\},$$

$$\psi = (\mathcal{A}',\ \{\psi_j\,(x)\}_{j \in I_1'},\ \{\psi_j\,(y)\}_{j \in I_2'}), \quad \mathcal{A}' = \{U_j'\}$$

be two elements of L_Σ. Consider the following four functions on M^1:

$$\varphi_H(x, y) = \sum_{\substack{j\in I_1 \\ (x, y)\in U_j}} \varphi_j(x), \quad \varphi_0(x, y) = \sum_{\substack{j\in I_2 \\ (x, y)\in U_j}} \varphi_j(y),$$

$$\psi_H(x, y) = \sum_{\substack{j\in I_1' \\ (x, y)\in U_j'}} \psi_j(x), \quad \psi_0(x, y) = \sum_{\substack{j\in I_2' \\ (x, y)\in U_j'}} \psi_j(y).$$

We shall say that the elements φ and ψ are equivalent and we shall write $\varphi \equiv \psi$ if such elements safisfy the following conditions:

(a) there exists a finite set Δ of the pairs (U, U'), where $U \in \mathscr{A}_H$, $U' \in \mathscr{A}_0$ such that for two different pairs (U_1, U_1'), (U_2, U_2') in Δ the intersection $U_1 \cap U_1' \cap U_2 \cap U_2'$ will be empty and

$$\varphi_H - \psi_H = \sum_{(U, U')\in\Delta} \alpha_{U, U'}, \quad \varphi_0 - \psi_0 = \sum_{(U, U')\in\Delta} \beta_{U, U'},$$

where $\alpha_{U, U'}$ and $\beta_{U, U'}$ are infinitely differentiable functions with support in $U \cap U'$;

(b) for any pair $(U, U') \in \Delta$ the following equality is valid:

$$\alpha_{U, U'}(x, y_U(x)) + V_{U, U'}[\beta_{U, U'}(x_{U'}(y), y)](x) = 0.$$

Denote by $\mathscr{L}$ the factor-set L_Σ according to the equivalence relation $\equiv$.

Introduce the structure of a vector space on $\mathscr{L}$. First of all, we note that if $\varphi, \psi \in L_\Sigma$ then there exist $\varphi', \psi' \in L_\Sigma$ such that $\varphi \equiv \varphi'$ and $\psi \equiv \psi'$ and the atlases corresponding to the elements φ' and ψ' coincide. The elements φ' and ψ' can be added. Define the sum of the classes of the equivalency $\{\varphi\}$ and $\{\psi\}$ by the formula

$$\{\varphi\} + \{\psi\} = \{\varphi' + \psi'\}. \tag{7.2}$$

It is easy to see that the sum is well-defined by (7.2). The product of the class $\{\varphi\}$ by a scalar will be defined in the ordinary way: $\lambda\{\varphi\} \overset{\text{def}}{=} \{\lambda\varphi\}$.

The introduced vector space $\mathscr{L}$ will be called the space of V-objects on the curve M^1.

Example. The functions on the curve as V-objects.

Consider a V-object on M^1 which corresponds to the following family of the "transition operators" $V_{U, U'}$: if U is a non-singular patch of the atlas $\mathscr{A}_\infty$ and U' is a singular patch intersecting with it, then

$$\begin{aligned} V_{U, U'}\varphi(y) &= \varphi(y_U(x)), \\ V_{U', U}\varphi(x) &= \varphi(x_{U'}(y)). \end{aligned} \tag{7.3}$$

Let φ be a V-object which corresponds to the family of transition operators (7.3) and constitutes the class of the equivalency contai-

ning the element

$$(\mathscr{A}, \{\varphi_j(x)\}, \{\varphi_j(y)\})$$

of the set L_Σ. Bring the object φ in correspondence with the function $\overline{\varphi}$ on M^1 defined by the formula

$$\overline{\varphi}(x_0, y_0) = \sum_{j \in K_1} \varphi_j(x_0) + \sum_{j \in K_2} \varphi_j(y_0), \tag{7.4}$$

where K_1 is the set of numbers of non-singular patches and K_2 is the set of numbers of singular patches (of the atlas $\mathscr{A}$) containing the point $(x_0, y_0) \in M_0^1$. From (7.3) it follows that the definition of the function does not depend on the choice of the representative V-object φ in L_Σ.

Conversely, let $\overline{\varphi}$ be an infinitely differentiable function with a compact support in M^1 and let $\mathscr{A}$ be a finite atlas in M^1. Consider some partition of unity $\{e_j\}$ corresponding to the atlas $\mathscr{A}$:

$$e_j \in C^\infty(M), \ \operatorname{supp} e_j \subset U_j, \quad \sum_{j=1}^{N} e_j = 1;$$

then every function $\overline{\varphi}_j = e_j\overline{\varphi}$ is supported in U_j. If U_j is a non-singular patch of the atlas $\mathscr{A}$ then set

$$\varphi_j(x) = \overline{\varphi}_j(x, y_{U_j}(x)), \tag{7.5}$$

and if U_j is a singular patch then

$$\varphi_j(y) = \overline{\varphi}_j(x_{U_j}(y), y). \tag{7.6}$$

The formulas (7.5), (7.6) bring the function $\overline{\varphi}$ in correspondence with an element $(\mathscr{A}, \{\varphi_j(x)\}, \{\varphi_j(y)\})$ of the set L_Σ such that (7.4) is valid. This establishes a one-to-one correspondence between the space $\mathscr{L}$ of V-objects corresponding to transition operators (7.4) and the space $C_0^\infty(M^1)$. Below we shall identify the V-object φ and the corresponding function $\overline{\varphi}$.

3. Linear operators on the space of V-objects. We shall now describe the method whereby linear operators on the space $\mathscr{L}$ of V-objects will be given. Since $\mathscr{L}$ is a factor-set it is natural to first set some operator A_0 on the set L_Σ and then, as usual, to bring an operator A on $\mathscr{L}$ in correspondence with A_0 according to the formula

$$A\{f\} = \{A_0 f\}. \tag{7.7}$$

Of course, for (7.7) actually to define the operator A on $\mathscr{L}$ it is necessary that the operator A_0 should be in agreement with the relation of the equivalency $\equiv$:

$$(f \equiv g) \to (A_0 f \equiv A_0 g).$$

To set a linear operator on $\mathscr{L}$ in the usual way one needs to set a linear operator A_0 on L_Σ. However L_Σ does not possess a structure of the vector space. For this reason the operator A_0 acting on L_Σ will be called a linear one if for every finite atlas $\mathscr{A}$ of a curve M^1 the restriction of the operator A_0 on $L_{\mathscr{A}}$ is a linear operator acting on $L_{\mathscr{A}}$. A structure of the vector space on $\mathscr{L}$ is introduced in such a way that the above-mentioned linear operator A_0 on L_Σ corresponds to the linear operator A on $\mathscr{L}$.

The above method of setting the linear operators on $\mathscr{L}$ is not yet versatile enough. One may now consider the operators A_0 defined on a part of L_Σ. It suffices to suppose that the operator A_0 is defined on the join of a family $\{L'_{\mathscr{A}}\}$ of linear subspaces such that $A_0 L'_{\mathscr{A}} \subset \subset L_{\mathscr{A}}$, $L'_{\mathscr{A}} \subset L_{\mathscr{A}}$, where for any $\varphi \subset \mathscr{L}$ there exists a representative in one of the subspaces $L'_{\mathscr{A}}$.

Example. A differential operator on the curve.

Consider again the space $C_0^\infty(M^1)$ as a space of V-objects on M^1. Fix an arbitrary finite atlas $\mathscr{A}$ on M^1 and the partition of unity $\{e_j\}$ corresponding to this atlas. This partition will be called a weighted partition of unity.

In the space $L_{\mathscr{A}}$ consider a linear subspace $L'_{\mathscr{A}}$ consisting of elements of the following form:

$$(\mathscr{A}, \{e_j(x, y_{U_j}(x))\, \varphi(x, y_{U_j}(x))\}, \{e_j(x_{U_j}(y), y)\, \varphi(x_{U_j}(y), y)\}),$$

where φ is an arbitrary function in $C_0^\infty(M^1)$. It is obvious that for any function $\varphi \in C_0^\infty(M')$ there exists its representative in $L'_{\mathscr{A}}$. Let I_1 be the set of numbers of non-singular patches and I_2 be the set of numbers of singular patches of the atlas $\mathscr{A}$. A differential operator L acting on $C_0^\infty(M^1)$ will be defined by means of the sets of operator $\{L_j\}_{j\in I_1}$ and $\{L_j\}_{j\in I_2}$ with differentiable coefficients according to the formulas

$$(\mathscr{A}, \{\psi_j(x)\}, \{\psi_j(y)\}) = L(\mathscr{A}, \{\varphi_j(x)\}, \{\varphi_j(y)\}), \tag{7.8}$$

$$\begin{aligned} \psi_j(x) &= L_j \varphi_j(x), \quad j \in I_1, \\ \psi_j(y) &= L_j \varphi_j(y), \quad j \in I_2 \end{aligned} \tag{7.9}$$

In order to define pseudodifferential operators on M^1, we introduce Sobolev spaces $W_2^k(M^1)$. Let again the finite atlas $\mathscr{A}$ and the weighted partition of unity $\{e_j\}$ be fixed. For any function $\varphi \in \in C_0^\infty(M^1)$ we set

$$\|\varphi\|_{W_2^k(M^1)} = \sum_{j\in I_1} \|\varphi_j(x)\|_{W_2^k(\mathbf{R})} + \sum_{j\in I_2} \|\varphi_j(y)\|_{W_2^k(\mathbf{R})}, \tag{7.10}$$

where $\varphi_j(x)$ and $\varphi_j(y)$ are the components of a representative of the function φ in the subspace $L'_{\mathcal{A}}$.

Problem. Prove that in the case when the curve M^1 is compact the norms of the form (7.10) corresponding to the different finite canonical atlases and to the different weighted partitions of unity are equivalent.

Example. Pseudodifferential operators on a curve.

Define the space $W_2^k(M^1)$ as the completion of the vector space $C_0^\infty(M^1)$ in norm (7.10). The elements of spaces $W_2^k(M^1)$ will be called generalized functions in M^1.

Let $\varphi \in C_0^\infty(M^1)$, $(\mathcal{A}, \{\varphi_j(x)\}, \{\varphi_j(y)\})$ be the representative of φ in $L'_{\mathcal{A}}$. Consider the two families of pseudodifferential operators

$$\left\{H_j\left(\overset{2}{x}, -i\overset{1}{\frac{d}{dx}}\right)\right\}_{j\in I_1}, \quad \left\{H_j\left(\overset{2}{y}, -i\overset{1}{\frac{d}{dy}}\right)\right\}_{j\in I_2} \tag{7.11}$$

whose symbols H_j satisfy the condition

$$\frac{H_j(\xi, \eta)}{(\eta+i)^m}\chi(\xi) \in \mathscr{B}_{S,0}(\mathbf{R}^2) \tag{7.12}$$

for any $\chi \in C_0^\infty(\mathbf{R})$. The operators of this family translate the space S in itself. Suppose that the operators $H_j\left(\overset{2}{x}, -i\overset{1}{d/dx}\right)$ and $H_j\left(\overset{2}{y}, -i\overset{1}{d/dy}\right)$ do not extend the support of the function on which they act (e.g. differential operators, the operator with the symbol $H(\xi, \eta) = e^{\eta}$ does not fit in). Bring the families of functions $\{\varphi_j(x)\}_{j\in I_1}$ and $\{\varphi_j(y)\}_{j\in I_2}$ in correspondence with the two new families $\{\psi_j(x)\}_{j\in I_1}$ and $\{\psi_j(y)\}_{j\in I_2}$ according to the formulas

$$\psi_j(x) = H_j\left(\overset{2}{x}, -i\overset{1}{\frac{d}{dx}}\right)\varphi_j(x),$$

$$\psi_j(y) = H_j\left(\overset{2}{y}, -i\overset{1}{\frac{d}{dy}}\right)\varphi_j(y).$$

Then

$$(\mathcal{A}, \{\psi_j(x)\}_{j\in I_1}, \{\psi_j(y)\}_{j\in I_2})$$

belongs to the space $L_{\mathcal{A}}$. Thus some operator H: $C_0^\infty(M^1) \to C_0^\infty(M^1)$ is defined. From the conditions (7.12) it follows that H is bounded as an operator acting on $W_2^{l+m}(M^1)$ to $W_2^l(M^1)$, where $|l| \leqslant s$. It is natural to consider the extension of the operator H to the homomorphism $W_2^{l+m}(M^1) \to W_2^l(M^1)$ as a pseudodifferential operator corresponding to the family of symbols $H_j(\xi, \eta)$.

Sec. 8. The Canonical Operator on the Family of Unclosed Curves

In this section the class of canonical operators will be constructed correct to $O(1/\omega^N)$ where N is any number.

1. V-objects for the canonical operator. Let the equations

$$x = x(\alpha, \beta, \omega), \quad p = p(\alpha, \beta, \omega)$$

define the family $\{M^1(\beta, \omega)\}$ of infinitely smooth curves on the phase plane x, p which satisfy the conditions mentioned in Sec. 6. Let C be a compact set in the space of parameters α, β, ω which does not contain points where $\omega = 0$. In Sec. 6 we introduced the concept of the canonical atlas of a neighborhood of the set C in the space of parameters. The set of all singular and non-singular patches of all canonical atlases corresponding to the totality of compact sets C will be called a complete canonical atlas of the space of parameters.

Proceed now to the definition of objects to which the canonical operator $\mathscr{K}_l$ will be applied. These will be some V-objects in the space of parameters. We allow some deviation from the general definitions given in the preceding section. This corresponds to our wish to avoid cumbersome details in the definitions of Sec. 7 in order to get them across more clearly. Now describe the mentioned deviations. In the definitions of the previous section the complete canonical atlas of the curve is to be replaced by the complete canonical atlas of the space of parameters and the finite canonical atlas of the curve by the canonical atlas of a neighborhood of the compact set C in the space of parameters. The elements of the space $L_{\mathscr{A}}$ will have function components

$$\begin{aligned} &\varphi_j(x, \beta, \omega), \ j \in I_1, \\ &\varphi_j(p, \beta, \omega), \ j \in I_2, \end{aligned} \tag{8.1}$$

where I_1 is the set of numbers of non-singular patches and I_2 is the set of numbers of singular patches of the atlas $\mathscr{A}$. Here we shall identify those functions (8.1) which differ only by $O(\omega^{-l-1})$, i.e., whose difference is the product of ω^{-l-1} by an infinitely differentiable function; for a patch containing the point $\omega = \infty$ we shall identify any two functions (8.1) if the patch does not contain points where $\omega = \infty$. In other words, we make a corresponding factorization in every space $L_{\mathscr{A}}$. By virtue of the preliminary factorization the conditions defining the relation of equivalency $\equiv$ are to be changed in the obvious way.

Define the transition operators $V_{U, U'}$ which translate the functions of x, β and ω into the functions of p, β and ω if U' is a non-singular patch and U is singular, and the functions of p, β and ω

into the functions of x, β and ω if U' is singular and U is non-singular. Let U be a non-singular patch and U' be a singular patch, both belonging to the atlas $\mathcal{A}_\infty$. The patch U corresponds to the function $\alpha_U(x, \beta, \omega)$ which satisfies the equation $x = x(\alpha_U(x, \beta, \omega), \beta, \omega)$ and the patch U' corresponds to the function $\alpha_{U'}(p, \beta, \omega)$ which satisfies the equation $p(\alpha_{U'}(p, \beta, \omega), \beta, \omega) = p$. Let $\varphi(\alpha, \beta, \omega)$ be an infinitely differentiable function with support contained in $U' \cap U$,

$$\bar{\varphi}(p, \beta, \omega) = \varphi(\alpha_U(p, \beta, \omega), \beta, \omega).$$

Consider the integral

$$I(x, \beta, \omega) = \sqrt{\frac{\omega}{-2\pi i}} \int_{-\infty}^{\infty} e^{i\omega p \cdot x - i\omega \tilde{S}(\alpha, \beta, \omega)} \times$$

$$\times \left| \frac{\partial p}{\partial \alpha} \right|^{-\frac{1}{2}} \varphi(\alpha, \beta, \omega) \big|_{\alpha = \alpha_{U'}(p, \beta, \omega)} dp,$$

where the action $\tilde{S}(\alpha, \beta, \omega)$ is defined by (6.3). On calculating the integral $I(x, \beta, \omega)$ by the method of stationary phase (for $x = x(\alpha, \beta, \omega)$, $(\alpha, \beta, \omega) \in U \cap U'$) we obtain

$$I(x, \beta, \omega) = (i \operatorname{sgn} \omega)^\sigma \sum_{j=0}^{l} \left\{ \left(\frac{i\sigma_0}{2\omega} \right)^j e^{i\omega S(\alpha_U(x, \beta, \omega), \beta, \omega)} \times \right.$$

$$\left. \times \left| \frac{\partial x}{\partial \alpha} \right|^{-\frac{1}{2}}_{\alpha = \alpha_U(x, \beta, \omega)} \chi_j(x, \beta, \omega) \right\} + r_{l+1}(x, \beta, \omega) \times$$

$$\times e^{i\omega S(\alpha_U(x, \beta, \omega), \beta, \omega)},$$

where

$$\sigma = \frac{1}{2}\left(1 - \operatorname{sgn}\left(\frac{\partial p}{\partial \alpha}\right) \Big/ \left(\frac{\partial x}{\partial \alpha}\right)\right)\Big|_{(\alpha, \beta, \omega) \in U \cap U'},$$

$$\sigma_0 = -\operatorname{sgn}\left(\frac{\partial p}{\partial \alpha}\right) \Big/ \left(\frac{\partial x}{\partial \alpha}\right)\Big|_{(\alpha, \beta, \omega) \in U \cap U'}$$

and the function $\omega^{l+1} r_{l+1}(x, \beta, \omega)$ belongs to any of the spaces $\mathcal{B}_{s_1, \ldots, s_{n+r}}(\mathbf{R}^{n+2})$. Concerning the functions χ_j we may say that by virtue of the method of stationary phase the functions $\chi_j(x, \beta, \omega)$ for a fixed function $\tilde{S}$ depends only on values of the function φ and its derivatives in α up to order l inclusive at the point $(\alpha_U(x, \beta, \omega), \beta, \omega)$.

We define the operator $V_{U,U'}$ in such a way that the following equality is valid:

$$I(x, \beta, \omega) = e^{i\omega S(\alpha_U(x, \beta, \omega), \beta, \omega)} \Big\{ (i \operatorname{sgn} \omega)^\sigma \times$$
$$\times \left| \frac{\partial x}{\partial \alpha} \right|^{-\frac{1}{2}}_{\alpha=\alpha_U(x, \beta, \omega)} [V_{U,U'}\bar{\varphi}](x, \beta, \omega) + r_{l+1}(x, \beta, \omega) \Big\}.$$

In other words, we set

$$[V_{U,U'}\bar{\varphi}](x, \beta, \omega) = \sum_{j=0}^{l} \left(\frac{i\sigma_0}{2\omega}\right)^j \chi_j(x, \beta, \omega). \tag{8.2}$$

2. The construction of the canonical element. Consider the space $\mathscr{L}$ of V-objects in the space of parameters α, β, ω corresponding to the family of transition operators $V_{U,U'}$ defined in the present section. The vectors of the space $\mathscr{L}$ will be called canonical elements.

We shall describe a standard method for constructing a canonical element. Let $\varphi(\alpha, \beta, \omega)$ be an infinitely differentiable function supported in a compact set C, let $\mathscr{A} = (U_j)_{j\in I}$ be the canonical atlas of a neighborhood of the set C and $\{e_j\}$ be a partition of unity called a weighted one corresponding to the atlas $\mathscr{A}$:

$$e_j \in C^\infty, \quad \operatorname{supp} e_j \subset U_j,$$
$$\sum e_j(\alpha, \beta, \omega) = 1, \quad (\alpha, \beta, \omega) \in C.$$

The function φ, the atlas $\mathscr{A}$ and the weighted partition of unity $\{e_j\}$ are brought in correspondence with the canonical element by means of the following procedure: let $\mathscr{A}' = \{U'_j\}_{j\in I'}$ be an arbitrary canonical atlas of a neighborhood of the set C and let $\{e'_j\}$ be the partition of unity corresponding to the atlas $\mathscr{A}'$. We set

$$\begin{aligned} \varphi_j(x, \beta, \omega) &= e'_j(\alpha, \beta, \omega)\, e_H(\alpha, \beta, \omega) \times \\ &\quad \times \varphi(\alpha, \beta, \omega)\,|_{\alpha=\alpha_{U'_j}(x, \beta, \omega),\ j\in I'_1} \\ \varphi_j(p, \beta, \omega) &= e'_j(\alpha, \beta, \omega)\, e_0(\alpha, \beta, \omega) \times \\ &\quad \times \varphi(\alpha, \beta, \omega)\,|_{\alpha=\alpha_{U_j}(p, \beta, \omega),\ j\in I'_2}, \end{aligned} \tag{8.3}$$

where

$$e_H = \sum_{j\in I_1} e_j, \qquad e_0 = \sum_{j\in I_2} e_j,$$

I_1 is the set of numbers of non-singular patches of the atlas $\mathscr{A}$, I_2 is the set of numbers of singular patches of the atlas $\mathscr{A}$ and I'_1 (I'_2) is the set of numbers of non-singular (singular) patches of the atlas $\mathscr{A}'$. Thus for every canonical atlas $\mathscr{A}'$ of a neighborhood C we define the element

$$\varphi_{\mathscr{A}'} = (\mathscr{A}', \{\varphi_j(x)\}_{j\in I'_1}, \quad \{\varphi_j(p)\}_{j\in I'_2})$$

of the space $L_{\mathscr{A}'}$. It is left to the reader to verify that for any canonical atlases $\mathscr{A}'$ and $\mathscr{A}''$ there exists an equivalency $\varphi_{\mathscr{A}'} \equiv \varphi_{\mathscr{A}''}$ so that all elements of the form $\varphi_{\mathscr{A}'}$ define one and the same class $\{\varphi_{\mathscr{A}'}\} \in \mathscr{L}$.

Let $k_1, \ldots, k_{n+2}$ be such non-negative integers that $l > k_1 + \ldots + k_{n+1}$. Intoduce in the space of infinitely differentiable functions of x, β and ω the following relation of equivalency $\approx$: $\varphi(x, \beta, \omega) \approx \psi(x, \beta, \omega)$ if and only if

$$\omega^{l-k_1-\ldots-k_{n+1}}[\varphi(x, \beta, \omega)-\psi(x, \beta, \omega)] \in \mathscr{B}_{k_1, \ldots, k_{n+2}}(\mathbf{R}^{n+2}).$$

The quotient space corresponding to the relation of equivalency $\approx$ will be denoted by $S^{k_1, \ldots, k_{n+2}, l}$. Allowing for the common inaccuracy in the notation we shall deal with the elements of the space $S^{k_1, \ldots, k_{n+2}, l}$ as with functions of x, β and ω, keeping in mind their actual representatives in $C^\infty(\mathbf{R}^{n+2})$. In the same way we shall often use notation and expressions as if the components $\varphi_j(x)$ and $\varphi_j(p)$ of an element of L_Σ were functions and not classes of equivalent functions.

3. The definition of the canonical operator. We may now define the canonical operator K_l on the family of the curves $M^1(\beta, \omega)$ with a fixed parametrization and a fixed "initial" point P_0 in the space of parameters, the operator acting on L_Σ to $S^{k_1, \ldots, k_{n+2}, l}$. Let $\mathscr{A}$ be the canonical atlas of a neighborhood of a compact set C in the space of parameters, I_1 be the set of numbers of non-singular patches of the atlas $\mathscr{A}$ and I_2 be the set of numbers of singular patches of this atlas. Let P_0 belong to a non-singular patch $U_{j_0} \in \mathscr{A}$ and γ be the index of the chain of patches of the atlas $\mathscr{A}$ which starts with the patch U_{j_0} and finishes with the patch U_j.

Definition. *For any element*

$$\varphi = (\mathscr{A}, \{\varphi_j(x, \beta, \omega)\}_{j\in I_1}, \{\varphi_j(p, \beta, \omega)\}_{j\in I_2})$$

of the space $L_{\mathscr{A}}$ *we set*

$$\begin{aligned}[K_l\varphi](x, \beta, \omega) = {} & \sum_{j\in I_1} (i \operatorname{sgn} \omega)^{\nu_j} e^{i\omega S(\alpha, \beta, \omega)} \times \\ & \times \left|\frac{\partial x}{\partial \alpha}\right|^{-\frac{1}{2}}\Bigg|_{\alpha=\alpha_{U_j}(x, \beta, \omega)} \varphi_j(x) + \sum_{j\in I_2} (i \operatorname{sgn} \omega)^{\nu_j} \times \\ & \times \sqrt{\frac{\omega}{-2\pi i}} \int_{-\infty}^{\infty} e^{i\omega((px)-\tilde{S}(\alpha, \beta, \omega))} \times \\ & \times \left|\frac{\partial p}{\partial \alpha}\right|^{-\frac{1}{2}}\Bigg|_{\alpha=\alpha_{U_j}(p, \beta, \omega)} \varphi_j(p)\, dp. \end{aligned} \tag{8.4}$$

Problem. Show that the definition is correct in a sense that if

$$\varphi' = (\mathcal{A}, \{\varphi'_j(x, \beta, \omega)\}, \{\varphi'_j(p, \beta, \omega)\})$$

and

$$\varphi'' = (\mathcal{A}, \{\varphi''_j(x, \beta, \omega)\}, \{\varphi''_j(p, \beta, \omega)\})$$

are two representatives of one and the same element $\varphi \in L_{\mathcal{A}}$ then $K_l\varphi' = K_l\varphi''$.

Theorem 8.1. *Let* $\varphi, \psi \in L_\Sigma$ *and* $\varphi \equiv \psi$. *Then*

$$K_l\varphi = K_l\psi.$$

Proof. First we note that without loss of generality one may consider that φ and ψ belong to one and the same space $L_{\mathcal{A}}$. Let

$$\varphi = (\mathcal{A}, \{\varphi_j(x, \beta, \omega)\}_{j\in I_1}, \{\varphi_j(p, \beta, \omega)\}_{j\in I_2}),$$
$$\psi = (\mathcal{A}, \{\psi_j(x, \beta, \omega)\}_{j\in I_1}, \{\psi_j(p, \beta, \omega)\}_{j\in I_2}).$$

Then

$$[K_l\varphi - K_l\psi](x, \beta, \omega) = [K_l(\varphi - \psi)](x, \beta, \omega),$$

where $\varphi - \psi \equiv 0$. Hence it is necessary to show

$$(\varphi \equiv 0) \Rightarrow (K_l\varphi = 0).$$

Let

$$\varphi = (\mathcal{A}, \{\varphi_j(x, \beta, \omega)\}_{j\in I_1}, \{\varphi_j(p, \beta, \omega)\}_{j\in I_2}), \qquad \varphi \equiv 0.$$

According to the definition of the relation of equivalency $\equiv$ consider the following two functions

$$\varphi_H(\alpha, \beta, \omega) = \sum_{j\in I_1} \varphi_j(x(\alpha, \beta, \omega), \beta, \omega),$$

$$\varphi_0(\alpha, \beta, \omega) = \sum_{j\in I_2} \varphi_j(p(\alpha, \beta, \omega), \beta, \omega).$$

Then there exists a finite set Δ of the pairs (U, U'), where U is a non-singular patch and U' is a singular one, both belonging to the atlas $\mathcal{A}_\infty$ such that the following conditions are fulfilled:

(a) for any two different elements (U_1, U'_1) and (U_2, U'_2) of the set Δ the intersection $U_1 \cap U'_1 \cap U_2 \cap U'_2$ is empty;

$$\text{(b)}\ \varphi_n(\alpha, \beta, \omega) = \sum_{(U, U')\in\Delta} \mu_{U, U'}(\alpha, \beta, \omega),$$

$$\varphi_s(\alpha, \beta, \omega) = \sum_{(U, U')\in\Delta} \nu_{U, U'}(\alpha, \beta, \omega),$$

where $\mu_{U,\,U'}$ and $\nu_{U,\,U'}$ are infinitely differentiable functions with support contained in $U \cap U'$,

(c) for any pair $(U, U') \in \Delta$

$$\mu_{U,\,U'}(\alpha_U(x, \beta, \omega), \beta, \omega) + \\ + [V_{U,\,U'}\nu_{U,\,U'}(\alpha_{U'}(p, \beta, \omega), \beta, \omega)](x, \beta, \omega) = 0.$$

Let $\gamma(U)$ be the index of the chain of patches starting with the patch U_{j_0} and finishing with the patch U. Then

$$[K_l^0\varphi](x, \beta, \omega) = \sum_{(U,\,U')\in\Delta} \Big\{ (i \operatorname{sgn} \omega)^{\gamma(U)} e^{i\omega S(\alpha, \beta, \omega)} \times \\ \times \left|\frac{\partial x}{\partial \alpha}\right|^{-\frac{1}{2}} \mu_{U,\,U'}(\alpha, \beta\ \omega)\,|_{\alpha=\alpha_U(x,\,\beta,\,\omega)} + \\ + (i \operatorname{sgn} \omega)^{\gamma(U')} \sqrt{\frac{\omega}{-2\pi i}} \int_{-\infty}^{\infty} e^{i\omega(px - \tilde{S}(\alpha,\,\beta,\,\omega))} \times \\ \times \left|\frac{\partial p}{\partial \alpha}\right|^{-\frac{1}{2}} \nu_{U,\,U'}(\alpha, \beta, \omega)\,|_{\alpha=\alpha_{U'}(p,\,\beta,\,\omega)}\, dp \Big\}. \tag{8.5}$$

According to the method of stationary phase every summand in the right-hand member of (8.5) is of the following form:

$$(i \operatorname{sgn} \omega)^{\gamma(U)} e^{i\omega S(\alpha, \beta, \omega)} \left|\frac{\partial x}{\partial \alpha}\right|^{-\frac{1}{2}} \Big|_{\alpha=\alpha_U(x,\,\beta,\,\omega)} \times \\ \times \{\mu_{U,\,U'}(\alpha_U(x, \beta, \omega), \beta, \omega) + \\ + [V_{U,\,U'}\gamma_{U,\,U'}(\alpha_{U'}(p, \beta, \omega), \beta, \omega)](x, \beta, \omega)\} + \\ + e^{i\omega S(\alpha_U(x,\,\beta,\,\omega),\,\beta,\,\omega)} r_1(x, \beta, \omega) + r_2(x, \beta, \omega),$$

where the function $r_1(x, \beta, \omega)$ is supported in the domain of the function $\alpha_U(x, \beta, \omega)$ and the functions r_1 and r_2 satisfy the condition

$$\omega^{l+1} r_j(x, \beta, \omega) \in \mathscr{B}_{k_1, \ldots, k_{n+2}}(\mathbf{R}^{n+2}),$$

where $j = 1, 2$ for any $k_1, \ldots, k_{n+2}$. Hence the function

$$\omega^{l-k_1-\ldots-k_{n+1}} [e^{i\omega S(\alpha_U(x,\,\beta,\,\omega),\,\beta,\,\omega)} r_1(x, \beta, \omega) + r_2(x, \beta, \omega)]$$

belongs to the space $\mathscr{B}_{k_1, \ldots, k_{n+2}}(\mathbf{R}^{n+2})$, i.e., it is equivalent to zero.

The theorem is proved.

From this theorem it follows that the operator K_l induces some operator on $\mathscr{L}$ to $S^{k_1, \ldots, k_{n+2}, l}$. This operator will be called *a canonical operator* $\mathscr{K}_l$.

Now let $A_1, \ldots, A_{n+2}$ be generators of degree $k_1, \ldots, k_{n+2}$, respectively, acting on a Banach space B and defined on a dense linear manifold D. The operator

$$[K_l\varphi]\begin{pmatrix} s & 2 & & s-1 & s+1 & & n+2 & 1 \\ A_1, & A_2, & \ldots, & A_{s-1}, & A_s, & \ldots, & A_{n+1}, & A_{n+2}\end{pmatrix} A_{n+2}^{l-k_1-\ldots-k_{n+1}}$$

is bounded in B if $\varphi\equiv 0$. Let $\widetilde{\widetilde{\mathrm{Op}}}$ be the space of linear operators in B defined on D and factorized according to the following relation of equivalence $\simeq$: $A' \simeq A''$ if and only if the operator $(A' - A'')\times \times A_{n+2}^{l-k_1-\ldots-k_{n+1}}$ is bounded. If the elements φ' and φ'' of L_Σ are representatives of one and the same class $\varphi\in\mathscr{L}$ then

$$[K_l\varphi']\begin{pmatrix} s & 2 & & s-1 & s+1 & & n+2 & 1 \\ A_1, & A_2, & \ldots, & A_{s-1}, & A_s, & \ldots, & A_{n+1}, & A_{n+2}\end{pmatrix} \simeq$$
$$\simeq [K_l\varphi'']\begin{pmatrix} s & 2 & & s-1 & s+1 & & n+2 & 1 \\ A_1, & A_2, & \ldots, & A_{s-1}, & A_s, & \ldots, & A_{n+1}, & A_{n+2}\end{pmatrix}. \tag{8.6}$$

We shall denote the class of equivalent operators on $\widetilde{\widetilde{\mathrm{Op}}}$ which the left and right members of (8.6) belong to by

$$[K_l\varphi]\begin{pmatrix} s & 2 & & s-1 & s+2 & & n+2 & 1 \\ A_1, & A_2, & \ldots, & A_{s-1}, & A_s, & \ldots, & A_{n+1}, & A_{n+2}\end{pmatrix}.$$

Sec. 9. The Canonical Operator on the Family of Closed Curves

1. The definition of the canonical operator K_l on $L_{\mathscr{A}}$. Consider the family of closed curves $M^1(\beta, \omega)$ in the phase plane (x, p) defined by the equations

$$x = x(\alpha, \beta, \omega), \quad p = p(\alpha, \beta, \omega),$$

where α is a point on the circle, $\beta\in\mathbf{R}^n$, ω belongs to the extended numerical axis and the functions $x(\alpha, \beta, \omega)$ and $p(\alpha, \beta, \omega)$ are supposed to be infinitely differentiable, for $\omega = \infty$ as well.

For the reader who is unfamiliar with the theory of manifolds we shall explain some topological concepts related to the space of parameters α, β and ω considered in this section. The convergence of a sequence of points on the circle will be understood as the convergence of this sequence on the plane where the circle is lying. The convergence to ∞ of the sequence of points of an extended numerical axis is understood in the obvious way. We shall say that the sequence $\{(\alpha_m, \beta_m, \omega_m)\}$ of points of the space of parameters converges to the point (α, β, ω) if

$$\lim_{m\to\infty}\alpha_m = \alpha, \qquad \lim_{m\to\infty}\beta_m = \beta \qquad \text{and} \lim_{m\to\infty}\omega_m = \omega.$$

Next, the concepts of a closed set and of a closure of a set in the space of parameters (α, β, ω) are defined in the same way as in $\mathbf{R}^n$. A compact set in the space of parameters (α, β, ω) is a closed set contained in some set of the form $|\beta| < \text{const}$. An open set is the complement of a closed set. If the parameter t is introduced on the circle in a neighborhood of the point α_0 then in the neighborhood of the point $(\alpha_0, \beta_0, \omega_0)$ of the space of parameters every function of α, β and ω may be considered as a function of t, β and ω. This enables us to introduce the concept of a differentiable function defined on the space of parameters α, β and ω. Below we shall assume that the points where $\omega = 0$ are excluded from the space.

As before, a complete canonical atlas $\mathscr{A}'_\infty$ and a canonical atlas $\mathscr{A}'$ of a neighborhood of the compact set are introduced.

Let $(\alpha_0, \beta_0, \omega_0)$ be a fixed point in the space of parameters and $C = (U_1, \ldots, U_m)$ be the chain of patches, where $(\alpha_0, \beta_0, \omega_0) \in U_1$. Consider the function $S_1(\alpha, \beta, \omega)$ defined in the patch U_1 by the formulas

$$dS_1(\alpha, \beta, \omega) = p(\alpha, \beta, \omega) \frac{\partial x}{\partial \alpha}(\alpha, \beta, \omega)\, d\alpha,$$

$$S_1(\alpha_0, \beta_0, \omega_0) = 0.$$

In order to interpret the symbols $\frac{\partial}{\partial \alpha}$ and $d\alpha$ from now on we shall identify a moving point α on the circle with the polar angle of this point.

Let $(\alpha_1, \beta_1, \omega_1) \in U_1 \cap U_2$. In the patch U_2 define a function $S_2(\alpha, \beta, \omega)$ by the formulas

$$dS_2(\alpha, \beta, \omega) = p(\alpha, \beta, \omega) \frac{\partial x}{\partial \alpha}(\alpha, \beta, \omega)\, d\alpha,$$

$$S_2(\alpha_1, \beta_1, \omega_1) = S_1(\alpha_1, \beta_1, \omega_1).$$

Continuing this procedure we shall come to a function $S_m(\alpha, \beta, \omega)$ in U_m, which will be denoted by $S_C(\alpha, \beta, \omega)$. Define a function $\widetilde{S}_C(\alpha, \beta, \omega)$ by the formula

$$\widetilde{S}_C(\alpha, \beta, \omega) = -S_C(\alpha, \beta, \omega) + p(\alpha, \beta, \omega)\, x(\alpha, \beta, \omega).$$

Define the index γ_C of the chain of patches C as before in Sec. 6. Besides, introduce V-objects in the space of parameters in the same way as in the previous section.

Bring every patch U_j of the atlas $\mathscr{A}$ in correspondence with the chain C_j of patches of this atlas starting with the patch U_{j_0} containing the point $(\alpha_0, \beta_0, \omega_0)$ and finishing with the patch U_j. Denote $\gamma_{C_j} = \gamma_j$, $S_{C_j} = S_j$ and $\widetilde{S}_{C_j} = \widetilde{S}_j$. Define on $L_{\mathscr{A}}$ the canonical operator K_l in the following way: let

$$\varphi = (\mathscr{A}, \{\varphi_j(x, \beta, \omega)\}_{j \in I_1}, \{\varphi_j(p, \beta, \omega)\}_{j \in I_2});$$

then

$$[K_l\varphi](x, \beta, \omega) \overset{\mathrm{def}}{=} \sum_{j\in I_1} (i \operatorname{sgn} \omega)^{\gamma_j} e^{i\omega S_j(\alpha, \beta, \omega)} \times$$

$$\times \left|\frac{\partial x}{\partial \alpha}\right|^{-\frac{1}{2}}\Bigg|_{\alpha=\alpha_{U_j}(x, \beta, \omega)} \varphi_j(x, \beta, \omega) +$$

$$+ \sum_{j\in I_2} (i \operatorname{sgn} \omega)^{\gamma_j} \sqrt{\frac{\omega}{-2\pi i}} \int_{-\infty}^{\infty} \Bigg[e^{i\omega(px-\widetilde{S}_j(\alpha, \beta, \omega))} \times$$

$$\times \left|\frac{\partial p}{\partial \alpha}\right|^{-\frac{1}{2}}\Bigg]_{\alpha=\alpha_{U_j}(p, \beta, \omega)} \varphi_j(p, \beta, \omega)\, dp.$$

2. The condition of quantization. The following lemma is valid.

Lemma 9.1. *Let $C = (U_1, \ldots, U_m)$, where $U_m = U_1$ be the closed chain of patches of the atlas $\mathcal{A}_\infty$. Then γ_C is even.*

Proof. The index of a closed path in the space of parameters is equal to the index of the circle

$$\beta = \text{const}, \qquad \omega = \text{const},$$
$$[x(\alpha, \beta, \omega) - x_0]^2 + [p(\alpha, \beta, \omega) - p_0]^2 = R^2$$

which passes on the phase plane integral number of times clockwise and counterclockwise. Calculate the index of the circle, it is equal to 2. The lemma is proved.

Suppose now, for simplicity, that β is a one-dimensional parameter. From the lemma it follows that the function

$$[K_l\varphi](x, \beta, \omega)$$

does not depend on the choice of chains of patches C_j at the points satisfying the conditions

$$\omega \int_{M^1(\beta, \omega)} p\, dx - \pi \equiv 0 \pmod{2\pi}. \tag{9.1}$$

Equation (9.1) correlates the variables β and ω when the family $\{M^1(\beta, \omega)\}$ is given. Equation (9.1) is known as the "condition of quantization".

Example. Let the curve $M^1(\beta, \omega)$ have the equation $x^2 + p^2 = \beta$. Then the condition of quantization is of the form

$$\omega\pi\beta - \pi \equiv 0 \pmod{2\pi}. \tag{9.2}$$

Let ε be an arbitrary real number. Set $\beta(\omega) = \dfrac{2[\varepsilon\omega]+1}{\omega}$, where $[\varepsilon\omega]$ is the integral part of $\varepsilon\omega$.

If $\beta = \beta(\omega)$ is substituted in (9.2) then this condition is identically satisfied. Note that $\beta(\omega)$ is a bounded function with discontinuities at isolated points. This situation is typical.

Proceed to the general case. Let the equation

$$\oint\limits_{M^1(\beta, \omega)} p\,dx = \sigma$$

with respect to unknown β have a unique solution

$$\beta = \bar{\beta}(\sigma, \omega),$$

where $\bar{\beta}(\sigma, \omega)$ is an infinitely differentiable function, and the derivatives of $\bar{\beta}$ in ω of all orders are bounded. Set

$$\sigma(\varepsilon, \omega) = \frac{2[\varepsilon\omega]+1}{\omega},$$

where $[\varepsilon\omega]$ is the integral part of the number $\varepsilon\omega$ and denote

$$\beta(\varepsilon, \omega) = \bar{\beta}(\sigma(\varepsilon, \omega), \omega).$$

Then the substitution $\beta = \beta(\varepsilon, \omega)$ turns the condition of quantization (9.1) into an identity.

The operator K_{quan} defined by the formula

$$[K_{\text{quan}}\varphi](x, \varepsilon, \omega) = [K_l\varphi](x, \beta,(\varepsilon, \omega), \omega)$$

will be called *a quantized canonical operator*. The quantized canonical operator does not depend on the choice of chains C_j.

3. The regularization of the canonical operator. The function $[K_{\text{quan}}\varphi](x, \varepsilon, \omega)$ is not a smooth function. For this reason it can be used for the construction of a function of operators. In order to avoid this difficulty we shall regularize the canonical operator. Let r be an integer. We shall correlate the function $K_{\text{reg}}\varphi$ defined in $\mathbf{R}^3$ with the element $\varphi \in L_\Sigma$:

$$[K_{\text{reg}}\varphi](x, \varepsilon, \omega) = \omega^{-r}\int_{-\infty}^{\infty} p(\varepsilon-\varepsilon')[K_{\text{quan}}\varphi](x, \varepsilon', \omega)\,d\varepsilon', \qquad (9.3)$$

where $p(\varepsilon) \in C_0^\infty(\mathbf{R})$. The operator K_{reg} will be called *a regularized canonical operator*.

Lemma 9.2. *Let* $r \geqslant s_1 + s_3 + 1$. *Then*

$$[K_{\text{reg}}\varphi] \in \mathscr{B}_{s_1, s_2, s_3}(\mathbf{R}^3).$$

Proof. Denote $f(x, \varepsilon, \omega) = [K_{\text{quan}}\varphi](x, \varepsilon, \omega)$,

$$f(x, \varepsilon, \omega) = [K_{\text{reg}}\varphi](x, \varepsilon, \omega) = \\ = \frac{1}{\omega^r} \int_{-\infty}^{\infty} p(\varepsilon - \varepsilon') f(x, \varepsilon', \omega)\, d\varepsilon'.$$

The function $f(x, \varepsilon, \omega)$ is of the form

$$f(x, \varepsilon, \omega) = h(x, \beta, \omega)\,|_{\beta = \bar{\bar{\beta}}([\varepsilon\omega], \omega)}, \tag{9.4}$$

where $h(x, \beta, \omega) = [\mathscr{K}_l\varphi](x, \beta, \omega)$, $\bar{\bar{\beta}}([\varepsilon\omega], \omega) = \bar{\beta}\left(\frac{2[\varepsilon\omega]+1}{\omega}, \omega\right)$ and the functions h and $\bar{\bar{\beta}}$ satisfy the following conditions:

(a) h is infinitely differentiable;

(b) $h(x, \beta, \omega) = 0$ for sufficiently large $|\beta|$ and $h(x, \beta, \omega) = 0$ for sufficiently small $|\omega|$;

(c) for any function $e(x) \in C_0^\infty(\mathbf{R})$ which is equal to unity in a sufficiently large neighborhood of zero, the function $[1 - e(x)] \times \times h(x, \beta, \omega)$ belongs to any Sobolev space;

(d) for any function $e(x) \in C_0^\infty(\mathbf{R})$ the function

$$\left| \left(\frac{\partial}{\partial\beta}\right)^r \left(\frac{\partial}{\partial x}\right)^{r_2} \left(\frac{\partial}{\partial\omega}\right)^{r_3} \frac{e(x)\, h(x, \beta, \omega)}{\omega^r} \right| \leqslant \frac{c}{\sqrt{\omega^2+1}}$$

for $r \geqslant r_1 + r_2 + 1$ and any r_3 ($c = \text{const}$ depending only on r_1. r_2, r_3, r);

(e) the function $\bar{\bar{\beta}}(\xi, \omega)$ is infinitely differentiable;

(f) $\bar{\bar{\beta}}([\varepsilon\omega], \omega) \to \infty$ for $\varepsilon \to \infty$ uniformly in ω;

(g) the derivatives of $\bar{\bar{\beta}}(\xi, \omega)$ in ω of all orders are bounded in any set, where $\bar{\bar{\beta}}(\xi, \omega)$ is bounded.

In (9.3) we shall make a change of variables $\xi = \varepsilon'\omega$. Then we obtain

$$g(x, \varepsilon, \omega) = \frac{1}{\omega^{r+1}} \int_{-\infty}^{\infty} p\left(\varepsilon - \frac{\xi}{\omega}\right) h(x, \bar{\bar{\beta}}([\xi], \omega), \omega)\, d\xi. \tag{9.5}$$

On differentiating the right-hand member of (9.5) and by using the properties (a)-(g) we obtain the proof of the lemma. Note that $g(x, \varepsilon, \omega) = 0$ for sufficiently large $|\varepsilon|$.

Consider also the operator K'_{reg} defined by the formula

$$[K_{\text{reg}}\varphi](x, \varepsilon, \omega) = \frac{\chi(\varepsilon)}{\omega^r} \int_{-\infty}^{\infty} e^{-(\varepsilon-\varepsilon')^2\omega^{2s}} [K_{\text{quan}}\varphi](x, \varepsilon', \omega)\, d\varepsilon', \tag{9.6}$$

where $\chi(\varepsilon) \in C_0^\infty(\mathbf{R})$.

Lemma 9.3. *Let* $r \geqslant s_1 + rss_2 + rss_3 + rs - s_3 + 1$ *if* $s \geqslant 1$ *and* $r \geqslant s_1 + s_3 + 1$ *if* $s = 0$.

Then

$$[K'_{\text{reg}}\varphi] \in \mathscr{B}_{s_1, s_2, s_3}(\mathbf{R}^3).$$

Proof. Make a change of variables of integration

$$\xi = \varepsilon' \omega$$

Then we obtain

$$[K'_{\text{reg}}\varphi](x, \varepsilon, \omega) = \frac{\chi(\varepsilon)}{\omega^{r+1}} \int_{-\infty}^{\infty} e^{-\left(\varepsilon - \frac{\xi}{\omega}\right)\omega^{2s}} \times$$

$$\times h(x, \overline{\overline{\beta}}([\xi\ \omega], \omega), \omega)\, d\xi, \tag{9.7}$$

where the functions h, $\overline{\overline{\beta}}$ are the same as in the proof of the previous lemma. By differentiating the right-hand member of (9.7) we obtain

$$[K'_{\text{reg}}\varphi] \in W_2^{s'_1, s'_2, s'_3}(\mathbf{R}^3),$$

where $s'_i > s_i + \frac{1}{2}$, $i = 1, 2, 3$. Hence follows the proof of the lemma.

Consider in the space $\mathscr{B}_{s_1, s_2, s_3}(\mathbf{R}^3)$ the following relation of equivalency:

$$\varphi(x, \varepsilon, \omega) \overset{l}{\approx} \psi(x, \varepsilon, \omega)$$

if and only if

$$\omega^{l+1}[\varphi(x, \varepsilon, \omega) - \psi(x, \varepsilon, \omega)] \in \mathscr{B}_{s_1, s_2, s_3}.$$

Theorem 9.1. *Let* $\varphi, \psi \in L_\Sigma$ *and* $\varphi \equiv \psi$. *Then for* $r \geqslant s_1 + s_3 + 1$

$$K_{\text{reg}}\varphi \overset{l}{\approx} K_{\text{reg}}\psi;$$

for $r \geqslant \max\{s_1 + s_3 + 1,\ s_1 + rss_2 + rss_3 + rs - s_3\}$

$$K'_{\text{reg}}\varphi \overset{l}{\approx} K'_{\text{reg}}\psi.$$

Proof. Following the proof of Theorem 8.1 it suffices to show that if $\varphi \equiv 0$ then $K_{\text{reg}}\varphi \overset{l}{\approx} 0$ and $K'_{\text{reg}}\varphi \overset{l}{\approx} 0$. Let $\varphi \equiv 0$. As in the previous section we obtain

$$[K_{\text{reg}}\varphi](x, \varepsilon, \omega) = \frac{1}{\omega^{r+l+2}} \int_{-\infty}^{\infty} \rho\left(\varepsilon - \frac{\xi}{\omega}\right) h(x, \overline{\overline{\beta}}([\xi], \omega), \omega)\, d\xi,$$

$$[K'_{\text{reg}}\varphi](x, \varepsilon, \omega) = \frac{1}{\omega^{r+l+2}} \int_{-\infty}^{\infty} e^{-\left(\varepsilon - \frac{\xi}{\omega}\right)^2 \omega^{2s}} h(x, \overline{\overline{\beta}}([\xi], \omega), \omega)\, d\xi,$$

where the functions h and $\overline{\overline{\beta}}$ satisfy the above conditions (a)-(g), Q.E.D.

Now let A_1, A_2, A_3 be a generator of degree s_1, s_2, s_3 acting on a Banach space B and defined on a linear manifold D dense everywhere.

In the space Hom (B, B) introduce the relation of equivalency $\simeq$: $A' \simeq A''$ if and only if the operator $(A' - A'')\, A_3^{l+1}$ is bounded. The factor-space Hom $(B, B) \mid_{\simeq}$ will be denoted by $\widetilde{\widetilde{\mathrm{Op}}}\,(B)$.

From Theorem 9.1 it follows that if the elements φ' and φ'' of L_Σ are representatives of the same class $\varphi \in \mathscr{L}$ and r satisfies the condition of Theorem 9.1, then

$$[K_{\mathrm{reg}}\varphi']\left(\overset{3}{A}_1, \overset{2}{A}_2, \overset{1}{A}_3\right) \simeq [K_{\mathrm{reg}}\varphi'']\left(\overset{3}{A}_1, \overset{2}{A}_2, \overset{1}{A}_3\right), \tag{9.8}$$

$$[K'_{\mathrm{reg}}\varphi']\left(\overset{3}{A}_1, \overset{2}{A}_2, \overset{1}{A}_3\right) \simeq [K'_{\mathrm{reg}}\varphi'']\left(\overset{3}{A}_1, \overset{2}{A}_2, \overset{1}{A}_3\right). \tag{9.9}$$

The class of equivalent operators on $\widetilde{\widetilde{\mathrm{Op}}}\,(B)$ which the right and left members of (9.8) belong to will be denoted by $[\mathscr{K}_{\mathrm{reg}}\varphi]\left(\overset{3}{A}_1, \overset{2}{A}_2, \overset{1}{A}_3\right)$; and the class which the left and right members of (9.9) belong to will be denoted by

$$[\mathscr{K}'_{\mathrm{reg}}\varphi]\left(\overset{3}{A}_1, \overset{2}{A}_2, \overset{1}{A}_3\right).$$

In the same way, the regularized canonical operators corresponding to other methods of ordering the triple A_1, A_2 and A_3 such as

$$[\mathscr{K}_{\mathrm{reg}}\varphi]\left(\overset{2}{A}_1, \overset{3}{A}_2, \overset{1}{A}_3\right), \quad [\mathscr{K}'_{\mathrm{reg}}\varphi]\left(\overset{2}{A}_1, \overset{3}{A}_2, \overset{1}{A}_3\right)$$

are defined, where φ is an arbitrary canonical element.

Sec. 10. An Example of Commutation of a Canonical Operator with a Hamiltonian

In this section we shall consider the simplest case of a curve $M^1(\beta, \omega)$ which does not depend on parameters β and ω. For the convenience of notation we assume that $\omega = 1/h$. In order to illustrate general situation we shall first consider the following example.

Example. Find a "nearly" exact solution of the equation $\frac{h^2}{2}y'' + xy = 0$ for $h > 0$ in the region $|x| < a$ to an accuracy of $O(h^3)$, i. e. find such a function $y(x)$ which on substitution in the given equation yields a residual term $O(h^3)$.

This problem is trivial because after the $\frac{1}{h}$-Fourier transform we arrive at the equation $\frac{p^2}{2}\tilde{y} - ih\frac{d\tilde{y}}{dp} = 0$ which is a linear equation of first order. Nevertheless we shall construct its solution by means

of the canonical operator so as to illustrate the general situation. For this reason we shall not use the simplest canonical atlases.

Seek y in the form $y = K_1\varphi$, where K_1 is a canonical operator (mod $O(h^2)$) on the curve $\Lambda = \{x = \frac{\alpha^2}{2},\ p = \alpha\}$ associated with the Hamiltonian $-\frac{h^2}{2}\frac{d^2}{dx^2} - x$. $\left(\text{In the canonical operator we make the substitution } \omega = \frac{1}{h}.\right)$ Consider the canonical atlas on Λ which consists of non-singular patches U_1 and U_2 and a singular patch U_3, where the patch U_1 corresponds to the region $\alpha > 1$, U_2 to the region $\alpha < -1$ and U_3 to $|\alpha| < 2$. The situation in question is essentially general.

The canonical element φ which the operator K acts on is defined by the three functions: $\varphi_1(\alpha)$, $\varphi_2(\alpha)$, and $\varphi_3(\alpha)$ also depending on h with support contained in the patches U_1, U_2, U_3, respectively. These functions are defined correct to modulo $O(h^2)$. The other triple of functions $\psi_1(\alpha)$, $\psi_2(\alpha)$ and $\psi_3(\alpha)$ defines the same canonical element φ if the following conditions are fulfilled (modulo $O(h^2)$):

$$\begin{aligned}
&\varphi_1(\alpha) = \psi_1(\alpha) \text{ for } \alpha > 2,\\
&\varphi_2(\alpha) = \psi_2(\alpha) \text{ for } \alpha < -2,\\
&\varphi_3(\alpha) = \psi_3(\alpha) \text{ for } |\alpha| < 1,\\
&\varphi_1(\alpha) + V_{1,3}\varphi_3(\alpha) = \psi_1(\alpha) + V_{1,3}\psi_3(\alpha) \text{ for } \alpha \in U_1 \cap U_3,\\
&\varphi_2(\alpha) + V_{2,3}\varphi_3(\alpha) = \psi_2(\alpha) + V_{2,3}\psi_3(\alpha) \text{ for } \alpha \in U_2 \cap U_3,
\end{aligned}$$

where $V_{1,3}$ $(V_{2,3})$ is a transition operator on the patch U_3 to the patch U_1 (U_2). In the case in question according to (8.2) it is of the form:

$$V_{1,3} = V_{2,3} \overset{\text{def}}{=} V = \\ = 1 - ih\left[\frac{1}{2\alpha}\frac{d^2}{d\alpha^2} - \frac{1}{2\alpha^2}\frac{d}{d\alpha} + \frac{5}{24\alpha^3}\right] \bmod O(h^2).$$

The canonical operator is of the form

$$K\varphi = \frac{e^{\frac{i}{h}S_1(x)}}{\sqrt{y_1(x)}}\varphi_1(\sqrt{2x}) + i\,\frac{e^{\frac{i}{h}S_2(x)}}{\sqrt{|y_2(x)|}}\varphi_2(-\sqrt{2x}) + \\ + \sqrt{\frac{1}{-2\pi ih}}\int_{-\infty}^{\infty} e^{\frac{i}{h}\left(px - \frac{p^3}{6}\right)}\varphi_3(p)\,dp = K_1\varphi_1 + K_2\varphi_2 + K_3\varphi_3,$$

where

$$S_1(x) = \frac{(2x)^{\frac{3}{2}}}{3}, \qquad S_2(x) = -\frac{(2x)^{\frac{3}{2}}}{3},$$
$$y_1(x) = \sqrt{2x}, \qquad y_2(x) = -\sqrt{2x}.$$

Denote $H = -\frac{h^2}{2}\frac{d^2}{dx^2}$. We have

$$HK_1 = -ih\, K_1P_1,\ HK_2 = -ih\, K_2P_2,\ HK_3 = -ih\, K_3P_3,$$

where

$$P_2 = P_1 = \frac{d}{d\alpha} - ih\left\{\frac{5}{8\alpha^4} - \frac{1}{\alpha^3}\frac{d}{d\alpha} + \frac{1}{2\alpha^2}\frac{d^2}{d\alpha^2}\right\},\quad P_3 = \frac{d}{d\alpha}.$$

The triple of operators P_1, P_2 and P_3 defines an operator P acting in the space of canonical elements since the following conditions are fulfilled

$$VP_3 = P_1V \bmod O\,(h^2),\qquad VP_3 = P_2V \bmod O\,(h^2).$$

Thus, there exists the following formula of commutation of a canonical operator with a Hamiltonian

$$HK = -ihKP.$$

Therefore the equation $P\varphi = 0$ is to be solved in a neighborhood of the region $\alpha^2 \leqslant 2a$, i.e., it is necessary to find such a canonical element φ that the element $\psi = P\varphi$ will be defined by the functions $\psi_1(\alpha)$, $\psi_2(\alpha)$ and $\psi_3(\alpha)$, where $\psi_i(\alpha) = 0$ for $|\alpha| = \sqrt{2a} + \varepsilon$, ε being an arbitrary positive number. The element φ is defined by the three functions: $\varphi_1(\alpha)$, $\varphi_2(\alpha)$ and $\varphi_3(\alpha)$. The equation $P\varphi = 0$ for $|\alpha| < \sqrt{2a} + \varepsilon$ will be satisfied if the functions φ_i satisfy the following conditions:

(a) $P_1\varphi_1(\alpha) = 0$ for $2 < \alpha < \sqrt{2a} + \varepsilon$,

(b) $P_1\varphi_1(\alpha) + VP_3\varphi_3(\alpha) = 0$ for $1 < \alpha < 2$,

(c) $P_3\varphi_3(\alpha) = 0$ for $|\alpha| < 1$,

(d) $P_2\varphi_2(\alpha) + VP_3\varphi_3(\alpha) = 0$ for $-2 < \alpha < -1$,

(e) $P_2\varphi_2(\alpha) = 0$ for $-\sqrt{2a} - \varepsilon < \alpha < -2$.

Since P_1, P_2 and P_3 define the operator P conditions (b) and (d) may be rewritten in the form

(b′) $P_1(\varphi_1(\alpha) + V\varphi_3(\alpha)) = 0$ for $1 < \alpha < 2$,

(d′) $P_2(\varphi_2(\alpha) + V\varphi_3(\alpha)) = 0$ for $-2 < \alpha < -1$.

Start with equation (c)

$$\varphi_3'(\alpha) = 0.$$

The solution of this equation (correct within an arbitrary constant factor) is as follows

$$\varphi_3(\alpha) = 1 \quad \text{for } |\alpha| < 1.$$

Now denote

$$\varphi_1(\alpha) + V\varphi_3(\alpha) = \chi(\alpha), \qquad 1 < \alpha < 2.$$

The equation (b′) becomes

$$\frac{d\chi}{d\alpha} - ih\left\{\frac{5}{8\alpha^4} - \frac{1}{\alpha^3}\frac{d}{d\alpha} + \frac{1}{2\alpha^2}\frac{d^2}{d\alpha^2}\right\}\chi(\alpha) = 0,$$

$$\chi(1) = V\varphi_3(1) = 1 - \frac{5}{24}ih.$$

Solve this equation by means of the perturbation theory

$$\chi = \chi^{(0)} + h\chi^{(1)},$$

$$\frac{d\chi^{(0)}}{d\alpha} = 0, \quad \chi^{0}(\alpha) = \chi^{0}(1) = 1,$$

$$\frac{d\chi^{(1)}}{d\alpha} = +i\left\{\frac{5}{8\alpha^4} - \frac{1}{\alpha^3}\frac{d}{d\alpha} + \frac{1}{2\alpha^2}\frac{d^2}{d\alpha^2}\right\}\chi^{(0)}(\alpha),$$

$$\frac{d\chi^{(1)}}{d\alpha} = \frac{5i}{8\alpha^4}, \quad \chi^{(1)}(1) = -\frac{5i}{24}.$$

Hence we obtain

$$\chi^{(1)}(\alpha) = -\frac{5i}{24\alpha^3}, \quad \chi(\alpha) = 1 - \frac{5ih}{24\alpha^3}.$$

Let $e(\alpha) = 1$ in a neighborhood of 1 and $e(\alpha) = 0$ in a neighborhood of 2. We can set

$$V\varphi_3(\alpha) = e(\alpha)\,\chi(\alpha), \ \varphi_1(\alpha) = (1 - e(\alpha))\,\chi(\alpha),$$
$$1 < \alpha < 2.$$

Hence we find

$$\varphi_3(\alpha) = V^{-1}e(\alpha)\left(1 - \frac{5ih}{24\alpha^3}\right) =$$
$$= \left[1 + ih\left\{\frac{1}{2\alpha}\frac{d^2}{d\alpha^2} - \frac{1}{2\alpha^2}\frac{d}{d\alpha} + \frac{5}{24\alpha^3}\right\}\right]e(\alpha)\left(1 - \frac{5ih}{24\alpha^3}\right) =$$
$$= e(\alpha) + ih\left[\frac{1}{2\alpha}e''(\alpha) - \frac{1}{2\alpha^2}e'(\alpha)\right], \quad 1 < \alpha < 2.$$

At $\alpha = 2$ we obtain the following initial condition for $\varphi_1(\alpha)$:

$$\varphi_1(2) = \chi(2) = 1 - \frac{5ih}{192}.$$

Next we solve a Cauchy problem:

$$\frac{d\varphi_1^{(0)}}{d\alpha} = 0, \quad \varphi_1^{(0)}(2) = 1,$$

$$\frac{d\varphi_1^{(1)}}{d\alpha} = i\left\{\frac{5}{8\alpha^4} - \frac{1}{\alpha^3}\frac{d}{d\alpha} + \frac{1}{2\alpha^2}\frac{d^2}{d\alpha^2}\right\}\varphi_1^{(0)}(\alpha),$$

$$\varphi_1^{(1)}(2) = -\frac{5i}{192}, \quad 2 < \alpha < \sqrt{2a} + \varepsilon.$$

We obtain

$$\varphi_1^{(0)}(\alpha)=1, \quad \varphi_1^{(1)}(\alpha)=-\frac{5i}{24\alpha^3},$$

$$\varphi_1(\alpha)=1-\frac{5ih}{24\alpha^3} \text{ for } 2<\alpha<\sqrt{2a}+\varepsilon.$$

In the same way we find

$$\varphi_2(\alpha)+V\varphi_3(\alpha)=1-\frac{5ih}{24\alpha^3},$$

$$\varphi_3(\alpha)=\bar{e}(\alpha)+ih\left[\frac{1}{2\alpha}\bar{e}''(\alpha)-\frac{1}{2\alpha^2}\bar{e}'(\alpha)\right]$$

$$\text{for } -2<\alpha<1,$$

where $\bar{e}(\alpha)=1$ in a neighborhood of 1 and $\bar{e}(\alpha)=0$ in a neighborhood of -2,

$$\varphi_2(\alpha)=(1-\bar{e}(\alpha))\left(1-\frac{5ih}{24\alpha^3}\right) \text{ for } -2<\alpha<1$$

and

$$\varphi_2(\alpha)=1-\frac{5ih}{24\alpha^3} \text{ for } -\sqrt{2a}-\varepsilon<\alpha<-2.$$

Now it is easy to write the solution and find out how it is related to the exact solution which can be obtained for the case in question.

Now let M^1 be an arbitrary smooth curve in the phase plane (x, p) and K be a canonical operator mod $O(h^{l+1})$ on the curve $M^1=\{x(\alpha), p(\alpha)\}$. The operator acts on canonical elements. Let $\mathcal{A}=\{U_j\}_{j\in I}$ be a canonical atlas on the curve M^1. The canonical element φ is given by the family of functions $\{\varphi_j(\alpha)\}_{j\in I}$, where supp $\varphi_j \in U_j$. The functions φ_j are defined by mod $O(h^{l+1})$, $\varphi_j \in C_0^\infty(U_j)$.

Let $\mathcal{A}'=\{U_j'\}_{j\in I'}$ be anotner canonical atlas. The family of functions $\psi_j(\alpha)\in C_0^\infty(U_j')$, $j\in I'$ defines the same element φ as the family $\{\varphi_j(\alpha)\}_{j\in I}$ if the following conditions are met:

(a) in a sufficiently small neighborhood of any non-local point on M^1 the following equality is valid:

$$\sum_{j\in I_1}\varphi_j(\alpha)+\sum_{j\in I_2}V\varphi_j(\alpha)=\sum_{j\in I_1'}\psi_j(\alpha)+\sum_{j\in I_2'}V\psi_j(\alpha),$$

where

I_1 is the set of numbers of non-singular patches of the atlas $\mathcal{A}$,
I_1' is the set of numbers of non-singular patches of the atlas $\mathcal{A}'$,
I_2 is the set of numbers of singular patches of the atlas $\mathcal{A}$,
I_2' is the set of numbers of singular patches of the atlas $\mathcal{A}'$,
V is a transition operator from singular patches to non-singular ones (it is defined in a neighborhood of any non-focal point and

has the form

$$V = 1 + \sum_{k=1}^{l} (ih)^k L_k \mod O(h^{l+1}),$$

where L_k is a differential operator of order $2k$);

(b) in a sufficiently small neighborhood of any focal point on M^1 the following equality is valid:

$$\sum_{j \in I_2} \varphi_j(\alpha) = \sum_{j \in I_2'} \psi_j(\alpha).$$

The canonical operator K acts according to the formula

$$K\varphi = \sum_{j \in I} i^{\gamma_j} K_j \varphi_j(\alpha),$$

where

$$K_j\varphi_j(\alpha) = \left. \frac{e^{\frac{i}{h} S(\alpha)}}{\sqrt{\left|\frac{dx}{d\alpha}\right|}} \varphi_j(\alpha) \right|_{\alpha = \alpha_j(x)}$$

for $j \in I_1$ and

$$K_j\varphi_j(\alpha) = \frac{1}{\sqrt{-2\pi i h}} \int_{-\infty}^{\infty} \left. \frac{e^{\frac{i}{h}(px - \widetilde{S}(\alpha))} \varphi_j(\alpha)}{\sqrt{\left|\frac{dp}{d\alpha}\right|}} \right|_{\alpha = \alpha_j(p)} dp$$

for $j \in I_2$.

Let H be a Hamiltonian or a pseudodifferential operator of the form

$$H = \mathscr{H}\left(\overset{2}{x}, \overset{1}{\hat{p}}, h\right),$$

where $\hat{p} = -ih \frac{d}{dx}$ (the operator h commutes with x and $\hat{p}$, hence it is not important in which order it acts). Let the curve M^1 be associated with the Hamiltonian H, i.e.,

$$\mathscr{H}(x(\alpha), p(\alpha), 0) = 0,$$

and the parameter α be chosen in such a way that the relations are carried out on M^1

$$d\alpha = \frac{dx}{H_p(x, p, 0)} = -\frac{dp}{H_x(x, p, 0)}.$$

Then for any $j \in I$ we have

$$HK_j\varphi_j(\alpha) = -ihK_jP_j\varphi_j(\alpha),$$

where P_j is of the form

$$P_j = \sum_{k=0}^{l} (ih)^k P_j^{(k)} \bmod O(h^{l+1}).$$

Here $P_j^{(k)}$ are differential operators and

$$P_j^{(0)} = \frac{d}{dx} + f(\alpha),$$

where $f(\alpha)$ does not depend on j. Hence in a neighborhood of any point on M^1 the operator P_j depends only on whether U_j is non-singular or singular.

The family of operators $\{P_j\}$ defines some operator P acting on the space of canonical elements. In other words, if U_{j_1} is non-singular and U_{j_2} is singular, then for any function $\chi \in C_0^\infty(U_{j_1} \cap U_{j_2})$ the following equality is valid:

$$P_{j_1} V\chi = VP_{j_2}\chi.$$

In fact we have

$$HK\varphi = -ihK\psi,$$

where the canonical element ψ is defined by the family of functions $\psi_j = P_j\varphi_j$. It remains to be shown that the canonical element ψ is defined uniquely by the canonical element φ or, what is the same, that the canonical operator is monomorphic:

$$(K\psi = 0) \Rightarrow (\psi = 0).$$

Let the canonical element ψ be non-zero; prove that $K\psi \neq 0$. For example, let $\psi(\bar{\alpha}) \neq 0$. Denote $\bar{x} = x(\bar{\alpha})$ and consider the function $K\psi$ in a sufficiently small neighborhood of the point $\bar{x}$. Choose on M^1 such a canonical atlas $\mathcal{A}$ that any two patches of the atlas whose projections onto the axis x contain the point $\bar{x}$ have non-intersecting projections onto the axis p. There exists a canonical element Ψ defined by the family of functions $\{\psi_j\}$ which correspond to the atlas $\mathcal{A}$ such that the following conditions are fulfilled:

(a) $\psi_j = 0$ if the projection of the patch U_j onto the axis x does not contain the point $\bar{x}$,

(b) $K\Psi = K\psi$ in a neighborhood of the point $\bar{x}$. It suffices to verify that $K\Psi \neq 0$. Denote $K\Psi(x) = \Phi(x)$. In a sufficiently small neighborhood of a point $\bar{x}$ we have

$$\tilde{\Phi}(p) = i^{\gamma_j} e^{-\frac{i}{h}\tilde{S}(\alpha)} \left. \frac{\psi_j(\alpha)}{\sqrt{\frac{dp}{d\alpha}}} \right|_{\alpha=\alpha_j(p)} \bmod O(h^\infty),$$

where U_j is a patch containing the point $\bar{\alpha}$. Thus $\tilde{\Phi}(p) \neq 0$, Q.E.D

Consider the problem: find a function $y(x)$ such that the equation $Hy = 0$ is satisfied for $|x| < a$ with an error $O(h^{l+2})$. We shall seek y in the form

$$y = K\varphi.$$

For a canonical element φ we obtain the condition

$$\operatorname{supp} P\varphi \cap M_a = \varnothing,$$

where M_a is an image of the segment $|x| < a$ by projection of M^1 onto the axis x. The canonical element φ satisfying this condition can be constructed by means of the perturbation theory in the same way as in the above example.

Sec. 11. Commutation of a Hamiltonian with a Canonical Operator

Let $\{\rho_1(\alpha), \rho_2(\alpha)\}$ be a weighted partition of unity on M^1, i.e. $\rho_1, \rho_2 \in C^\infty$, let $\operatorname{supp} \rho_1$ not contain points which do not belong to any non-singular patch of the complete canonical atlas $\mathcal{A}_\infty$ of the curve M^1, and let $\operatorname{supp} \rho_2$ be contained in the union of singular patches of the atlas $\mathcal{A}_\infty$; $\rho_1 + \rho_2 = 1$. The weighted partition of unity $\{\rho_1, \rho_2\}$ corresponds to the operator $\mathfrak{X}$ translating the function φ (defined by mod $O(h^{l+1})$) on M^1 into a canonical element on M^1 which is described in the canonical atlas $\mathcal{A} = \{U_j\}_{j \in I}$ by the formulas

$$\varphi_j(\alpha) = e_j(\alpha)\rho_1(\alpha)\varphi(\alpha) + Ve_j(\alpha)\rho_2(\alpha)\varphi(\alpha),$$

if U_j is a non-singular patch and

$$\varphi_j(\alpha) = V^{-1}e_j(\alpha)\rho_1(\alpha)\varphi(\alpha) + e_j(\alpha)\rho_2(\alpha)\varphi(\alpha),$$

if U_j is a singular patch. Here $\{e_j\}$ is the partition of unity corresponding to the atlas $\mathcal{A}$. Denote

$$S = K\mathfrak{X}.$$

Lemma 11.1. *The operator S is invertible.*

The proof is analogous to that of the invertibility of the canonical operator K.

Lemma 11.2. *Let the curve M^1 be associated with the Hamiltonian H. Then for any function φ of the domain of the operator S the function $HS\varphi$ is of the form $-ihS\psi$, where ψ is a function on M^1.*

Proof. We have $HS\varphi = HK\mathfrak{X}\varphi = -ihKP\mathfrak{X}\varphi$. Let in the atlas $\mathcal{A} = \{U_j\}_{j \in I}$ a canonical element $P\mathfrak{X}\varphi$ be defined by the family of functions $\{\psi_j(\alpha)\}$. It suffices to show that for any $j \in I$ there exists such a function χ_j on M^1 that the element $\mathfrak{X}\chi_j$ is defined

by the family of functions

$$\overline{\psi}_k = \psi_k \delta_{kj}.$$

Let U_j be a non-singular patch. Then the function χ_j is defined by the equation

$$(\rho_1 + V\rho_0)\,\chi_j = \psi_j. \tag{11.1}$$

The operator V has the form

$$V = 1 + \sum_{k=1}^{l} (ih)^k L_k.$$

For this reason (11.1) can be rewritten in the form

$$\left[1 + \sum_{k=1}^{l} (ih)^k L_k\right] \chi_j = \psi_j.$$

The function χ_j can now be determined by means of the perturbation theory. In the case when U_j is a singular patch the functions χ_j are found in an analogous manner. The lemma is proved.

Theorem 11.1. *Under the assumptions of Lemma* 11.2 *there exists the following commutative relation*:

$$HS = -ihSP_\Phi, \tag{11.2}$$

$$P_\Phi = \frac{d}{d\alpha} + r(\alpha) + \sum_{j=1}^{l} (-ih)^j P_\Phi^{(j)}, \tag{11.3}$$

where $r(\alpha)$ *is a function and* P_Φ^j *are differential operators on* M^1.

Proof. According to Lemmas 11.1 and 11.2 there exists the commutation relation (11.2) with an operator P_Φ acting on functions defined on M^1 correct to modulo $O(h^{l+1})$. It remains to be shown that P_Φ is of the form (11.3). For this it suffices to consider the action of this operator on a function with support in a single patch of the canonical atlas. For example let $\varphi(\alpha)$ be a function with support in a non-singular patch U_j. Then the canonical element $\mathfrak{X}\varphi$ may be determined by the set of functions: $\varphi_k(\alpha) = 0$ for $k \neq j$ and

$$\varphi_j(\alpha) = \rho_1(\alpha)\,\varphi(\alpha) + V\rho_2\varphi(\alpha).$$

By applying the operator P to the element $\mathfrak{X}\varphi$ we obtain the element $P\mathfrak{X}\varphi$ which may be determined by the following family of functions $\{\psi_k(\alpha)\}$:

$\psi_k(\alpha) = 0$ for $k \neq j$ and

$$\psi_j(\alpha) = \frac{d\varphi_j(\alpha)}{d\alpha} + r(\alpha)\,\varphi_j(\alpha) + \sum_{\mu=1}^{l} (-ih)^\mu P_j^{(\mu)}\varphi_j(\alpha),$$

where $r(\alpha)$ is a function, $P_j^{(\mu)}$ are differential operators. On calculating now the function $\psi = P_\Phi \varphi = \mathfrak{X}^{-1} P \mathfrak{X} \varphi$ by means of the perturbation theory in a manner analogous to the proof of Lemma 11.2 we obtain formula (11.3) for the case in question.

The case when $\varphi(\alpha)$ is a function with support in a singular patch can be treated in a similar manner, Q.E.D.

Theorem 11.1 enables us to reduce the equation $Hy = 0$ to the equation $P_\Phi \varphi = 0$. The solution of the latter equation can easily be obtained by means of the perturbation theory.

Sec. 12. The General Canonical Transformation of the Pseudodifferential Operator

Let $\overset{1}{\omega} = ih \frac{\overset{1}{\partial}}{\partial x}$ and $T = [K\varphi]\left(\overset{2}{x}, \overset{1}{\omega}\right)$, where K is a canonical operator on a family of curves depending on the parameter ω and φ is a real function. Consider the transformation of the pseudodifferential operator $L = f\left(\overset{2}{x}, \overset{1}{\omega}, \overset{3}{\omega}\right)$ by the formula

$$L \to T^* L T. \tag{12.1}$$

This transformation contains the canonical transformations of Sec. 1 as particular cases. In this section an asymptotic expansion correct to $O(h^2)$ of the transformation (12.1) will be obtained.

Lemma 12.1. *Let M^1: $\{x = x(\alpha),\ p = p(\alpha)\}$ be a smooth curve in the phase plane (x, p), K be a canonical operator (correct to $O(\omega^{-1})$) on M^1, $f(x)$ be a smooth function. Then for any function $\varphi_1(\alpha)$, $\varphi_2(\alpha) \in C_0^\infty(M^1)$ the following equality is valid:*

$$(f(x)[K\varphi_1](x, \omega),\ [K\varphi_2](x, \omega))_{L_2(x)} =$$
$$= (f(x(\alpha))\varphi_1(\alpha),\ \varphi_2(\alpha))_{L_2(\alpha)} + O\left(\frac{1}{\omega}\right).$$

Proof. Obviously it suffices to consider the case when the support of the function f is sufficiently small. Then on the curve M^1 such a canonical atlas $\mathcal{A}$ can be chosen that if the projections of two patches U_j and U_k, $j \neq k$ of the atlas $\mathcal{A}$ onto the axis x intersect with the support of the function f, then the projections of these patches onto the axis p do not intersect. Let $\{U_j\}_{j \in J_0}$ be a family of patches of the atlas $\mathcal{A}$ whose projections onto the axis x intersect with supp f. Denote by U_j' the set $\pi^{-1}[(\operatorname{supp} f) \cap (\pi U_j)]$, where π is a projection onto the axis x. Then correct to $O(\omega^{-\infty})$ we obtain

$$K\varphi_1 = \sum_{j \in J_0} K\varphi_{1j}, \quad K\varphi_2 = \sum_{j \in J_0} K\varphi_{2j}, \quad \text{on support } f,$$

where $\operatorname{supp}\varphi_{1j} \subset U_j$, $\operatorname{supp}\varphi_{2j} \subset U_j$ and $\varphi_{1j}(\alpha) = \varphi_2(\alpha)$ for $\alpha \in U'_j$ and $\varphi_{2j}(\alpha) = \varphi_2(\alpha)$ where $\alpha \in U'_j$. If $j \neq k$ then $(f(x(\alpha))\,\varphi_{1j}(\alpha),\ \varphi_{2k}(\alpha))_{L_2(\alpha)} = 0$. Besides

$$(f(x)\,[K\varphi_{1j}](x, \omega),\ [K\varphi_{2k}](x, \omega)) = O(\omega^{-1}). \tag{12.2}$$

In fact, denote

$$\psi_{1j}(x) = [K\varphi_{1j}](x, \omega),\ \psi_{2k}(x) = \\ = [K\varphi_{2k}](x, \omega),\ \psi_{1j}(x)\,f(x) = \Psi_{1j}(x)$$

(the functions ψ_{1j}, ψ_{2k} depend on ω as on a parameter). We have

$$\widetilde{\psi}_{1j}(p) = e^{-i\omega\widetilde{S}(\alpha_j(p))}\chi_{1j}(p), \quad \widetilde{\psi}_{2k} = e^{-i\omega\widetilde{S}(\alpha_k(p))}\chi_{2k}(p),$$

where the functions χ_{1j} and χ_{2k} have non-intersecting supports. Denote $\widetilde{S}_j(p) = \widetilde{S}(\alpha_j(p))$. We have

$$\Psi_{1j}(p) = f\left(\frac{i}{\omega}\frac{\partial}{\partial p}\right) e^{-i\omega\widetilde{S}_j(p)}\chi_{1j}(p) = \\ = e^{-i\omega\widetilde{S}_j(p)} f\left(\frac{i}{\omega}\frac{\partial}{\partial p} + \widetilde{S}'_j(p)\right)\chi_{1j}(p) = \\ = e^{-i\omega\widetilde{S}_j(p)} f(\widetilde{S}'_j(p))\,\chi_{1j}(p) + O(\omega^{-1})$$

so that the function $\Psi_{1j}(p)$ is equal to $O(\omega^{-1})$ on the support of the function $\widetilde{\psi}_{2k}$.

By virtue of (12.1) it suffices to prove the lemma in the case when $\varphi_1(\alpha) = \varphi_{1j}(\alpha)$ and $\varphi_2(\alpha) = \varphi_{2j}(\alpha)$. Then, however, the proof of the lemma directly follows from the definition of the canonical operator.

Lemma 12.2. *Let M^1: $\{x = x(\alpha),\ p = p(\alpha)\}$ be a smooth curve on the phase plane (x, p), K_1 be a canonical operator on Λ (to an accuracy of $O(\omega^{-2})$), $f(x)$ be a smooth function and $\mathfrak{X}$ be the operator translating smooth functions on M^1 into canonical elements in the usual way and corresponding to a real weighted partition of unity. Then for any real function $\varphi(\alpha) \in \overset{\circ}{C}{}^{\infty}_{\theta}(M^1)$ the following equality is valid:*

$$(f(x)\,[K_1\mathfrak{X}\varphi](x, \omega),\ f(x)\,[K_1\mathfrak{X}\varphi](x, \omega))_{L_2(x)} = \\ = (|f(x(\alpha))|^2\,\varphi(\alpha),\ \varphi(\alpha))_{L_2(\alpha)} + O(\omega^{-2}). \tag{*}$$

The proof of the lemma is analogous to that of Lemma 12.1. We shall note only a few additional features which distinguish the proof of Lemma 12.2 from that of Lemma 12.1.

(1) Let the function φ have support in a non-singular patch U. Then as is easily verified by using the definition of operators $V_{U,U'}$

we obtain

$$[K_1\mathfrak{X}\varphi](x, \omega) = ce^{i\omega S(\alpha(x))}\left|\frac{dx}{d\alpha}\right|^{-\frac{1}{2}}\left[\left(1+\frac{i}{\omega}L\right)\varphi(\alpha)\right]_{\alpha=\alpha(x)},$$

where L is a differential operator (of second order) with real coefficients and $|c| = 1$. The statement of the lemma is valid in this case because

$$\left|\left(1+\frac{1}{\omega}L\right)\varphi(\alpha)\right|^2 = |\varphi(\alpha)|^2 + O(\omega^{-2}).$$

(2) An analogous formula exists for the ω-Fourier transform of the function $K_1\mathfrak{X}\varphi$, if U is a singular patch.

(3) Let U be a singular patch. Denote $\psi(x) = [K_1\mathfrak{X}\varphi](x, \omega)$,

$$\Psi(x) = f(x)\psi(x), \quad \widetilde{S}_1(p) = \widetilde{S}(\alpha(p)), \quad \frac{dp}{d\alpha} = J_1(p(\alpha)).$$

By expanding the function

$$\widetilde{\Psi}(p) = C_1 e^{-i\omega\widetilde{S}(p)} f\left(\frac{i}{\omega}\frac{\partial}{\partial p} + \widetilde{S}_1'(p)\right)|J_1(p)|^{-\frac{1}{2}} \times$$

$$\times\left[\left(1+\frac{1}{\omega}L_1\right)\varphi(\alpha)\Big|_{\alpha=\alpha(p)}\right] \tag{12.3}$$

in powers of ω we obtain that in equality (12.3) we can replace the operator $f\left(\frac{i}{\omega}\frac{\partial}{\partial p} + \widetilde{S}_1'(p)\right)$ correct to $O(\omega^{-2})$ by the following operator:

$$f(\widetilde{S}_1'(p)) + \frac{i}{\omega}L_2,$$

where L_2 is a differential operator with the real coefficients in the case of a real function f (the latter can be assumed without loss of generality). Consider the set of the operators depending on $\omega \to \infty$ which are of the form

$$A_1 + \frac{i}{\omega}A_2 + O(\omega^{-2}), \tag{12.4}$$

where A_1 and A_2 are continuous operators acting on C_0^∞ and translating real functions into real ones. The set of operators of the type (12.4) form a circle $\varkappa$ relative to the composition of operators

$$\left(A_1 + \frac{i}{\omega}A_2 + O(\omega^{-2})\right)\left(A_3 + \frac{i}{\omega}A_4 + O(\omega^{-2})\right) =$$

$$= A_1A_3 + \frac{i}{\omega}A_5 + O(\omega^{-2}).$$

Consequently,

$$\widetilde{\Psi}(p(\alpha)) = c_1 e^{-i\omega\widetilde{S}(\alpha)}\left|\frac{dp}{d\alpha}\right|^{-\frac{1}{2}}\left(f(x(\alpha)) + \frac{i}{\omega}L_2\right)\varphi(\alpha)$$

which leads to the proof of the lemma in the case in question.

Note. Lemma 12.2 remains valid if in the left-hand side of (*) $f(x)$ is replaced by the operator $f\left(\overset{2}{x}, -\frac{i}{\omega}\overset{1}{\partial/\partial x}\right)$, and in the right-hand side of (*) $f(x(\alpha))$ is replaced by $f(x(\alpha), p(\alpha))$. The proof is completely analogous.

Theorem 12.1. *Let $\Gamma_\omega = \{(x, p)\colon x = X(\alpha, \omega),\ p = P(\alpha, \omega)\}$ be a one-parameter family of curves in the phase space $\mathbf{R}_x^n \times \mathbf{R}_p^n$, where $D(X, P+\omega)/D(\alpha, p) = 1$.*

Consider a canonical operator $[K\varphi](x, \omega)$ on the family Γ_ω applied to a real function $\varphi(\alpha, \omega) \in C_0^\infty$.

Let $T = [K\varphi]\left(\overset{2}{x}, \overset{1}{\omega}\right)$, where $\overset{1}{\omega} = -ih\overset{1}{\frac{\partial}{\partial x}}$. Then in the space $L_2(\mathbf{R})$

$$T^*T = \varphi\left(\overset{2}{x}, \overset{3}{\omega}\right)\varphi\left(\overset{2}{x}, \overset{1}{\omega}\right) + h^2 R_2(h),$$

where the operator R_2 is bounded in L_2 uniformly in h: $\| R_2(h) \| \leqslant C$.

Proof. From the definition of the canonical operator it follows that $[K\varphi](x, \omega)$ has compact support Q. Let Q_ω be a projection of Q onto the axis ω and

$$Q_1 = \{(x, \omega', \omega)\colon (x, \omega) \in Q \text{ and } (x, \omega') \in Q\}.$$

Consider in a neighborhood Q_1 a finite partition of unity $\{f_\mu(x, \omega', \omega)\} \subset C_0^\infty$ in which the supports f_μ are sufficiently small. Then

$$T^*T = \sum_\mu f_\mu\left(\overset{2}{x}, \overset{1}{\omega}, \overset{3}{\omega}\right)[K\varphi]\left(\overset{2}{x}, \overset{1}{\omega}\right)\overline{[K\varphi]}\left(\overset{2}{x}, \overset{3}{\omega}\right),$$

where the line means complex conjugation. Prove that

$$f_\mu\left(\overset{2}{x}, \overset{1}{\omega}, \overset{3}{\omega}\right)[K\varphi]\left(\overset{2}{x}, \overset{1}{\omega}\right)\overline{[K\varphi]}\left(\overset{2}{x}, \overset{3}{\omega}\right) = \\ = \varphi\left(\overset{2}{x}, \overset{3}{\omega}\right)\varphi\left(\overset{2}{x}, \overset{1}{\omega}\right) f_\mu\left(\frac{1}{2}\left(X\left(\overset{2}{x}, \overset{3}{\omega}\right) + \right.\right. \\ \left.\left. + X\left(\overset{2}{x}, \overset{1}{\omega}\right)\right), \overset{1}{\omega}, \overset{3}{\omega}\right) + O(h^2). \tag{12.5}$$

Summing up these operators with respect to μ we obtain the proof of the theorem.

Let π $(\tilde{\pi})$ be a projection of the curve Γ_ω onto the axis x (or p). Take into account that the support f_μ is arbitrarily small. Then if the projections π^{-1} $(\operatorname*{supp}_x f_\mu(x, \omega, \omega'))$ for $\omega, \omega' \in Q_\omega$ intersect with the patches $U_1, \ldots, U_l$ of the canonical atlas Γ_ω then we

may consider that

$$e_i\,(\tilde{\pi}^{-1}\,(p+\omega-\omega',\,\omega),\,\omega)\;e_j\,(\tilde{\pi}^{-1}\,(p,\,\omega''),\,\omega'')=0$$

for $i\neq j$ and $\omega,\,\omega',\,\omega''\in Q_\omega$, where $\{e_i\,(\alpha,\,\omega)\}$ is the partition of unity corresponding to the canonical atlas. (If this condition is not fulfilled, then, for a given f_μ, we may choose another canonical atlas and use the invariance property of the canonical operator when changing the atlas.)

On the support f_μ the canonical operator is of the form

$$[K\varphi]\,(x,\;\omega)=\sum_{j=1}^{l}\,[K\,(\varphi e_j)]\,(x,\;\omega)$$

and

$$f_\mu\begin{pmatrix}2&1&3\\x,&\omega,&\omega\end{pmatrix}[K\varphi]\begin{pmatrix}2&1\\x,&\omega\end{pmatrix}\overline{[K\varphi]}\begin{pmatrix}2&3\\x,&\omega\end{pmatrix}=$$

$$=\sum_{i,\,j=1}^{l}\,[\overline{K\,(\varphi e_i)}]\begin{pmatrix}2&3\\x,&\omega\end{pmatrix}[K\,(\varphi e_j)]\begin{pmatrix}2&1\\x,&\omega\end{pmatrix}f_\mu\begin{pmatrix}2&1&3\\x,&\omega,&\omega\end{pmatrix}. \qquad (12.6)$$

Prove that

$$[\overline{K\,(\varphi e_i)}]\begin{pmatrix}2&3\\x,&\omega\end{pmatrix}[K\,(\varphi e_j)]\begin{pmatrix}2&1\\x,&\omega\end{pmatrix}f_\mu\begin{pmatrix}2&1&3\\x,&\omega,&\omega\end{pmatrix}=h^2R_2\,(h), \qquad (12.7)$$

$$\text{for } i\neq j,$$

where $\|\,R_2\,(h)\,\|\leqslant c$. Indeed we have

$$F_{x\to p}\,[K\,(\varphi e_i)]\,(x,\;\omega)=e^{-\frac{i}{h}\tilde{S}_i(p,\,\omega)}\psi_i\,(p,\;\omega), \qquad (12.8)$$

$$F_{x\to p}\,[K\,(\varphi e_j)]\,(x,\,\omega')=e^{-\frac{i}{h}\tilde{S}_j(p,\,\omega')}\,\psi_j\,(p,\,\omega'),$$

where

$$\psi_i\,(p,\,\omega)\;\psi_j\,(p+\omega-\omega',\,\omega')=0\quad\text{for }\omega,\,\omega'\in Q_\omega. \qquad (12.9)$$

Since the supports ψ_i, ψ_j do not intersect we can assume that in (12.8)

$$\tilde{S}_i\,(p,\,\omega)=\tilde{S}\,(p,\,\omega),\qquad \tilde{S}_j\,(p,\,\omega)=\tilde{S}\,(p,\,\omega),$$

where $\tilde{S}\,(p,\,\omega)$ is a smooth real function. Then

$$[\overline{K\,(\varphi e_i)}]\begin{pmatrix}2&3\\x,&\omega\end{pmatrix}[K\,(\varphi e_j)]\begin{pmatrix}2&1\\x,&\omega\end{pmatrix}f_\mu\begin{pmatrix}2&1&3\\x,&\omega,&\omega\end{pmatrix}=$$

$$=[\overline{K\,(\varphi e_i)}]\begin{pmatrix}2&3\\x,&\omega\end{pmatrix}\int e^{\frac{i}{h}p\overset{2}{x}}f_\mu\left(ih\frac{\partial}{\partial p},\;\overset{1}{\omega},\;\overset{3}{\omega}\right)\times$$

$$\times\,e^{-\frac{i}{h}\tilde{S}\left(p,\,\overset{1}{\omega}\right)}\,\psi_i\left(p,\;\overset{1}{\omega}\right)dp=T^*_{\psi_i}T_{\chi_j}, \qquad (12.10)$$

where

$$T_g \overset{\text{def}}{=} \int e^{\frac{i}{h}\left[\overset{2}{p x}-\widetilde{S}\left(p, \overset{1}{\omega}\right)\right]} g\left(p, \overset{1}{\omega}, \overset{3}{\omega}\right) dp,$$

$$\chi_j(p, \omega', \omega) = f_\mu\left(ih\frac{\partial}{\partial p} + \frac{\partial \widetilde{S}}{\partial p}(p, \omega'), \omega', \omega\right)\psi_j(p, \omega').$$

Expanding the operator in the right-hand side of the equality in powers of h we obtain

$$\chi_j = f_\mu\left(\frac{\partial \widetilde{S}}{\partial p}(p, \omega'), \omega', \omega\right)\psi_j(p, \omega') +$$
$$+ \sum_{n=1}^{m-1}(-ih)^k V^{(k)}\psi_j(p, \omega') + (-ih)^m Q_j^{(m)}(p, \omega', \omega),$$

where $V^{(1)}, \ldots, V^{(m-1)}$ are known differential operators, and $Q_j^{(m)} \in W_2^r(\mathbf{R}^3)$ for all r.

By virtue of (12.9)

$$\overline{\psi_j(p, \omega)}\frac{\partial^k}{\partial p^k}\psi_j(p+\omega-\omega', \omega') = 0 \quad (i \neq j).$$

Consequently, according to the results of Sec. 1 we have

$$T^*_{\psi_i} T_{\chi_j} = (-ih)^m T^*_{\psi_i} T_{Q_j^{(m)}} (i \neq j).$$

Therefore, by virtue of (12.10) to obtain the proof of (12.7) we must estimate in $\mathscr{B}_0(\mathbf{R}^3)$ the norm of the symbol of the operator $T^*_{\psi_i} T_{Q_j^{(m)}}$, i.e., the norm of the function

$$\eta(x, \omega', \omega) = (-ih)^m \int\int dp\, dp' \left\{ e^{\frac{i}{h}(p'-p)x} \times \right.$$
$$\left. \times e^{\frac{i}{h}[\widetilde{S}(p, \omega)-\widetilde{S}(p', \omega')]}\overline{\psi_i(p, \omega)}\, Q_j^{(m)}(p', \omega', \omega)\right\}.$$

We have

$$\|\eta\|_{\mathscr{B}_0(\mathbf{R}_x\times\mathbf{R}_{\omega'}\times\mathbf{R}_\omega)} \leqslant c_1 h^m \int\int dp\, dp' \times$$
$$\times \left\| e^{\frac{i}{h}[\widetilde{S}(p, \omega)-\widetilde{S}(p', \omega')]}\overline{\psi_i(p, \omega)}\, Q_j^{(m)}(p', \omega', \omega)\right\|_{\mathscr{B}_0(\mathbf{R}_{\omega'}\times\mathbf{R}_\omega)} \leqslant$$
$$\leqslant c_2 h^m \int\int dp\, dp' \left\| e^{\frac{i}{h}[\widetilde{S}(p, \omega)-\widetilde{S}(p', \omega')]} \times \right.$$
$$\left. \times \overline{\psi_i(p, \omega)}\, Q_j^{(m)}(p', \omega', \omega)\right\|_{W_2^1(\mathbf{R}_{\omega'}\times\mathbf{R}_\omega)} \leqslant c_3 h^{m-2}.$$

This completes the proof of (12.7). Then from (12.6) it follows

$$f_\mu\begin{pmatrix}2 & 1 & 3\\ x, & \omega, & \omega\end{pmatrix}[K\varphi]\begin{pmatrix}2 & 1\\ x, & \omega\end{pmatrix}[\overline{K\varphi}]\begin{pmatrix}2 & 3\\ x, & \omega\end{pmatrix} =$$

$$= \sum_{j=1}^{l}[K(\varphi e_j)]\begin{pmatrix}2 & 3\\ x, & \omega\end{pmatrix}[K(\varphi e_j)]\begin{pmatrix}2 & 1\\ x, & \omega\end{pmatrix} f_\mu\begin{pmatrix}2 & 1 & 3\\ x, & \omega, & \omega\end{pmatrix} +$$

$$+ O(h^2).$$

According to the theorems of Sec. 1 this operator is transformed in the following manner:

$$\sum_{j=1}^{l}(\varphi e_j)\begin{pmatrix}2 & 3\\ x, & \omega\end{pmatrix}(\varphi e_j)\begin{pmatrix}2 & 1\\ x, & \omega\end{pmatrix}\times$$

$$\times f_\mu\left(\frac{1}{2}X\begin{pmatrix}2 & 1\\ x, & \omega\end{pmatrix}+\frac{1}{2}X\begin{pmatrix}2 & 3\\ x, & \omega\end{pmatrix}, \overset{1}{\omega}, \overset{3}{\omega}\right)+O(h^2). \quad (12.11)$$

Next we have

$$(\varphi e_i)(x, \omega)(\varphi e_j)(x, \omega') = 0 \text{ for } i \neq j \text{ and } \omega, \omega' \in Q_\omega,$$

because we have assumed, in particular, that

$$\operatorname*{supp}_x e_i(x, \omega) \cap \operatorname*{supp}_x e_j(x, \omega') = \varnothing$$

$$\text{for } i \neq j;\ \omega, \omega' \in Q_\omega.$$

The expression (12.11) then coincides with the right-hand member of (12.5) which completes the proof of (12.5), Q.E.D.

Corollary. *Let* $f \in C^\infty(\mathbf{R}_x \times \mathbf{R}_{\omega'} \times \mathbf{R}_\omega)$ *and*

$$\varphi = \varphi_0(x, \omega) + (-ih)\varphi_1(x, \omega) + h^2\varphi_2(x, \omega, h),$$

where φ_0, φ_1 *are real functions,* $\varphi_0, \varphi_1, \varphi_2 \in C_0^\infty$, $\sup\limits_{0\leqslant h\leqslant 1}\|\varphi_2\|_{W_2^r(\mathbf{R}_x\times\mathbf{R}_\omega)} \leqslant$ $\leqslant c_r$ *for sufficiently large* r.

Then if the conditions of the theorem are fulfilled the following equality is valid:

$$[\![[\overline{K\varphi}]\begin{pmatrix}2 & 1\\ x, & \omega\end{pmatrix}]\!] f\begin{pmatrix}2 & 1 & 3\\ x, & \omega, & \omega\end{pmatrix}[\![[K\varphi]\begin{pmatrix}2 & 1\\ x, & \omega\end{pmatrix}]\!] =$$

$$= \varphi_0\begin{pmatrix}2 & 3\\ x, & \omega\end{pmatrix}\varphi_0\begin{pmatrix}2 & 1\\ x, & \omega\end{pmatrix} f\left(\frac{1}{2}\left(X\begin{pmatrix}2 & 1\\ x, & \omega\end{pmatrix}+\right.\right.$$

$$\left.\left.+X\begin{pmatrix}2 & 3\\ x, & \omega\end{pmatrix}\right), \overset{1}{\omega}, \overset{3}{\omega}\right)+h^2\mathbf{R}_2(h),$$

where $\|R_2(h)\| \leqslant c$.

IV. GENERALIZED HAMILTON-JACOBI EQUATIONS

Notation. Let $g(x, y)$, $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, be a C^∞ function. Then, by g_{xy} we denote the $n \times m$ matrix $\left(\frac{\partial^2 g}{\partial x_i \partial y_i}\right)$.

Let $y(x)$ be a C^∞ function with values in $\mathbf{R}^m$, defined on an open subset of $\mathbf{R}^n$. Then we shall use the notation $\partial y/\partial x = y_x$ for the Jacobi matrix: $\left(\frac{\partial y}{\partial x}\right)_{ij} = \frac{\partial y_i}{\partial x_j}$. If $m = n$, then Dy/Dx denotes the Jacobian $\det(\partial y/\partial x)$.

Let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ be two ordered n-tuples and let I and K be two disjoint subsets of $\{1, 2, \ldots, n\}$. Then $\{u_I, v_K\}$ will denote the following subsequence $\{w_i\}_{i \in I \cup K}$ of the finite sequence $(u_1, v_1, u_2, v_2, \ldots, u_n, v_n)$:

$$w_i = \begin{cases} u_i & \text{for } i \in I \\ v_i & \text{for } i \in K. \end{cases}$$

For instance, if $I = \{3, 1\}$ and $K = \{2\}$, then $\{u_I, v_K\} = (u_1, v_2, u_3)$. We shall mostly deal with $K = \bar{I}$, where $\bar{I} = \{1, \ldots, n\} \setminus I$, or $K = \varnothing$. In the latter case we shall write $\{u_I, v_K\} = u_I$.

One should bear in mind (see Introduction) that

$$\langle u, v \rangle \overset{\text{def}}{=} \sum_{i=1}^{n} u_i v_i.$$

Let α be a multi-index, say $\alpha = (\alpha_1, \ldots, \alpha_k)$. Then

$$\alpha! \overset{\text{def}}{=} \alpha_1! \ldots \alpha_k!.$$

For two multi-indices $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_k)$, we write $\alpha \leqslant \beta$ if $\alpha_i \leqslant \beta_i$, $i = 1, \ldots, k$. If $x = (x_1, \ldots, x_k)$ and α is a multi-index, then we set

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k}.$$

Now let $\xi(x)$ and $\varphi(x)$, $x \in \mathbf{R}^n$, be two smooth complex functions, the matrix $C = \partial\xi/\partial x$ being non-degenerate. Then we define the *formal derivative* $\partial\varphi/\partial\xi$ as follows:

$$\frac{\partial\varphi}{\partial\xi} = {}^tC^{-1}\frac{\partial\varphi}{\partial x}.$$

Second-order formal derivatives are given by

$$\frac{\partial^2\varphi}{\partial\xi^2} = {}^tC^{-1}\frac{\partial^2\varphi}{\partial x^2}C^{-1} + \sum_{m=1}^{n}\left({}^tC^{-1}\frac{\partial\varphi}{\partial x}\right)_m {}^tC^{-1}\frac{\partial^2\xi_m}{\partial x^2}C^{-1}.$$

In particular,

$$\frac{\partial^2\varphi}{\partial\xi_i\,\partial\xi_j} = \frac{\partial^2\varphi}{\partial\xi_j\,\partial\xi_i}$$

for every smooth function φ.

Sec. 1. Hamilton-Jacobi Equations with Dissipation

The examples of the wave equation, the crystal oscillation equation and the Schrödinger equation for an oscillator (see Sec. 8 of Introduction) show that the problem of finding asymptotic solutions of these equations leads in "zero approximation" to a first-order partial differential equation called the characteristic equation or the Hamilton-Jacobi equation. The latter equation does not involve the parameter in which the asymptotics for the initial problem is to be found. Its solution $S(x, t)$ is real. In case of the Schrödinger equation, for example, this function determines the phase of oscillations and the solution can be expanded, after factoring out $\exp\left\{\frac{i}{h}S(x, t)\right\}$, into an asymptotic power series in the small parameter h. For a number of examples (involving the "absorption" phenomenon), however, the analogous function is complex, so that $\exp\left[\frac{i}{h}S(x, t)\right]$ tends rapidly to zero in the domain

$$S_2 \stackrel{\text{def}}{=} \operatorname{Im} S(x, t) > 0$$

when $h \to 0$. If the initial problem is to be solved modulo $O(h^N)$, then it is clear that the value of $S_1 \stackrel{\text{def}}{=} \operatorname{Re} S(x, t)$ in the domain $S_2(x, t) > 0$ is inessential. Therefore, the asymptotic solutions of this kind lead to the more complicated statement of the problem for a first-order equation than those we dealt with in the above examples.

Now state the problem.

Definition 1.1. *Let $f(x)$, $x \in \mathbf{R}^n$, be a non-negative function and $g(x)$ be a smooth function. We shall write $g = O_f(h^\alpha)$ if*

$$\left|\frac{\partial^{|l|} g(x)}{\partial x^l}\right| e^{\frac{-f(x)}{h}} = o\left(h^{\alpha - \frac{|l|}{2}}\right)$$

for $\alpha - \frac{|l|}{2} > 0$, $h \to 0$, O-estimate being locally uniform in x. Similarly we define $o(h^\alpha)$.

Problem 1.1. Let $f \geqslant 0$, $\alpha > 0$. Then the inequalities $|g|\exp\left(-\frac{f}{h}\right) \leqslant ch^\alpha$ and $|g| \leqslant c_1 f^\alpha$, where $c_1 = \alpha^{-\alpha} e^\alpha c$, are equivalent. Now let two real smooth functions $f(p, \tilde{p}, x, t)$ and $\tilde{f}(p, \tilde{p}, x, t)$ satisfy the conditions

$$f_{\tilde{p}} = -\tilde{f}_p, \quad \tilde{f}_{\tilde{p}} = f_p, \quad f_{\tilde{p}\tilde{p}} = -f_{pp}, \qquad \tilde{f}_{\tilde{p}\tilde{p}} = -\tilde{f}_{pp}, \tag{1.1}$$
$$\tilde{f} \leqslant 0 \text{ for } \tilde{p} = 0.$$

The problem is to find such two real smooth functions $S_1(x, t)$ and $S_2(x, t)$ that

$$\begin{gathered}\frac{\partial S_1}{\partial t} + f\left(\frac{\partial S_1}{\partial x}, \frac{\partial S_2}{\partial x}, x, t\right) = o_{S_2}(h), \\ \frac{\partial S_2}{\partial t} + \tilde{f}\left(\frac{\partial S_1}{\partial x}, \frac{\partial S_2}{\partial x}, x, t\right) = o_{S_2}(h), \\ t \in (0, T).\end{gathered} \tag{1.2}$$

We shall call (1.2) a *dissipative Hamilton-Jacobi system.*

Note that if

$$\tilde{f} = \langle \tilde{p}, H_p \rangle, \qquad f = -\langle \tilde{p}, H_{pp}\tilde{p} \rangle + H,$$

where $H = H(p, x, t)$ is a real function, then the system (1.2) with the initial condition $S_2(x, t) = 0$ reduces to the Hamilton-Jacobi equation

$$\frac{\partial S_1}{\partial t} + H\left(\frac{\partial S_1}{\partial x}, x, t\right) = 0.$$

The Hamilton-Jacobi equation is characterized by the Hamilton function $H(p, x, t)$. We shall transform the dissipative Hamilton-Jacobi system to such a form which will be also characterized by a single function $\mathcal{H}(p, x, t)$, this function being complex.

First we prove two lemmas, which will be used frequently in the treatise.

Lemma 1.1. (The Gårding inequality.) *Let $F(x) \in C^2(\mathbf{R}^n)$ be a non-negative function, then*

$$\left|\frac{\partial F}{\partial x}(x)\right|^2 \leqslant cF(x) \cdot \sup_{k,\, x} |F_{x_k x_k}(x)|, \quad c = \text{const}.$$

Proof. We have

$$0 \leqslant F(x_1, \ldots, x_{k-1}, x_k + t, x_{k+1}, \ldots, x_n) \leqslant$$
$$\leqslant F(x_1, \ldots, x_n) + tF_{x_k}(x_1, \ldots, x_n) +$$
$$+ \frac{t^2}{2} \sup_x |F_{x_k x_k}|.$$

Since the non-negativity condition for a quadratic trinomial consists in the non-positivity of its discriminant, we obtain

$$\left| \frac{\partial F}{\partial x_k} \right|^2 \leqslant 2F \sup_x |F_{x_k x_k}|,$$

which proves the lemma.

Corollary 1.1. *Let $F \in C^\infty(\mathbf{R}^n)$ be a non-negative function and let P_N be a homogeneous polynomial of degree N in $\partial F/\partial x$ with coefficients smoothly dependent on x. Then $P_N = O_F(h^{N/2})$.*

Proof. In view of locality of O_F-estimates we may assume without loss of generality that $F \in C_0^\infty(\mathbf{R}^n)$. Then, by Lemma 1.1 we have

$$|F_x| \leqslant \text{const} \cdot F^{1/2}.$$

Hence, for $|l| < N$,

$$\frac{\partial^{|l|}}{\partial x^l} P_N(x) \leqslant \text{const}\, [F(x)]^{\frac{N-|l|}{2}}.$$

According to the result of Problem 1.1 this proves the required assertion.

Lemma 1.2. (A representation of the remainder term in Taylor's formula.) *If $f \in C^\infty(\mathbf{R}^n)$, then*

$$f(x) = f(0) + \langle x, f_x(0)\rangle + \frac{1}{2}\langle x, f_{xx}(0)\, x\rangle + \sum_{i,j,k=1}^{n} x_i x_j x_k g_{ijk}(x),$$

where $g_{ijk} \in C^\infty(\mathbf{R}^n)$.

The proof of this lemma follows from the identity

$$f(x) = f(0) + \int_0^1 \frac{df(tx)}{dt}\, dt = f(0) + \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f(tx)}{\partial x_i}\, dt. \tag{1.3}$$

Note. There is a similar representation of the remainder term in Taylor's formula containing any number of terms.

Now we shall turn directly to transforming the system (1.2).

Set

$$r(p, \tilde{p}, x. t) = f(p, \tilde{p}, x, t) - f(p, 0, x, t) -$$
$$- \langle \tilde{p}, f_p(p, 0, x, t)\rangle - \frac{1}{2}\langle \tilde{p}, f_{\tilde{p}\tilde{p}}(p, 0, x, t)\tilde{p}\rangle,$$
$$\tilde{r}(p, \tilde{p}, x, t) = \tilde{f}(p, \tilde{p}, x, t) - \tilde{f}(p, 0, x, t) -$$
$$- \langle \tilde{p}, \tilde{f}_p(p, 0, x, t)\rangle - \frac{1}{2}\langle \tilde{p}, \tilde{f}_{pp}(p, 0, x, t)\tilde{p}\rangle.$$

Lemma 1.2 and Corollary 1.1 imply that

$$r\left(\frac{\partial S_1}{\partial x}, \frac{\partial S_2}{\partial x}, x, t\right) = O_{S_2}(h^{3/2}),$$
$$\tilde{r}\left(\frac{\partial S_1}{\partial x}, \frac{\partial S_2}{\partial x}, x, t\right) = O_{S_2}(h^{3/2}).$$

Therefore (1.2) is equivalent to the following system:

$$\begin{gathered}\frac{\partial S_1}{\partial t} + f\left(\frac{\partial S_1}{\partial x}, 0, x, t\right) + \left\langle \frac{\partial S_2}{\partial x}, f_{\tilde{p}}\left(\frac{\partial S_1}{\partial x}, 0, x, t\right)\right\rangle + \\ + \frac{1}{2}\left\langle \frac{\partial S_2}{\partial x}, f_{\tilde{p}\tilde{p}}\left(\frac{\partial S_1}{\partial x}, 0, x, t\right)\frac{\partial S_2}{\partial x}\right\rangle = o_{S_2}(h), \\ \frac{\partial S_2}{\partial t} + \tilde{f}\left(\frac{\partial S_1}{\partial x}, 0, x, t\right) + \left\langle \frac{\partial S_2}{\partial x}, \tilde{f}_{\tilde{p}}\left(\frac{\partial S_1}{\partial x}, 0, x, t\right)\right\rangle + \\ + \frac{1}{2}\left\langle \frac{\partial S_2}{\partial x}, \tilde{f}_{\tilde{p}\tilde{p}}\left(\frac{\partial S_1}{\partial x}, 0, x, t\right)\frac{\partial S_2}{\partial x}\right\rangle = o_{S_2}(h).\end{gathered} \tag{1.4}$$

Set

$$H(p, x, t) = f(p, 0, x, t), \; \tilde{H}(p, x, t) = \tilde{f}(p, 0, x, t),$$
$$\mathscr{H}(p, x, t) = H(p, x, t) + i\tilde{H}(p, x, t).$$

Now we multiply the second equation of the system by i and add it to the first one. Making use of (1.1) we arrive at

$$\begin{gathered}F_{\mathscr{H}}[S_1 + iS_2] \stackrel{\text{def}}{=} \frac{\partial(S_1 + iS_2)}{\partial t} + \mathscr{H}\left(\frac{\partial S_1}{\partial x}, x, t\right) + \\ + i\left\langle \mathscr{H}_p\left(\frac{\partial S_1}{\partial x}, x, t\right), \frac{\partial S_2}{\partial x}\right\rangle - \\ - \frac{1}{2}\left\langle \frac{\partial S_2}{\partial x}, \mathscr{H}_{pp}\left(\frac{\partial S_1}{\partial x}, x, t\right)\frac{\partial S_2}{\partial x}\right\rangle = o_{S_2}(h).\end{gathered} \tag{1.5}$$

It is easily seen that each pair of functions satisfying (1.5) satisfies (1.4) as well.

The relation (1.5) will be called the *Hamilton-Jacobi equation with dissipation* associated with the Hamilton function $\mathscr{H}(p, q, t)$.

Sec. 2. The Lagrangean Manifold with a Complex Germ

In this section we shall develop a geometric theory which will be used for solving Hamilton-Jacobi equations with dissipation.

Consider the $2n$-dimensional Euclidean space $\mathbf{R}^n \times \mathbf{R}^n$ with coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n$. This space will be called the *phase space.*

Definition 2.1. *An n-dimensional submanifold* Λ^n *of the phase space is called a Lagrangean manifold if it satisfies the conditions*

$$[p, q]_{jk} \overset{\text{def}}{=} \left\langle \frac{\partial p}{\partial \alpha_j}, \frac{\partial q}{\partial \alpha_k} \right\rangle - \left\langle \frac{\partial p}{\partial \alpha_k}, \frac{\partial q}{\partial \alpha_j} \right\rangle = 0, \quad j, k = 1, \ldots, n, \tag{2.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ *are local coordinates on* Λ^n *and* $q = q(\alpha)$, $p = p(\alpha)$ *are local equations of* Λ^n. *The expressions* $[p, q]_{ij}$ *are called the Lagrange brackets for the vector functions* $p(\alpha)$ *and* $q(\alpha)$.

(2.1) implies that the form pdq on Λ^n is locally the differential of a function. Therefore $\int_l p\, dq$, where l is any path on Λ^n, does not change under continuous deformation of l with fixed initial and final points. If it is additionally assumed that $\oint p\, dq = 0$ for each closed path on Λ^n, then there will be a function s on Λ^n satisfying the equation $ds = p\, dq$. We shall call such a function an s-action on Λ^n.

Let $D(\alpha)$ be a given non-negative C^∞ function on Λ^n, called the *dissipation*. We shall denote by Γ the set of all its zeroes.

Definition 2.2. *A pair* $r^n = (\omega(\alpha), z(\alpha))$ *of smooth n-dimensional complex vector functions on* Λ^n *is called a complex germ, if the following conditions* (the complex germ axioms) *are satisfied*:

g 1) $z(\alpha) = O_D(h^{1/2}), \qquad \omega(\alpha) = O_D(h^{1/2})$

g 2)

$$\operatorname{rank} \begin{pmatrix} \dfrac{\partial (p+w)}{\partial \alpha} \\ \dfrac{\partial (q+z)}{\partial \alpha} \end{pmatrix}_{\alpha \in \Gamma} = n$$

g 3) *there exists a smooth function* $E(\alpha) = O_D(h)$ *on* Λ^n, *called a potential, such that*

$$dE = \langle w, dq \rangle - \langle z, dp \rangle + \frac{\langle w, dz \rangle - \langle z, dw \rangle}{2} + \langle f, d\alpha \rangle,$$

where $f = O_D(h)$.

Note 2.1. We shall consider a Lagrangean manifold with a complex germ (or simply germ) to be given if all the four objects: Λ^n, r^n, D and E are given.

Note 2.2. If there is an s-action on Λ^n, then the existence of a potential on Λ^n is equivalent to that of a function $W(\alpha)$ on Λ^n, satisfying the condition

$$dW = \langle p + w,\ d(q + z)\rangle + O_D(h).$$

This function will be called an *enthalpy of the complex germ*. The potential and the enthalpy of the same complex germ are related by the formula

$$W = E + s + \left\langle p + \frac{1}{2} w,\ z\right\rangle + O_D(h^{3/2}) + \text{const}.$$

Let $H(p, q, t) \in C^\infty(\mathbf{R}^{2n+1})$ be a real function. Denote by $p(\alpha, t)$, $q(\alpha, t)$ the solution of the Hamiltonian system

$$\frac{dp}{dt} = -Hq, \quad \frac{dq}{dt} = Hp, \tag{2.2}$$

satisfying the initial condition $p(\alpha, 0) = p(\alpha)$, $q(\alpha, 0) = q(\alpha)$, where $(p(\alpha), q(\alpha)) \in \Lambda^n$. The equations

$$p = p(\alpha, t), \quad q = q(\alpha, t)$$

determine for a fixed t a Lagrangean manifold which we shall denote by Λ^n_t.

Problem 2.1. Verify that (2.1) is satisfied on Λ^n_t.

Let $p(p_0, q_0, t)$, $q(p_0, q_0, t)$ be the solution of the system (2.2) with the initial condition $p(p_0, q_0, 0) = p_0$, $q(p_0, q_0, 0) = q_0$. The mapping

$$(p_0, q_0) \to (p(p_0, q_0, t),\ q(p_0, q_0, t)) \stackrel{\text{def}}{=\!=} g^t_H(p_0, q_0)$$

will be called the canonical transformation of the phase space associated with the Hamiltonian function H.

There is an $(n + 1)$-dimensional submanifold (with a border) $M^{2n+1} \subset \mathbf{R}^{2n} \times \mathbf{R}^1$ naturally associated with the family $\{\Lambda^n_t\}$. As a rule we shall identify this family with M^{n+1}.

Let α be local coordinates in Λ^n_0, then (α, t) are local coordinates in M^{n+1}. It is these ones that will be used constantly. We shall denote by (α, t) the corresponding point of M^{n+1} as well, and we hope there will be no confusion in using the same symbols.

Problem 2.2. Suppose that there is an s-action on the Lagrangean manifold

$$\Lambda^n_0\colon \{p = p(\alpha),\ q = q(\alpha)\}$$

and let $\Lambda_t^n = g_H^t \Lambda_0^n$ be given by $p = p(\alpha, t)$, $q = q(\alpha, t)$. Then the function $s(\alpha, t)$ defined by

$$s(\alpha, t) = s_0(\alpha) + \int_0^t [\langle p(\alpha, \tau), H_p(p(\alpha, \tau), q(\alpha, \tau), \tau)\rangle - \\ - H(p(\alpha, \tau), q(\alpha, \tau), \tau)]\, d\tau \tag{2.3}$$

is an s-action on Λ_t^n.

Note that s-action on a Lagrangean manifold is not unique but defined only up to an additive constant. When dealing with the family $\Lambda_t^n = g_H^t \Lambda_0^n$ of Lagrangean manifods we always assume the s-actions on these manifolds to be co-ordinated by (2.3).

Let $H(p, q, t)$ and $\widetilde{H}(p, q, t)$ be real C^∞ functions and $\mathscr{H} = H + i\widetilde{H}$. We shall correlate with $\mathscr{H}$ the following t-dependent transformation of $\mathbf{R}^{2n} \times \mathbf{C}^{2n}$:

$$(p_0, q_0, w_0, z_0) \to (p, q, w, z),$$

where $(p, q) = g_H^t(p_0, q_0)$ and the functions $w = w(p_0, q_0, w_0, z_0, t)$ and $z = z(p_0, q_0, w_0, z_0, t)$ satisfy the system

$$\begin{aligned} \frac{dw}{dt} &= -i\widetilde{H}_q - \mathscr{H}_{qq} z - \mathscr{H}_{qp} w, \\ \frac{dz}{dt} &= i\widetilde{H}_p + \mathscr{H}_{pq} z + \mathscr{H}_{pp} w \end{aligned} \tag{2.4}$$

with the initial condition $w|_{t=0} = w_0$, $z|_{t=0} = z_0$. Denote by $dg_{H+i\widetilde{H}}^t$ the mapping $(p_0, q_0, w_0, z_0) \to (w, z)$; allowing for a slight inaccuracy, we shall write $(w, z) = dg_{H+i\widetilde{H}}^t(w_0, z_0)$.

Definition 2.3. *The transformation $(g_H^t, dg_{H+i\widetilde{H}}^t)$ will be called the complex canonical transformation associated with the complex Hamiltonian function $\mathscr{H} = H + i\widetilde{H}$. This transformation will be said to be dissipative if the condition $\widetilde{H} \leqslant 0$ is satisfied.*

In what follows all the canonical transformations under consideration are assumed to be dissipative.

Note 2.3. Let $\widetilde{H} = 0$. Then $dg_{H+i\widetilde{H}}^t = dg_H^t$ is the differential of the mapping g_H^t, (2.3) being the variation system for (2.2).

Let

$$(p(\alpha, t), q(\alpha, t)) = g_H^t(p_0(\alpha), q_0(\alpha)),$$
$$(w(\alpha, t), z(\alpha, t)) = dg_{H+i\widetilde{H}}^t(w_0(\alpha), z_0(\alpha)),$$

and let $F(p, q, w, z, t)$ be a function. The integral

$$\int_0^t F(p(\alpha, \tau), q(\alpha, \tau), w(\alpha, \tau), \tau)\, d\tau$$

will be called an integral along the $\mathscr{H}$-trajectory starting from α, $\int\limits_{(\alpha,0)}^{(\alpha,t)} F dt$ or $\int\limits_0^t F dt$ in symbol.

We shall use also the notation dF/dt or $\dot{F}$ for the derivative of the composite function

$$F(p(\alpha, t),\ q(\alpha, t),\ w(\alpha, t),\ z(\alpha, t),\ t)$$

with respect to t.

Theorem 2.1. *Let (Λ_0^n, r_0^n) be a Lagrangean manifold with a complex germ and let D_0 and E_0 be the dissipation and the potential of this germ. Let us define a dissipation $D(\alpha, t)$ on $\Lambda_t^n = g_H^t \Lambda_0^n$ by*

$$D(\alpha, t) = D_0(\alpha) - \int\limits_0^t \widetilde{H}\, dt.$$

Then

$$r^n(\alpha, t) = dg^t_{H+i\widetilde{H}} r_0^n(\alpha)$$

is a complex germ on Λ_t^n and

$$E(\alpha, t) = E_0(\alpha) - i \int\limits_0^t \left(\widetilde{H} + \frac{\langle w, \widetilde{H}_p\rangle + \langle z, \widetilde{H}_q\rangle}{2}\right) dt$$

is a potential on Λ_t^n.

In order to prove this theorem we need the following three lemmas.

Lemma 2.1. *Let $u(y)$, $0 \leqslant y \leqslant a$, be an m-dimensional real vector C^1-function, and let u satisfy the following inequality:*

$$|u'(y)| \leqslant c\,[f(y) + |u(y)|],$$

where f is a continuous non-negative function and $c =$ const. Then

$$|u(y)| \leqslant c_1 \left[|u(0)| + \int\limits_0^y f(\xi)\, d\xi \right],$$

where $c_1 = c_1(c, a)$ is smooth in c and a.

Proof. We have

$$|u(y)| \leqslant |u(0)| + c \int\limits_0^y [f(\xi) + |u(\xi)|]\, d\xi.$$

Hence

$$|u(y)| \leqslant c_0 \left(F(y) + \int\limits_0^y |u(\xi)|\, d\xi \right),$$

where

$$c_0 = 1 + c, \quad F(y) = |u(0)| + \int_0^y f(\xi)\,d\xi.$$

Let I be the operator defined by

$$If(y) = \int_0^y f(\xi)\,d\xi.$$

Then we have

$$|u| \leqslant c_0 (F + I|u|) \leqslant c_0 (F + c_0 IF + c_0 I^2 |u|).$$

It follows by induction that

$$|u| \leqslant \left(\sum_{n=1}^{N} C_0^n I^{n-1} F\right) + C_0^N I^N |u|.$$

The identity

$$I^n f(y) = \frac{1}{(n-1)!} \int_0^y (y-\xi)^{n-1} f(\xi)\,d\xi$$

implies the inequality

$$|I^n f(y)| \leqslant \frac{y^n}{n!} \max_{0 \leqslant \xi \leqslant y} |f(\xi)| \leqslant \frac{a^n}{n!} \max_{0 \leqslant \xi \leqslant y} |f(\xi)|.$$

The function F being monotonic, we obtain the inequality

$$|u(y)| \leqslant \sum_{n=1}^{N} \frac{C_0^n a^{n-1}}{(n-1)!} F(y) + \frac{C_0^N a^N}{N!} \max_{0 \leqslant \xi \leqslant a} |u(\xi)|.$$

Passing to the limit for $N \to \infty$ finally gives

$$|u(y)| \leqslant c_0 e^{c_0 a} F(y),$$

which proves the lemma.

Lemma 2.2. *Let* $p(\alpha, t)$, $q(\alpha, t)$, $w(\alpha, t)$, $z(\alpha, t)$ *be a solution of the system* (2.2), (2.4) *smoothly dependent on a parameter* $a \in \mathbf{R}^{2n}$, *and let* $J = \frac{D(p+w,\, q+z)}{Da}$. *Then*

$$\frac{\partial J}{\partial t} = \langle \varphi, w \rangle + \langle \psi, z \rangle,$$

where $\varphi = \varphi(\alpha, t)$ *and* $\psi = \psi(\alpha, t)$ *are smooth vector functions.*

Proof. We have

$$\left.\begin{aligned} \dot{p}+\dot{w} &= -\mathscr{H}_q-\mathscr{H}_{qp}w-\mathscr{H}_{qq}z, \\ \dot{q}+\dot{z} &= \mathscr{H}_p+\mathscr{H}_{pp}w+\mathscr{H}_{pq}z. \end{aligned}\right\} \tag{2.5}$$

Differentiation of these equations with respect to a shows that

$$\begin{aligned} (\dot{p}+\dot{w})_a &= -\mathscr{H}_{qq}(q_a+z_a)-\mathscr{H}_{qp}(p_a+w_a)+ \\ &\quad +\sum_{j=1}^{n}(z_jA^{(j)}+w_jB^{(j)}), \\ (\dot{q}+\dot{z})_a &= \mathscr{H}_{pq}(q_a+z_a)+\mathscr{H}_{pp}(p_a+w_a)+ \\ &\quad +\sum_{j=1}^{n}(z_jC^{(j)}+w_jD^{(j)}), \end{aligned} \tag{2.6}$$

where $A^{(j)}$, $B^{(j)}$, $C^{(j)}$ and $D^{(j)}$ are smooth matrix functions of a and t.

By the rule of differentiation of a determinant

$$\frac{\partial J}{\partial t}=\sum_{k=1}^{2n}J_k,$$

where J_k is the determinant obtained by differentiating the kth row of J. It follows from (2.6) that

$$J_k=\begin{cases} -J\mathscr{H}_{q_kp_k}+\langle\psi_k,\ z\rangle+\langle\varphi_k,\ w\rangle, & k\leqslant n \\ J\mathscr{H}_{p_kq_k}+\langle\psi_k,\ z\rangle+\langle\varphi_k,\ w\rangle, & k>n, \end{cases}$$

where φ_k, ψ_k are smooth functions of a and t. Thus,

$$\frac{\partial J}{\partial t}=\sum_{k=1}^{2n}(\langle\psi_k,\ z\rangle+\langle\varphi_k,\ w\rangle),$$

which proves the lemma.

As a corollary of Lemma 2.2 we obtain the following classical result:

Theorem (Liouville). *Let g_H^t be the canonical transformation of the phase space associated with a Hamiltonian function H and let Ω_0 be a measurable subset of the phase space. Set $\Omega_t=g_H^t\Omega_0$. Then*

$$\int\limits_{\Omega_t}dq_1\ldots dq_n\,dp_1\ldots dp_n=\int\limits_{\Omega_0}dq_1\ldots dq_n\,dp_1\ldots dp_n.$$

To prove this theorem it suffices to put $\widetilde{H}=0$, $w=0$, $z=0$, $a=(p(0),\ q(0))$ in Lemma 2.2.

Lemma 2.3. *Let*

$$\Lambda_0^n : \{p = p_0(\alpha),\ q = q_0(\alpha)\},$$
$$\gamma_0^n : \{w = w_0(\alpha),\ z = z_0(\alpha)\}$$

be a Lagrangean manifold with a complex germ, let $p(\alpha, t)$, $q(\alpha, t)$, $w(\alpha, t)$, $z(\alpha, t)$ *be the solution of the system* (2.2), (2.4) *satisfying the initial condition* $p(\alpha, 0) = p_0(\alpha)$, $q(\alpha, 0) = q_0(\alpha)$, $w(\alpha, 0) = w_0(\alpha)$, $z(\alpha, 0) = z_0(\alpha)$ *and let* $\alpha_0 \in \Lambda_0^n$. *If*

$$\operatorname{rank} \begin{pmatrix} \dfrac{\partial (p_0 + w_0)}{\partial \alpha} \\ \dfrac{\partial (q_0 + z_0)}{\partial \alpha} \end{pmatrix}_{\alpha = \alpha_0} = n$$

and $w(\alpha_0, t) = z(\alpha_0, t) = 0$ *for* $0 \leqslant t \leqslant T$, *then*

$$\operatorname{rank} \begin{pmatrix} \dfrac{\partial (p + w)}{\partial \alpha} \\ \dfrac{\partial (q + z)}{\partial \alpha} \end{pmatrix}_{\substack{\alpha = \alpha_0 \\ t = T}} = n.$$

Proof. Let $\beta \in \mathbf{R}^n$. In a neighbourhood of the point $\alpha = \alpha_0$, $\beta = 0$ in $\mathbf{R}^{2n}$ we set

$$\left.\begin{aligned} p_0(\alpha, \beta) &= p_0(\alpha),\ q_0(\alpha, \beta) = q_0(\alpha), \\ w_0(\alpha, \beta) &= w_0(\alpha) + A\beta, \\ z_0(\alpha, \beta) &= z_0(\alpha) + B\beta, \end{aligned}\right\} \quad (2.7)$$

where A and B are $n \times n$ matrices. Obviously, we can choose A and B so that the matrix

$$\frac{\partial (p_0 + w_0,\ q_0 + z_0)}{\partial (\alpha, \beta)}$$

is non-singular at $(\alpha_0, 0)$.

Let $p(\alpha, \beta, t)$, $q(\alpha, \beta, t)$, $w(\alpha, \beta, t)$, $z(\alpha, \beta, t)$ be the solution of (2.2), (2.4) satisfying the initial condition (2.7). By applying Lemma 2.2 with $a = (\alpha, \beta)$, we have

$$\left.\frac{D(p + w,\ q + z)}{D(\alpha, \beta)}\right|_{\substack{\alpha = \alpha_0 \\ \beta = 0}} = \left.\frac{D(p_0 + w_0,\ q_0 + z_0)}{D(\alpha, \beta)}\right|_{\substack{\alpha = \alpha_0 \\ \beta = 0}} \neq 0$$

for $0 \leqslant t \leqslant T$, which proves the lemma.

Proof of Theorem 2.1. I. We first verify the validity of g 1). By Lemma 2.1,

$$|w(\alpha, t)| + |z(\alpha, t)| \leqslant$$
$$\leqslant c(\alpha, t) \left(|z_0(\alpha)| + |w_0(\alpha)| + \int_0^t (|\widetilde{H}_p| + |\widetilde{H}_q|)\, dt \right),$$

where $c \in C^\infty$.

Applying Lemma 1.1 now and the Schwartz inequality we have

$$\int_0^t (|\widetilde{H}_p|+|\widetilde{H}_q|)\,dt \leqslant c_1(\alpha,\ t)\int_0^t \sqrt{|\widetilde{H}|}\,dt \leqslant$$

$$\leqslant c_1(\alpha,\ t)\sqrt{t}\left(\int_0^t |\widetilde{H}|\,dt\right)^{1/2}, \quad c_1 \in C^\infty;$$

it follows that there exists such a smooth function $c_2(\alpha,\ t)$ that

$$|w|+|z| \leqslant c_2\left[D_0^{1/2}+\left(\int_0^t |\widetilde{H}|\,dt\right)\right] \leqslant$$

$$\leqslant \sqrt{2}\,c_2\left(D_0-\int_0^t \widetilde{H}\,dt\right)^{1/2} = \sqrt{2}\,c_2 D^{1/2}.$$

II. To verify the validity of g 3) we consider the following function:

$$f_j(\alpha,\ t) = E_{\alpha_j} - \langle w,\ q_{\alpha_j}\rangle - \frac{1}{2}\langle z,\ w_{\alpha_j}\rangle + \frac{1}{2}\langle w,\ z_{\alpha_j}\rangle.$$

It follows from the systems (2.2), (2.4), the equation

$$\dot{E} = -i\widetilde{H} - \frac{i}{2}\langle \widetilde{H}_p,\ w\rangle - \frac{i}{2}\langle \widetilde{H}_q,\ z\rangle$$

and the expressions of $\dot{q}_\alpha$, $\dot{p}_\alpha$, $\dot{w}_\alpha$, $\dot{z}_\alpha$, $\dot{E}_\alpha$ being obtained by differentiating these equations with respect to α that

$$\dot{f}_j(\alpha,\ t) = \langle w,\ a_1^{(j)} w\rangle + \langle w,\ a_2^{(j)} z\rangle + \langle z,\ a_3^{(j)} z\rangle,$$

where $a_{1,\,2,\,3}^{(j)}$ are smooth matrix functions of t and α. Thus,

$$\left|\int_0^t \dot{f}_j(\alpha,\ \tau)\,d\tau\right| \leqslant c(\alpha,\ t)\int_0^t (|w(\alpha,\ \tau)|^2+|z(\alpha,\ \tau)|^2)\,d\tau,$$

$$\left|\int_0^t \frac{\partial \dot{f}_j(\alpha,\ \tau)}{\partial\alpha}\,d\tau\right| \leqslant c(\alpha,\ t)\int_0^t (|w(\alpha,\ \tau)|+|z(\alpha,\ \tau)|)\,d\tau,$$

where $c(\alpha,\ t)$ is a smooth function. Since $D(\alpha,\ t)$ is monotonic in t and

$$z = O_D(h^{1/2}),\quad w = O_D(h^{1/2}),$$

we have

$$\int^t \dot{f}_j(\alpha,\ \tau)\,d\tau = O_D(h).$$

It remains to note that

$$f_j(\alpha, 0) = O_{D_0}(h),$$

for E_0 is a potential on Λ_0^n and it follows from

$$D_0(\alpha) \leqslant D(\alpha, t)$$

that

$$f_j(\alpha, t) = f_j(\alpha, 0) + \int_0^t \dot{f}_j(\alpha, \tau)\, d\tau = O_D(h).$$

The reader is offered to verify that $E = O_D(h)$ holds.

III. If $D(\alpha_0, t) = 0$, then $D(\alpha_0, \tau) = 0$ for $0 \leqslant \tau \leqslant t$, and therefore $z(\alpha_0, \tau) = w(\alpha_0, \tau) = 0$ for $0 \leqslant \tau \leqslant t$. Thus g 2) is valid by Lemma 2.3. The theorem is proved.

In what follows we shall always suppose that if a family of Lagrangean manifolds Λ_t^n with complex germs r_t^n is obtained from (Λ_0^n, r_0^n) by a complex canonical transformation, then the dissipation $D(\alpha, t)$ and the potential $E(\alpha, t)$ on Λ_t^n are correlated with $D_0(\alpha)$ and $E_0(\alpha)$ by the formulas of Theorem 2.1.

Example 2.1. Let $\mathscr{H}_I = H_I = \frac{1}{2}(p_I^2 + q_I^2)$ and let $(\Lambda_0^n, \gamma_0^n)$ be given by

$$p = p_0(\alpha), \quad q = q_0(\alpha),$$
$$w = w_0(\alpha), \quad z = z_0(\alpha).$$

Then (Λ_t^n, r_t^n) is given by

$$p_I(\alpha, t) = p_{I0}(\alpha),\ p_{\bar I}(\alpha, t) = p_{\bar I_0}(\alpha)\cos t - q_{\bar I 0}(\alpha)\sin t,$$
$$q_I(\alpha, t) = q_{I0}(\alpha), \quad q_{\bar I}(\alpha, t) = q_{\bar I 0}(\alpha)\cos t + p_{\bar I 0}(\alpha)\sin t,$$
$$w_I(\alpha, t) = w_{I0}(\alpha), \quad w_{\bar I}(\alpha, t) = w_{\bar I 0}(\alpha)\cos t - z_{\bar I 0}(\alpha)\sin t,$$
$$z_I(\alpha, t) = z_{I0}(\alpha), \quad z_{\bar I}(\alpha, t) = z_{\bar I 0}(\alpha)\cos t + w_{\bar I 0}(\alpha)\sin t,$$
$$D(\alpha, t) = D_0(\alpha),$$
$$E(\alpha, t) = E_0(\alpha),$$
$$s(\alpha, t) = s_0(\alpha) + \frac{1}{4}[p_{\bar I 0}^2 - q_{\bar I 0}^2]\sin 2t + \frac{1}{2}\langle q_{\bar I 0}, p_{\bar I 0}\rangle(\cos 2t - 1),$$

where $D_0(\alpha)$, $E_0(\alpha)$ and $s_0(\alpha)$ are the dissipation, the potential and the s-action on Λ_0^n, respectively.

In particular, we obtain for $t = \pm\pi/2$:

$$\Lambda^n_{\pm\frac{\pi}{2}} : \{p_I = p_{I0}(\alpha), \quad p_{\bar I} = \pm q_{\bar I 0}(\alpha),$$
$$q_I = q_{I0}(\alpha), \quad q_{\bar I} = \pm p_{\bar I 0}(\alpha)\},$$
$$r^n_{\pm\frac{\pi}{2}} : \{w_I = w_{I0}(\alpha), \quad w_{\bar I} = \mp z_{\bar I 0}(\alpha),$$
$$z_I = z_{I0}(\alpha), \quad z_{\bar I} = \pm w_{\bar I 0}(\alpha)\}.$$

Note that

$$\left\langle p\left(\alpha, \pm\frac{\pi}{2}\right), q\left(\alpha, \pm\frac{\pi}{2}\right)\right\rangle = \langle p_{I0}(\alpha), q_{I0}(\alpha)\rangle -$$
$$- \langle p_{\bar I 0}(\alpha), q_{\bar I 0}(\alpha)\rangle = \langle p_0(\alpha), q_0(\alpha)\rangle_I$$

which makes the notation $\langle u, v\rangle_I$ reasonable.

Definition 2.4. *Let I be a subset of $\{1, \ldots, n\}$ and let Λ^n: $\{p = p(\alpha), q = q(\alpha)\}$, $r^n = (w(\alpha), z(\alpha))$ be a Lagrangean manifold with a complex germ. The subset of Λ^n defined by*

$$J_I \overset{\text{def}}{=} \frac{D\{q_I + z_I, p_{\bar I} + w_{\bar I}\}}{D\alpha} \neq 0 \tag{2.8}$$

will be called the Ith *zone of* (Λ^n, r^n), Ω_I *in symbol. The zone* Ω_{I_0} *with* $I_0 = \{1, 2, \ldots, n\}$ *will be called non-singular*. By dealing with the non-singular zone we omit the index I_0: for example, we write Ω in place of Ω_{I_0}, E in place of E_{I_0} and so on.

It is obvious that Ω_I is an open subset of Λ^n. We shall show that the axioms g 2) and g 3) imply that the family of zones $\{\Omega_I\}$ with I running through all subsets of $\{1, 2, \ldots, n\}$ covers Γ.

Lemma 2.4. *Let $u(\alpha)$ and $v(\alpha)$ be smooth $\mathbf{C}^n$-valued functions defined in a neighborhood of a point $\alpha_0 \in \mathbf{R}^n$, and let the following conditions be satisfied for $\alpha = \alpha_0$:*

(i) $[u, v]_{ij} = 0, \quad i, j = 1, \ldots, n;$

(ii) $\operatorname{rank}\begin{pmatrix} \dfrac{\partial u}{\partial \alpha} \\ \dfrac{\partial v}{\partial \alpha} \end{pmatrix} = n.$

Then there exists an $I \subset \{1, \ldots, n\}$ such that the matrix

$$\frac{\partial\{u_I, v_{\bar I}\}}{\partial \alpha}$$

is non-degenerate at $\alpha = \alpha_0$.

Proof. Denote by A the matrix $\frac{\partial u}{\partial \alpha}(\alpha_0)$ and let rank $A = k$. If $k = n$, then the statement of the lemma is valid. Let $k < n$. We may assume without loss of generality that the first k rows of A are linearly independent. Let C be the matrix composed of the first k rows of A and the last $n - k$ rows of the matrix $B = \frac{\partial v}{\partial \alpha}(\alpha_0)$. We shall show that rank $C = n$, which means that the lemma is true. Suppose that this were not the case, i.e., rank $C = m < n$. Then rank $\begin{pmatrix} A \\ C \end{pmatrix} = m$. Let $\beta \in \mathbf{C}^n$ and $\alpha = M\beta$, where M is a complex non-degenerate matrix. Set

$$\frac{\partial}{\partial \beta_i} = \sum_{j=1}^{n} M_{ji} \frac{\partial}{\partial \alpha_j}, \quad A' = \frac{\partial u}{\partial \beta}(\alpha_0), \quad B' = \frac{\partial v}{\partial \beta}(\alpha_0),$$

and let C' be the matrix composed of the first k rows of A' and the last $n-k$ rows of B'. It is obvious that rank $\begin{pmatrix} A' \\ B' \end{pmatrix} = n$ and rank $\begin{pmatrix} A' \\ C' \end{pmatrix} = m$. Now me choose M so that the last $n - m$ columns of the matrix $\begin{pmatrix} A' \\ C' \end{pmatrix}$ vanish and det $(A'_{ij})_{i,\,j=1}^{k}$ does not vanish. The condition $[u, v]_{ij} = 0$ implies that

$$\left\langle \frac{\partial u}{\partial \beta_i}, \frac{\partial v}{\partial \beta_n} \right\rangle - \left\langle \frac{\partial u}{\partial \beta_n}, \frac{\partial v}{\partial \beta_i} \right\rangle = \sum_{r=1}^{k} \frac{\partial u_r}{\partial \beta_i} \frac{\partial v_r}{\partial \beta_n} = 0.$$

Since det $\left(\frac{\partial u_i}{\partial \beta_j}\right)_{i,\,j=1}^{k} \neq 0$, it follows that the last column of $\begin{pmatrix} A' \\ B' \end{pmatrix}$ vanishes, contradicting the condition rank $\begin{pmatrix} A' \\ B' \end{pmatrix} = n$. The lemma is proved.

Corollary. $\Gamma \in \bigcup_{I \subset \{1,2,\ldots,n\}} \Omega_I$.

In fact, set $u = q + z$, $v = p + w$ and let $\alpha \in \Gamma$. Then the condition (ii) of the lemma is identical with the axiom g 2) of the complex germ, and the condition (i) follows from the axiom g 3).

In what follows we shall consider only a little neighborhood of Γ in Λ^n, therefore we can always suppose that the *set of all zones covers* Γ.

Definition 2.5. *The function*

$$\mu(\alpha) = \frac{1}{2}\langle z(\alpha), \mathscr{E}(\alpha) z(\alpha)\rangle,$$

where

$$\mathscr{E}=BC^{-1},\ \ B=\frac{\partial (p+w)}{\partial \alpha}, \qquad C=\frac{\partial (q+z)}{\partial \alpha},$$

will be called the μ-action in the non-singular zone. The phase Φ of germ in the non-singular zone will be defined by

$$\Phi(\alpha)=-\frac{1}{2}\langle w(\alpha),\ z(\alpha)\rangle+\mu(\alpha)+s(\alpha)+E(\alpha).$$

Note that

$$\mathscr{E}=\frac{\partial (p+w)}{\partial (q+z)},$$

where $\partial/\partial (q+z)$ is the operator of formal differentiation with respect to $q+z$. Since an enthalpy W (see Note 2.2) satisfies the condition

$$p+w=\frac{\partial W}{\partial (q+z)}+O_D(h),$$

we must have

$$\mathscr{E}=\frac{\partial^2 W}{\partial (q+z)^2}+O_D(h^{1/2}).$$

An immediate consequence is that the matrix $\mathscr{E}$ is "almost symmetric":

$$\mathscr{E}-\overset{t}{\mathscr{E}}=O_D(h^{1/2}).$$

Now we introduce the s-action, the μ-action and the phase in the zone Ω_I. To this end, consider the canonical transformation $(g^{\pi/2}_{H_I},\ dg^{\pi/2}_{H_I})$ associated with the Hamiltonian function H_I introduced in Example 2.1. Note that the image of the zone Ω_I under $g^{\pi/2}_{H_I}$ lies in the non-singular zone of $g^{\pi/2}_{H_I}\Lambda^n$ because

$$\frac{\partial q\left(\alpha,\ \frac{\pi}{2}\right)}{\partial\alpha}=\frac{\partial\{q_I(\alpha),\ p_{\bar I}(\alpha)\}}{\partial\alpha}\neq 0 \quad \text{for } \alpha\in\Omega_I.$$

Definition 2.6. *We define the s-action, the μ-action and the phase in the zone* Ω_I *by the formulas*

$$s_I(\alpha)=s\left(\alpha,\ \frac{\pi}{2}\right),$$

$$\mu_I(\alpha)=\mu\left(\alpha,\ \frac{\pi}{2}\right),$$

$$\Phi_I(\alpha)=\Phi\left(\alpha,\ \frac{\pi}{2}\right),$$

where $s\left(\alpha, \frac{\pi}{2}\right)$, $\mu\left(\alpha, \frac{\pi}{2}\right)$ *and* $\Phi\left(\alpha, \frac{\pi}{2}\right)$ *are the s-action, the μ-action and the phase in the non-singular zone of* $g_{H_I}^{\pi/2}\Lambda^n$, *respectively.*

The result of Example 2.1 leads to the explicit formulas

$$s_I(\alpha) = s(\alpha) - \langle p_{\bar I}(\alpha),\ q_{\bar I}(\alpha)\rangle,$$

$$\mu_I(\alpha) = \frac{1}{2}\langle\{z_I,\ w_{\bar I}\},\ \mathscr{E}_I\{z_I,\ w_{\bar I}\}\rangle,$$

$$\Phi_I = -\frac{1}{2}\langle w,\ z\rangle_I + \mu_I + s_I + E,$$

where

$$\mathscr{E} = B_I C_I^{-1}, \quad B_I = \frac{\partial\{p_I + w_I,\ -q_{\bar I} - z_{\bar I}\}}{\partial\alpha},$$

$$C_I = \frac{\partial\{q_I + z_I,\ p_{\bar I} + w_{\bar I}\}}{\partial\alpha}.$$

Note that any s-action $s(\alpha, t)$ on a family of Lagrangean manifolds $\Lambda_t^n = \overset{t}{g}_H\Lambda^n$ satisfies the equation

$$ds_I = \langle p_I,\ dq_I\rangle - \langle q_{\bar I},\ dp_{\bar I}\rangle - H\,dt,$$

which is transformed into the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x_I},\ \xi_{\bar I},\ x_I,\ -\frac{\partial S}{\partial \xi_{\bar I}},\ t\right) = 0$$

under the change of variables

$$\left.\begin{aligned} &(\alpha,\ t) \to (x_I,\ \xi_{\bar I},\ t),\\ &x_I = q_I(\alpha,\ t), \quad \xi_{\bar I} = p_{\bar I}(\alpha,\ t),\\ &S(x_I,\ \xi_{\bar I},\ t) = s_I(\alpha,\ t). \end{aligned}\right\} \tag{2.9}$$

We shall see below that if $\operatorname{Im}\Phi_I \geqslant 0$, then $\Phi_I(\alpha, t)$ satisfies an equation which becomes a Hamilton-Jacobi equation with dissipation under the change (2.9).

Sec. 3. γ-Atlases and the Dissipativity Inequality

In this section we introduce a special sort of atlases of Λ^n which makes it possible to formulate a condition for a complex germ, called the dissipativity condition, essential in solving Hamilton-Jacobi equations with dissipation.

Lemma 3.1. *Let $\alpha_0 \in \Omega_I$. Then there exists a real n-dimensional vector function $\gamma(\alpha)$ such that each component of it is one of that of the*

$(2n+1)$*-dimensional vector*

$$(0, \ \operatorname{Re} z_I(\alpha), \quad \operatorname{Im} z_I(\alpha), \quad \operatorname{Re} w_{\bar{I}}(\alpha), \quad \operatorname{Im} w_{\bar{I}}(\alpha)),$$

and

$$\left.\frac{D(q_I+\gamma_I, \ p_{\bar{I}}+\gamma_{\bar{I}})}{D\alpha}\right|_{\alpha=\alpha_0} \neq 0$$

The following fact of linear algebra will be needed to prove this lemma.

Lemma 3.2. *Let* $u_1, \ldots, u_m, v_1, \ldots, v_m$ *be elements of a real vector space* V. *Then*

$$u_1 + iv_1, \ldots, u_m + iv_m$$

are linearly independent in the complexification $\mathbf{C}V$ *of* V *if and only if*

$$u_1 \oplus v_1, \ldots, u_m \oplus v_m, \ (-v_1) \oplus u_1, \ldots, (-v_m) \oplus u_m$$

are linearly independent in $V \oplus V$.

Proof. Let a_j, b_j be real numbers. Then

$$(a_j = b_j = 0, \ \text{all } j) \Leftrightarrow \Big(\sum_j (a_j + ib_j)(u_j + iv_j) = 0\Big) \Leftrightarrow$$

$$\Leftrightarrow \Big(\sum_j (a_j u_j - b_j v_j) = \sum_j (a_j v_j + b_j v_j) = 0\Big) \Leftrightarrow$$

$$\Leftrightarrow \Big(\sum_j a_j (u_j \oplus v_j) + \sum_j b_j ((-v_j) \oplus u_j) = 0\Big).$$

Proof of Lemma 3.1. Considering the canonical transformation $g_{H_I}^{\pi/2}$ of Example 2.1 shows that Ω_I may be assumed without loss of generality to be non-singular. Under this assumption we have $\det C(\alpha_0) \neq 0$. Fix $\alpha = \alpha_0$. Let $\operatorname{rank}(\partial q/\partial\alpha) = k$; we may, of course, assume that it is the first k rows of $\partial q/\partial\alpha$ which are linearly independent. Consider the matrix

$$M = \begin{pmatrix} \dfrac{\partial(q + \operatorname{Re} z)}{\partial\alpha} \\[2mm] \dfrac{\partial \operatorname{Im} z}{\partial\alpha} \end{pmatrix}.$$

By Lemma 3.2 non-degeneracy of C implies that

$$\det \begin{pmatrix} \dfrac{\partial(q + \operatorname{Re} z)}{\partial\alpha} & -\dfrac{\partial \operatorname{Im} z}{\partial\alpha} \\[2mm] \dfrac{\partial \operatorname{Im} z}{\partial\alpha} & \dfrac{\partial(q + \operatorname{Re} z)}{\partial\alpha} \end{pmatrix} \neq 0,$$

hence $\operatorname{rank} M = n$. Choose $l_{k+1}, \ldots, l_n$ so that the matrix M_1, obtained from $\partial q/\partial\alpha$ by replacing its last $n-k$ rows by those of M,

is non-degenerate. Set

$$\gamma_j = \begin{cases} 0 & \text{for } 1 \leqslant j \leqslant k, \\ \operatorname{Re} z_{l_j} & \text{for } j > k,\ l_j \leqslant n, \\ \operatorname{Im} z_{l_j} & \text{for } j > k,\ l_j > n, \end{cases}$$

then $\partial(q+\gamma)/\partial\alpha$ is non-degenerate. In fact,

$$\frac{\partial(q+\gamma)}{\partial\alpha} = \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix} M_1,$$

where A is an $(n-k)\times k$ matrix. The lemma is proved.

Corollary. *There exists such a neighbourhood $u_\gamma \ni \alpha_0$ that the restriction π^I_γ of the mapping*

$$\alpha \to \{q_I(\alpha) + \gamma_I(\alpha),\ p_{\bar I}(\alpha) + \gamma_{\bar I}(\alpha)\}$$

to u_j is a diffeomorphism into $\mathbf{R}^n$.

Definition 3.1. *We say that u_γ is a γ-domain, π^I_γ is a γ-diffeomorphism and (u_γ, π^I_γ) is a γ-patch of Λ^n.* Sometimes we shall use the term "γ-patch" as a synonym of "γ-domain".

Any set of γ-patches covering Λ^n will be called a γ-atlas of Λ^n. If $\gamma(\alpha) = Fv(\alpha)$, where F is a matrix and

$$v = (\operatorname{Re} z_I,\ \operatorname{Im} z_I,\ \operatorname{Re} w_{\bar I},\ \operatorname{Im} w_{\bar I}),$$

then the corresponding γ-patch of the zone Ω_I is said to be of type (I, F). A patch of type $(I, 0)$ is said to be non-singular.

Lemma 3.3. *Let $(u_{\bar\gamma}, \pi^I_{\bar\gamma})$ and $(u_{\bar{\bar\gamma}}, \pi^I_{\bar{\bar\gamma}})$ be γ-patches of the zone Ω_I. Then $\sigma = (\pi^I_{\bar\gamma})^{-1} \circ \pi^I_{\bar{\bar\gamma}}$ has the following asymptotic expansion:*

$$\sigma(\alpha) = \alpha + \sigma^{(1)}(\alpha) + \sigma^{(2)}(\alpha) + O_D(h^{3/2}),$$

where

$$\sigma^{(1)}(\alpha) = A^{-1}(\bar{\bar\gamma} - \bar\gamma),$$

$$\sigma^{(2)}(\alpha) = -\frac{1}{2}A^{-1}\sum_{i,j}\frac{\partial^2\{q_I + \bar\gamma_I,\ p_{\bar I} + \bar\gamma_{\bar I}\}}{\partial\alpha_i\,\partial\alpha_j}\sigma_i^{(1)}\sigma_j^{(1)},$$

$$A = \frac{\partial\{q_I + \bar\gamma_I,\ p_{\bar I} + \bar\gamma_{\bar I}\}}{\partial\alpha}.$$

Proof. Assume for simplicity (which does not affect the generality) that Ω_I is the non-singular zone. Then $\sigma(\alpha)$ satisfies the equa-

tion

$$q(\sigma(\alpha)) + \overline{\gamma}(\sigma(\alpha)) = q(\alpha) + \overline{\overline{\gamma}}(\alpha). \tag{3.1}$$

Set

$$\sigma(\alpha) = \alpha + \sigma^{(1)}(\alpha) + \sigma^{(2)}(\alpha) + \gamma(\alpha).$$

Substitution of the expansion of $q(\sigma(\alpha)) + \overline{\gamma}(\sigma(\alpha))$ in powers of $\sigma^{(1)} + \sigma^{(2)} + r$ into (3.1) leads to

$$(q_\alpha + \overline{\gamma}_\alpha)(\sigma^{(1)} + \sigma^{(2)} + r) = \frac{1}{2}\sum_{i,j} \frac{\partial^2 (q+\overline{\gamma})}{\partial\alpha_i\,\partial\alpha_j}(\sigma^{(1)} + \sigma^{(2)} + r)_i\,(\sigma^{(1)} +$$

$$+ \sigma^{(2)} + r)_j + F_3(\sigma^{(1)} + \sigma^{(2)} + r) = \overline{\overline{\gamma}}(\alpha) - \overline{\gamma}(\alpha),$$

where F_3 is a cubic form smoothly depending on α. Since $\sigma^{(1)}$ and $\sigma^{(2)}$ satisfy the equations

$$(q_\alpha + \overline{\gamma}_\alpha)\,\sigma^{(1)}(\alpha) = \overline{\overline{\gamma}}(\alpha) - \overline{\gamma}(\alpha),$$

$$(q_\alpha + \overline{\gamma}_\alpha)\,\sigma^{(2)}(\alpha) + \frac{1}{2}\sum_{i,j}\frac{\partial^2 (q+\overline{\gamma})}{\partial\alpha_i\,\partial\alpha_j}\,\sigma_i^{(1)}\sigma_j^{(1)},$$

and

$$\sigma^{(1)} = O_D(h^{1/2}) \qquad \sigma^{(2)} = O_D(h),\ \ r|_\Gamma = 0,$$

one has

$$A_1 r = O_D(h^{3/2})$$

with $A_1|_\Gamma = A_\Gamma$, so $\det A_1 \neq 0$ in a neighborhood of each $\alpha_0 \in \Gamma$. This proves the lemma.

Corollary. *$O_{D(\alpha)}$-estimate is equivalent to $O_{D(\sigma(\alpha))}$-estimate.*

Proof. Set $\widetilde{D}(\alpha) = D(\sigma(\alpha))$. Since $\sigma(\alpha) - \alpha = O_D(h^{1/2})$, the expansion

$$\widetilde{D}(\alpha) = D(\alpha) + \langle D_\alpha,\ (\sigma(\alpha) - \alpha)\rangle + \langle\sigma(\alpha) - \alpha,\ f(\alpha)(\sigma(\alpha) - \alpha)\rangle,$$

where $f(\alpha)$ is a smooth vector function, implies that $\widetilde{D} = O_D(h)$. The same argument with σ replaced by σ^{-1} shows that $D = O_{\widetilde{D}}(h)$.

Definition 3.2. *The function*

$$\Phi_I^\gamma = \Phi_I + \langle\{p_I,\ -q_{\bar{I}}\} + \eta^I,\ \gamma\rangle + \frac{1}{2}\langle\gamma,\ \mathscr{E}_I^s\gamma\rangle,$$

where

$$\eta^I = \{w_I,\ -z_{\bar{I}}\} - \mathscr{E}_I^s\{z_I,\ w_I\},$$

$$\mathscr{E}_I^s = \frac{1}{2}(\mathscr{E}_I + {}^t\mathscr{E}_I),$$

defined in a given γ-patch of the zone Ω_I, will be called the γ-phase of the complex germ in this γ-patch.

Lemma 3.4. *Let (u_γ, π_γ^I) and $(u_{\bar\gamma}, \pi_{\bar\gamma}^I)$ be two γ-patches of the zone Ω_I. Then*

$$\Phi_I^{\bar\gamma} = \Phi_I^\gamma \circ (\pi_\gamma^I)^{-1} \circ \pi_{\bar\gamma}^I + O_D(h^{3/2})$$

whenever $(\pi_\gamma^I)^{-1} \circ \pi_{\bar\gamma}^I$ is defined.

Note 3.1. $(\pi_\gamma^I)^{-1} \circ \pi_{\bar\gamma}^I$ is obviously defined in a neighbourhood of $u_\gamma \cap u_{\bar\gamma} \cap \Gamma$.

Note 3.2. Lemma 3.4 shows that if the system

$$q_I(\sigma(\alpha)) = q_I(\alpha) + \gamma_I(\alpha),$$
$$p_{\bar I}(\sigma(\alpha)) = p_{\bar I}(\alpha) + \gamma_{\bar I}(\alpha)$$

has a solution $\sigma(\alpha)$, then modulo $O_D(h^{3/2})$, the γ-phase $\Phi_I^\gamma(\alpha)$ is obtained from the phase $\Phi_I(\alpha)$ by a change of variables: $\Phi_I^\gamma(\alpha) \sim \Phi_I(\sigma(\alpha))$. Note, however, that Definition 3.2 defines Φ_I^γ on u_γ whether $\sigma(\alpha)$ exists or not.

Proof of Lemma 3.4. We may assume without loss of generality that Ω_I is non-singular. Set $\sigma = \pi_\gamma^{-1} \circ \pi_{\bar\gamma}$. Expanding $\Phi^\gamma(\sigma(\alpha))$ in powers of $(\sigma(\alpha) - \alpha)$ and using Lemma 3.3 we get

$$R(\alpha) \overset{\text{def}}{=} \Phi^\gamma(\sigma(\alpha)) - \Phi^{\bar\gamma}(\alpha) = \Phi^\gamma(\alpha) - \Phi^{\bar\gamma}(\alpha) + \langle \Phi_\alpha^\gamma, \sigma^{(1)}(\alpha) + $$
$$+ \sigma^{(2)}(\alpha)\rangle + \frac{1}{2}\langle \Phi_{\alpha\alpha}^\gamma \sigma^{(1)}(\alpha), \sigma^{(1)}(\alpha)\rangle + O_D(h^{3/2}). \tag{3.2}$$

To calculate the derivatives $\partial\Phi^\gamma/\partial\alpha$ and $\partial^2\Phi^\gamma/\partial\alpha^2$ it is convenient to express Φ^γ in terms of an enthalpy of the complex germ:

$$\Phi^\gamma = W - \left\langle p + w - \frac{1}{2}\mathscr{E}(z-\gamma),\ z-\gamma \right\rangle + O_D(h^{3/2}).$$

By the definition of an enthalpy

$$\frac{\partial W}{\partial\alpha} = {}^tC \frac{\partial W}{\partial(q+z)} = {}^tC(p+w) + O_D(h),$$

so

$$\Phi_\alpha^\gamma = {}^tC(p+w) - {}^tB(z-\gamma) + \frac{1}{2}\,{}^t(z-\gamma)_\alpha \mathscr{E}(z-\gamma) -$$
$$- {}^t(z-\gamma)_\alpha \left(p + w - \frac{1}{2}\mathscr{E}(z-\gamma)\right) + O_D(h) =$$
$$= {}^tA(p+w) - {}^tC\mathscr{E}(z-\gamma) + {}^t(z-\gamma)_\alpha \mathscr{E}(z-\gamma) +$$
$$+ O_D(h) = {}^tA[p + w - \mathscr{E}(z-\gamma)] + O_D(h), \tag{3.3}$$

where $A = \partial(q+\gamma)/\partial\alpha$. Differentiating once again and using the identity

$$1 - C^{-1}(z-\gamma)_\alpha = 1 - C^{-1}(q+z)_\alpha + C^{-1}(q+\gamma)_\alpha = C^{-1}A$$

we obtain

$$\frac{\partial^2\Phi^\gamma}{\partial\alpha^2} = \sum_{i,j}\left\langle \frac{\partial^2(q+\gamma)}{\partial\alpha_i\,\partial\alpha_j}, p\right\rangle + {}^tA\,(B - \mathscr{C}(z-\gamma)_\alpha) + O_D(h^{1/2}) =$$

$$= \sum_{i,j}\left\langle \frac{\partial^2(q+\gamma)}{\partial\alpha_i\,\partial\alpha_j}, p\right\rangle {}^tA\mathscr{C}A + O_D(h^{1/2}). \tag{3.4}$$

Substitution of (3.3), (3.4) and the formulas of Lemma 3.3 for $\sigma^{(1)}$ and $\sigma^{(2)}$ into (3.2) shows that $R = O_D(h^{3/2})$, which proves the lemma.

Definition 3.3. *Let u_γ be a γ-domain of the zone Ω_I of a complex germ. We say that the germ is dissipative in u_γ, if there exist smooth functions $\varepsilon(\alpha) > 0$ and $c(\alpha) \geqslant 0$ satisfying the condition*

$$\operatorname{Im}\Phi_I^\gamma + cD^{3/2} \geqslant \varepsilon D, \quad \alpha \in u_\gamma,$$

which will be called the dissipativity inequality.

Note 3.3. The condition of dissipativity for u_γ is equivalent to the following one: each $\alpha \in u_\gamma$ has a neighbourhood in which a dissipativity inequality holds. This statement is a consequence of the following fact:

Lemma 3.5. *Let Λ be a submanifold of $\mathbf{R}^m$, let $\{u_\beta\}$ be an open covering of Λ ($u_\beta \subset \Lambda$), and let $\{c_\beta\}$ be a family of real numbers. Set*

$$f(\alpha) = \inf_{u_\beta \ni \alpha} c_\beta.$$

Then there exists such a C^∞ function φ on Λ that $\varphi \geqslant f$.

Proof. Any submanifold of $\mathbf{R}^m$ is a locally compact space with a countable base of open sets. Therefore, there exists a sequence $\{K_n\}$ of compact subsets of Λ^n such that

$$K_n \subset \overset{\circ}{K}_{n+1}; \qquad \bigcup_n K_n = \Lambda^n$$

(the sign $\circ$ means the interior of a set). Since K_n is compact, it can be covered by a finite subfamily of $\{u_\beta\}$, which implies that there exists such a $d_n \geqslant 0$ that $f(\alpha) \leqslant d_n$ for $\alpha \in K_n$. Let $A_n = K_n \setminus \overset{\circ}{K}_{n-1}$. A_n is compact and

$$A_n \subset \overset{\circ}{K}_{n+1}, \quad A_n \cap K_{n-2} = \varnothing.$$

It follows that there exists such a smooth function φ_n on Λ that $0 \leqslant \varphi_n \leqslant d_n$ and

$$\varphi_n(\alpha)\begin{cases} d_n & \text{if } \alpha \in A_n, \\ 0 & \text{if } \alpha \notin \overset{\circ}{K}_{n+1} \text{ or } \alpha \in K_{n-2}. \end{cases}$$

The function $\varphi(\alpha) = \sum\limits_{n=1}^{\infty} \varphi_n(\alpha)$ is easily seen to satisfy the requirement of the lemma.

Note 3.4. We can choose ε and c in the dissipativity inequality so that $c(\alpha) = 0$ in a neighbourhood of the set $\Gamma \cap u_\gamma$.

Definition 3.2. *A complex germ γ^n on a Lagrangean manifold Λ^n with a fixed γ-atlas is called dissipative if it is dissipative in each γ-domain.*

Note 3.5. Since $s_I(\alpha)$ is real and the dissipativity condition does not involve the real part of the phase the concept of dissipativity of a germ can be used also even if there exists no s-action on Λ^n.

Theorem 3.1. *The property of a complex germ to be dissipative is independent of the choice of a γ-atlas on Λ^n.*

The result of this theorem may be interpreted to mean that if the inequality (3.4) holds in a neighbourhood $u \subset u_\gamma^I \cap u_\gamma^K$ of a point $\alpha_0 \in \Gamma$, where u_γ^I is a γ-domain of Ω_I and $u_{\gamma'}^K$ is a γ'-domain of Ω_K, then there exist a non-negative function c_1 and a positive function ε_1 such that

$$\operatorname{Im} \Phi_K^{\gamma'} = c_1 D^{3/2} \geqslant \varepsilon_1 D.$$

For $I = K$, this result follows immediately from Lemma 3.4 and the corollary to Lemma 3.3. In the general case, the theorem will be proved in Sec. 5.

From now on the complex germ will be assumed to be dissipative, the word "dissipative" being omitted.

Sec. 4. Solution of the Hamilton-Jacobi Equation with Dissipation

In this section we shall express the solution of the Hamilton-Jacobi equation with dissipation in terms of that of the Hamiltonian system (2.2) and the system (2.4).

Let

$$\Lambda_t^n\colon \{p = p(\alpha, t),\ q = q(\alpha, t)\}, \qquad 0 \leqslant t \leqslant T,$$

be the family of Lagrangean manifolds with complex germs

$$r_t^n\colon \{w = w(\alpha, t),\ z = z(\alpha, t)\},$$

obtained from (Δ_0^n, r_0^n) by the complex canonical transformation associated with a Hamiltonian function $\mathscr{H} = H + i\widetilde{H}$. Let M^{n+1} be the manifold associated with the family $\{\Lambda_t^n\}$. We define a *γ-atlas on* M^{n+1} as follows. Let U_γ be an open subset of M^{n+1} such that its intersection with any plane $t = \text{const}$ is a γ-domain on Λ_t^n (whenever it is non-empty), the type of this γ-domain being independent of t. Define a diffeomorphism

$$\Pi_\gamma^I: (\alpha, t) \to (x, t)$$

of U_γ into $\mathbf{R}^{n+1}$ by the formulas

$$x_I = q_I(\alpha, t) + \gamma_I(\alpha, t), \quad x_{\bar{I}} = p_{\bar{I}}(\alpha, t) + \gamma_{\bar{I}}(\alpha, t).$$

The pair (U_γ, Π_γ^I) will be called a γ-patch of type (I, F) of the manifold M^{n+1}.

Definition 4.1. *We say that a family $\{\Lambda_t^n, r_t^n\}$ is dissipative if for each γ-patch (U_γ, Π_γ^I) there exist smooth functions $\varepsilon > 0$ and $c \geqslant 0$ such that*

$$\varepsilon(\alpha, t) D(\alpha, t) \leqslant \operatorname{Im} \Phi_I^\gamma(\alpha, t) + c(\alpha, t)[D(\alpha, t)]^{3/2}, \ (\alpha, t) \in U_\gamma.$$

The last relation will be called a dissipativity inequality in the patch (U_γ, Π_γ^I).

Theorem 4.1. *If a complex germ r_0^n on a Lagrangean manifold Λ_0^n is dissipative, then the family $\{\Lambda_t^n, r_t^n\}$, where*

$$\Lambda_t^n = g_H^t \Lambda_0^n, \quad r_t^n = (dg_{H+i\widetilde{H}}^t) r_0^n,$$

is dissipative.

In particular, the property of a germ to be dissipative is invariant under complex canonical transformations.

The proof of this theorem will be given in Sec. 5.

Theorem 4.2. *Let M^{n+1} be the manifold associated with the family $\{\Lambda_t^n\}$, and suppose that there is a γ-atlas of M^{n+1} which consists of a single γ-patch (U_γ, Π_γ). Then the function*

$$S = (\Phi^\gamma + icD^{3/2}) \circ \Pi_\gamma^{-1},$$

where Φ^γ is the γ-phase of the germ and c is the same as in the dissipativity inequality, satisfies the relation (1.5).

The following lemma will be needed to prove this theorem.

Lemma 4.1. *There is a smooth function $g(\alpha, t)$ such that*

$$t \max_{0 \leqslant \tau \leqslant t} |\widetilde{H}(p(\alpha, \tau), q(\alpha, \tau), \tau)| \leqslant g(\alpha, t)[D(\alpha, t)]^{2/3}.$$

Proof. Omitting the argument α, set

$$f(t) = -\widetilde{H}(p(\alpha, t), q(\alpha, t), t).$$

Set

$$a = a(t) = \max_{0 \leqslant \tau \leqslant t} f(\tau), \quad a = f(\tau_0), \quad \tau_0 = \tau_0(t).$$

By the Gårding inequality,

$$|f'(\tau)| \leqslant g_1 \sqrt{f(\tau)}.$$

Set $\sqrt{f} = h$, then $|h'| \leqslant g_1/2$. It follows that

$$h(\tau) \geqslant \sqrt{a} - \frac{g_1}{2} |\tau - \tau_0|,$$

which implies

$$h(\tau) \geqslant \frac{\sqrt{a}}{2} \quad \text{for} \quad |\tau - \tau_0| \leqslant \frac{\sqrt{a}}{g_1},$$

that is,

$$f(\tau) \geqslant \frac{a}{4} \quad \text{for} \quad |\tau - \tau_0| \leqslant \frac{\sqrt{a}}{g_1}.$$

Consider the following two cases:

(1) if $\sqrt{a}/g_1 \leqslant t/2$, then

$$D(\alpha, t) \geqslant \frac{a}{4} \cdot \frac{\sqrt{a}}{g_1} = \frac{a^{3/2}}{4g_1},$$

so

$$t \max_{0 \leqslant \tau \leqslant t} f(\tau) \leqslant t (4g_1)^{2/3} [D(\alpha, t)]^{2/3};$$

(2) if $\sqrt{a}/g_1 > t/2$, then $D(\alpha, t) \geqslant \frac{1}{8} ta$, so

$$t \max_{0 \leqslant \tau \leqslant t} f(\tau) \leqslant 8D(\alpha, t) \leqslant c(\alpha, t) [D(\alpha, t)]^{2/3}.$$

The lemma is proved.

Proof of Theorem 4.2. I. Here we shall calculate Φ^γ. To do this rewrite Φ^γ in the form

$$\Phi^\gamma = \frac{1}{2} \langle z - \gamma, \mathscr{E}(z - \gamma) \rangle + \langle p + w, \gamma \rangle - \frac{1}{2} \langle w, z \rangle + s + E.$$

We have

$$\begin{aligned} \dot{\Phi}^\gamma = \frac{1}{2} \langle z - \gamma, \dot{\mathscr{E}}(z - \gamma) \rangle + \langle \dot{z} - \dot{\gamma}, \mathscr{E}(z - \gamma) \rangle + \\ + \langle p + w, \dot{\gamma} \rangle + \langle \dot{p} + \dot{w}, \gamma \rangle - \frac{1}{2} \langle \dot{w}, z \rangle - \\ - \frac{1}{2} \langle w, \dot{z} \rangle + \dot{s} + \dot{E} + \langle \dot{z} - \dot{\gamma}, O_D(h) \rangle. \end{aligned} \tag{4.1}$$

First we calculate

$$\dot{\mathscr{E}} = \dot{B}C^{-1} - BC^{-1}\dot{C}C^{-1}.$$

It follows from (2.6) that

$$\dot{B} = -\mathscr{H}_{qq}C - \mathscr{H}_{qp}B + O_D(h^{1/2}),$$

$$\dot{C} = \mathscr{H}_{pq}C + \mathscr{H}_{pp}B + O_D(h^{1/2}),$$

therefore

$$\dot{\mathscr{E}} = -\mathscr{H}_{qq} - \mathscr{H}_{qp}\mathscr{E} - \mathscr{E}\mathscr{H}_{pq} - \mathscr{E}\mathscr{H}_{pp}\mathscr{E} + O_D(h^{1/2})$$

Now recall that

$$\dot{E} = -i\tilde{H} - \frac{i}{2}[\langle w, \tilde{H}_p\rangle + \langle z, \tilde{H}_q\rangle],$$

$$\dot{s} = -H + \langle p, H_p\rangle$$

by the definitions of a potential and an s-action on Λ_t^n. Set $\xi = w - \mathscr{E}(z-\gamma)$. Substituting the above expressions of $\dot{\mathscr{E}}$, $\dot{s}$, $\dot{E}$ and those of $\dot{p}$, $\dot{w}$ and $\dot{z}$ (see (2.2), (2.4)) into (4.1) and, using the fact that $\mathscr{E}$ is "almost symmetric", we obtain

$$\begin{aligned}
\dot{\Phi}^\gamma = {} & \frac{1}{2}\langle z-\gamma, -\mathscr{H}_{qq}(z-\gamma) - 2\mathscr{H}_{qp}\mathscr{E}(z-\gamma) - \\
& - \mathscr{E}\mathscr{H}_{pp}\mathscr{E}(z-\gamma)\rangle + \langle \mathscr{E}(z-\gamma), i\tilde{H}_p + \mathscr{H}_{pq}z + \mathscr{H}_{pp}w\rangle + \\
& + \langle \dot{\gamma}, p+\xi\rangle - \langle \gamma, \mathscr{H}_q + \mathscr{H}_{qq}w + \mathscr{H}_{qq}z\rangle + \\
& + \frac{1}{2}\langle z, i\tilde{H}_q + \mathscr{H}_{qq}z + \mathscr{H}_{qp}w\rangle - \\
& - \frac{1}{2}\langle w, i\tilde{H}_p + \mathscr{H}_{pq}z + \mathscr{H}_{pp}w\rangle - \mathscr{H} + \langle p, H_p\rangle - \\
& - \frac{i}{2}[\langle w, \tilde{H}_p\rangle + \langle z, \tilde{H}_q\rangle] + \langle i\tilde{H}_p + \dot{\gamma}, O_D(h)\rangle + O_D(h^{3/2}) = \\
= {} & -\mathscr{H} - i\langle \tilde{H}_p, \xi\rangle - \langle \gamma, \mathscr{H}_q\rangle - \frac{1}{2}\langle \gamma, \mathscr{H}_{qq}\gamma\rangle - \\
& - \langle \mathscr{H}_{pq}\gamma, \xi\rangle - \frac{1}{2}\langle \xi, \mathscr{H}_{pp}\xi\rangle + \langle p, H_p\rangle + \\
& + \langle \dot{\gamma}, p+\xi\rangle + \langle i\tilde{H}_p + \dot{\gamma}, O_D(h)\rangle + O_D(h^{3/2}).
\end{aligned}$$

A simple calculation which uses Taylor's expansions of $\mathscr{H}(p, q+\gamma, t)$, $\mathscr{H}_p(p, q+\gamma, t)$ and $\mathscr{H}_{pp}(p, q+\gamma, t)$ in a neighbourhood of (p, q, t) and the relation $\xi = O_D(h^{1/2})$ yields

$$\begin{aligned}
\dot{\Phi}^\gamma = {} & -\mathscr{H}(p, q+\gamma, t) - \langle \mathscr{H}_p(p, q+\gamma, t), \xi\rangle - \\
& - \frac{1}{2}\langle \xi, \mathscr{H}_{pp}(p, q+\gamma, t)\xi\rangle + \langle H_p(p, q, t) + \dot{\gamma}, p+\xi\rangle + \\
& + \langle i\tilde{H}_p + \dot{\gamma}, O_D(h)\rangle + O_D(h^{3/2}).
\end{aligned}$$

Since

$$\operatorname{Re}\dot{z}=O_D(h^{1/2}),\ \operatorname{Im}\dot{z}=\widetilde{H}_p+O_D(h^{1/2})$$

by (2.4), it follows from the definition of $\gamma(\alpha, t)$ that

$$\langle\dot{\gamma},\ O_D(h)\rangle=\langle\widetilde{H}_p,\ O_D(h)\rangle+O_D(h^{3/2}). \tag{4.2}$$

Thus,

$$\begin{aligned}\dot{\Phi}^{\gamma}=&-\mathcal{H}(p,\ q+\gamma,\ t)-\langle\mathcal{H}_p(p,\ q+\gamma,\ t),\ \xi\rangle-\\&-\frac{1}{2}\langle\xi,\ \mathcal{H}_{pp}(p,\ q+\gamma,\ t)\xi\rangle+\\&+\langle H_p(p,\ q,\ t)+\dot{\gamma},\ p+\xi\rangle+\langle\widetilde{H}_p,\ O_D(h)\rangle+O_D(h^{3/2}).\end{aligned} \tag{4.3}$$

II. Given a smooth function φ on M^{n+1}, we shall denote by $\widetilde{\varphi}$ the composite function $\varphi_0\Pi_{\gamma}^{-1}$. Set also

$$\Phi_1^{\gamma}=\operatorname{Re}\Phi^{\gamma},\ \Phi_2^{\gamma}=\operatorname{Im}\Phi^{\gamma},\ \xi_1=\operatorname{Re}\xi,\ \xi_2=\operatorname{Im}\xi,$$

$$\widetilde{f}=\widetilde{\Phi}_x^{\gamma}-\widetilde{p}-\widetilde{\xi},\ f_1=\operatorname{Re}f,\ f_2=\operatorname{Im}f,\ r=cD^{3/2},$$

where c is the same as in the dissipativity inequality. We have:

$$\widetilde{r}_x=O_{\widetilde{D}}(h)\quad\text{by the Gårding inequality,}$$

$$\widetilde{\Phi}_{2x}^{\gamma}=O_{\widetilde{D}}(h^{1/2})\quad\text{by (3.3).}$$

Therefore,

$$\begin{aligned}F_{\mathcal{H}}[S(x,\ t)]=&\widetilde{\Phi}_t^{\gamma}+i\widetilde{r}_t+\mathcal{H}(\widetilde{\Phi}_{1x}^{\gamma},\ x,\ t)+\\&+i\langle\mathcal{H}_p(\widetilde{\Phi}_{1x}^{\gamma},\ x,\ t),\ \widetilde{\Phi}_{2x}^{\gamma}+\widetilde{r}_x\rangle-\\&-\frac{1}{2}\langle\widetilde{\Phi}_{2x}^{\gamma},\ \mathcal{H}_{pp}(\widetilde{\Phi}_{1x}^{\gamma},\ x,\ t)\widetilde{\Phi}_{2x}^{\gamma}\rangle+O_{\widetilde{D}}(h^{3/2}).\end{aligned}$$

Expand $\mathcal{H}$, $\mathcal{H}_p$ and $\mathcal{H}_{pp}$ in this formula in powers of $\widetilde{\xi}+\widetilde{f}_1$; since

$$\widetilde{\xi}=O_{\widetilde{D}}(h^{1/2}),\ \widetilde{\Phi}_{2x}^{\gamma}=\widetilde{\xi}_2+\widetilde{f}_2$$

and $\widetilde{f}=O_{\widetilde{D}}(h)$ by (3.3), we obtain

$$\begin{aligned}F_{\mathcal{H}}[S(x,\ t)]=&\widetilde{\Phi}_t^{\gamma}+i\widetilde{r}_t+\mathcal{H}(\widetilde{p},\ x,\ t)\ +\langle\mathcal{H}_p(\widetilde{p},\ x,\ t),\widetilde{\xi}_1+\widetilde{f}_1\rangle+\\&+\frac{1}{2}\langle\mathcal{H}_{pp}(\widetilde{p},\ x,\ t)\widetilde{\xi}_1,\ \widetilde{\xi}_1\rangle+i\langle\mathcal{H}_p(\widetilde{p},\ x,\ t),\ \widetilde{\xi}_2+\widetilde{f}_2+\widetilde{r}_x\rangle+\\&+i\langle\mathcal{H}_{pp}(\widetilde{p},\ x,\ t)\widetilde{\xi}_1,\ \widetilde{\xi}_2\rangle-\frac{1}{2}\langle\widetilde{\xi}_2,\ \mathcal{H}_{pp}(\widetilde{p},\ x,\ t)\widetilde{\xi}_2\rangle+\\&+O_{\widetilde{D}}(h^{3/2})=\widetilde{\Phi}_t^{\gamma}+\mathcal{H}(\widetilde{p},\ x,\ t)+\\&+\langle\mathcal{H}_p(\widetilde{p},\ x,\ t),\ \widetilde{\xi}+\widetilde{f}\rangle+\frac{1}{2}\langle\widetilde{\xi},\ \mathcal{H}_{pp}(p,\ x,\ t)\widetilde{\xi}\rangle+\\&+i(\widetilde{r}_t+\langle\mathcal{H}_p(\widetilde{p},\ x,\ t),\ \widetilde{r}_x\rangle)+O_{\widetilde{D}}(h^{3/2}).\end{aligned}$$

To calculate $\widetilde{\Phi}^\gamma$, we use (4.3), (4.2) and (3.3):

$$\begin{aligned}
\widetilde{\Phi}_t^\gamma &= \dot{\widetilde{\Phi}}^\gamma - \langle \Phi_x^\gamma, \dot{\widetilde{\gamma}} + H_p(\widetilde{p}, \widetilde{q}, \widetilde{t})\rangle = \\
&= -\mathscr{H}(\widetilde{p}, x, t) - \langle \mathscr{H}_p(\widetilde{p}, x, t), \widetilde{\xi}\rangle - \\
&- \frac{1}{2}\langle \widetilde{\xi}, \mathscr{H}_{pp}(\widetilde{p}, x, t)\widetilde{\xi}\rangle + \langle H_p(\widetilde{p}, \widetilde{q}, t) - \dot{\widetilde{\gamma}}, \widetilde{p} + \xi\rangle - \\
&- \langle \widetilde{\Phi}_x^\gamma, \dot{\widetilde{\gamma}} + H_p(\widetilde{p}, \widetilde{q}, t)\rangle + \langle \widetilde{H}_p(\widetilde{p}, \widetilde{q}, t), O_{\widetilde{D}}(h)\rangle + O_{\widetilde{D}}(h^{3/2}) = \\
&= -\mathscr{H}(\widetilde{p}, x, t) - \langle \mathscr{H}_p(\widetilde{p}, x, t), \widetilde{\xi}\rangle - \\
&- \frac{1}{2}\langle \widetilde{\xi}, \mathscr{H}_{pp}(\widetilde{p}, x, t)\widetilde{\xi}\rangle - \langle H_p(\widetilde{p}, \widetilde{q}, t), \widetilde{f}\rangle + \\
&+ \langle \widetilde{H}_p(\widetilde{p}, \widetilde{q}, t), O_{\widetilde{D}}(h)\rangle + O_{\widetilde{D}}(h^{3/2}).
\end{aligned}$$

Now we note that

$$\begin{aligned}
\widetilde{r}_t + \langle \mathscr{H}_p(\widetilde{p}, x, t), \widetilde{r}_x\rangle &= \widetilde{r}_t + \langle H_p(\widetilde{p}, \widetilde{q}, t), \widetilde{r}_x\rangle + \\
&+ \langle \widetilde{H}_p(\widetilde{p}, \widetilde{q}, t), O_{\widetilde{D}}(h)\rangle + O_{\widetilde{D}}(h^{3/2}) = \dot{\widetilde{r}} - \langle \widetilde{r}_x, \dot{\widetilde{\gamma}}\rangle + \\
&+ \langle \widetilde{H}_p(\widetilde{p}, \widetilde{q}, t), \quad O_{\widetilde{D}}(h)\rangle + O_{\widetilde{D}}(h^{3/2}) = \\
&= \dot{\widetilde{r}} + \langle \widetilde{H}_p(\widetilde{p}, \widetilde{q}, t), \quad O_{\widetilde{D}}(h)\rangle + O_{\widetilde{D}}(h^{3/2}).
\end{aligned}$$

The last three formulas give

$$F_{\mathscr{H}}[S(x, t)] = i\dot{\widetilde{r}} + \langle \widetilde{H}_p(\widetilde{p}, \widetilde{q}, t), O_{\widetilde{D}}(h)\rangle + O_{\widetilde{D}}(h^{3/2}). \tag{4.4}$$

For $t > 0$, the right-hand member of (4.4) is $o_{\widetilde{D}}(h)$. In fact, it follows from Lemma 4.1 that

$$\dot{\widetilde{r}} = -\frac{3}{2}\widetilde{c}\widetilde{D}^{1/2}\widetilde{H}(\widetilde{p}, x, t) + \dot{\widetilde{c}}\widetilde{D}^{3/2} = O_{\widetilde{D}}(h^{7/6}),$$

$$|\widetilde{H}_p(\widetilde{p}, \widetilde{q}, t)| \leqslant c_1(x, t)\sqrt{-\widetilde{H}(\widetilde{p}, \widetilde{q}, t)} \leqslant c_2(x, t)\widetilde{D}^{1/3}.$$

It should be noted that $O_{\widetilde{D}}$-estimate implies O_{S_2}-estimate because the germ is dissipative. The theorem is proved.

Now we turn to the Cauchy problem for the Hamilton-Jacobi equation with dissipation. Let a compact set $K_0 \subset \mathbf{R}^n$ be given, and let $S_0 = S_{10} + iS_{20}$ be a smooth function with the non-negative imaginary part S_{20}, defined in a neighbourhood of K_0. Let (Λ_0^n, r_0^n) be the Lagrangean manifold with the complex germ defined by the formulas

$$q = \alpha, \quad p = \frac{\partial S_{10}(\alpha)}{\partial \alpha}, \; z = 0, \; w = i\frac{\partial S_{20}(\alpha)}{\partial \alpha},$$

$$D(\alpha) = S_{20}(\alpha), \quad E(\alpha) = iS_{20}(\alpha)$$

(it is easy to verify that all the axioms of a complex germ are satisfied). The whole of this manifold lies in the non-singular zone and has a γ-atlas consisting of a single non-singular patch. Let $\mathscr{K}_0$ denote the inverse image of K_0 under π_0:

$$\mathscr{K}_0 = \pi_0^{-1} K_0 \subset \Lambda_0^n,$$

and let K_t be the image of

$$\mathscr{K}_t \overset{\text{def}}{=} g_H^t \mathscr{K}_0 \subset \Lambda_t^n$$

under the canonical projection of the phase space onto the plane $p = 0$.

Our explication of the Cauchy problem consists in finding a smooth function $S(x, t)$ which satisfies (1.5) in a neighbourhood of the set

$$K = \{(x, t) \mid 0 < t < T, \ x \in K_t\}$$

and the initial condition $S(x, 0) = S_0(x)$ in a neighbourhood of K_0. The pair (K_0, S_0) will be called *Cauchy data* for the Hamilton-Jacobi equation with dissipation.

Proposition 4.1. *Given a complex Hamiltonian function, suppose that the germ (Λ_0^n, r_0^n) generated by Cauchy data (K_0, S_0) for Hamilton-Jacobi equation with dissipation is such that the whole of $\mathscr{K}_t$ lies in the non-singular zone of Λ_t^n for $0 \leqslant t \leqslant T$.*

Let $\{U_j, \Pi_{\gamma_j}\}$, $j = 1, \ldots, N$, be a γ-atlas of a neighbourhood of the set

$$\mathscr{K} = \{(\alpha, t) \mid 0 \leqslant t \leqslant T, \ \alpha \in \mathscr{K}_t\} \subset M^{n+1}.$$

Set $V_j = \Pi_{\gamma_j}(U_j)$ and extend the family $\{V_j\}$ to an open covering of K by adding a set V_{N+1} which does not intersect the image of the set of all zeroes of the dissipation under the canonical projection onto the plane $p = 0$. Let $\{e_j\}$ be a partition of unity in K such that $\operatorname{supp} e_j \subset V_j$, *let $S^{(j)}$ be the solution of the Hamilton-Jacobi equation with dissipation given by Theorem 4.2 with (U_γ, Π_γ) replaced by (U_j, Π_{γ_j}) and let $S^{(N+1)}$ be any smooth function with the positive imaginary part.*

Then $S = \sum_{j=1}^{N+1} e_j S^{(j)}$ satisfies the Cauchy problem for the equation (1.5) *with the Cauchy data (K_0, S_0).*

This proposition is a direct consequence of the following lemma.

Lemma 4.2. *Let $S^{(1)}(x, t)$, $S^{(2)}(x, t)$, $\ldots$, $S^{(N)}(x, t)$ satisfy the relations $F_{\mathscr{H}}[S^{(j)}] = o_D(h)$, where D is a smooth non-negative func-*

tion, the relations

$$S^{(j)}-S^{(k)}=O_D(h^{3/2}),\ \operatorname{Im}\frac{\partial S^{(j)}}{\partial x}=O_D(h^{1/2})$$

being also satisfied, and let $e_1, e_2, \ldots, e_N$ *be smooth non-negative functions such that* $e_1+\ldots+e_N=1$. *Then*

$$F_{\mathscr{H}}[\sum_{j=1}^{N}e_jS^{(j)}]=o_D(h).$$

Proof is accomplished by induction on N. Let $N=2$. Set $\chi = S^{(2)}-S^{(1)}$, so $\chi=O_D(h^{3/2})$. We have

$$\frac{\partial}{\partial t}(e_1S^{(1)}+e_2S^{(2)})=S^{(1)}_t+e_2\chi_t+O_D(h^{3/2}),$$

$$\frac{\partial}{\partial t}(e_1S^{(1)}+e_2S^{(2)})=S^{(1)}_x+e_2\chi_x+O_D(h^{3/2}).$$

Set

$$S^{(j)}_1=\operatorname{Re}S^{(j)},\quad \chi_1=\operatorname{Re}\chi,$$

$$S=e_1S^{(1)}+e_2S^{(2)},\quad S_1=\operatorname{Re}S.$$

The expansion of $\mathscr{H}(S_{1x}, x, t)$, $\mathscr{H}_p(S_{1x}, x, t)$ and $\mathscr{H}_{pp}(S_{1x}, x, t)$ in powers of $S_{1x}-S^{(1)}_{1x}$ gives, by virtue of the relation $F_{\mathscr{H}}[S^{(1)}]=O_D(h)$,

$$F_{\mathscr{H}}[S]=\chi_te_2+\langle\mathscr{H}_p(S^{(1)}_{1x}, x, t), \chi_xe_2\rangle+O_D(h).$$

On the other hand, the expansion of $\mathscr{H}(S^{(2)}_{1x}, x, t)$, $\mathscr{H}(S^{(2)}_{1x}, x, t)$ and $\mathscr{H}_{pp}(S^{(2)}_{1x}, x, t)$ in powers of $S^{(2)}_{1x}-S^{(1)}_{1x}=\chi_{1x}$ leads, by using the relation $F_{\mathscr{H}}[S^{(2)}]=O_D(h)$, to

$$\chi_t+\langle\mathscr{H}_p(S^{(1)}_{1x}, x, t), \chi_x\rangle=O_D(h),$$

which implies that $F_{\mathscr{H}}[S]=O_D(h)$.

Now suppose that the assertion of the lemma is true for $N=N_0$ and prove it for $N=N_0+1$. We may assume that $e'\overset{\text{def}}{=}e_1+\ldots+e_{N_0}\neq 0$ (since the statement of the lemma is of local character). We have:

$$e_{N_0+1}+e'=1,$$

$$\sum_{j=1}^{N_0+1}e_jS^{(j)}=e_{N_0+1}S^{(N_0+1)}+e'\sum_{j=1}^{N_0}\frac{e_j}{e'}S^{(j)}.$$

Since $\sum_{j=1}^{N_0} e_j/e' = 1$, it follows from the induction hypothesis that

$$F_{\mathscr{H}}\left[\sum_{j=1}^{N_0} \frac{e_j}{e'} S^{(\gamma)}\right] = O_D(h).$$

To complete the proof use the result for $N = 2$.

Using the canonical transformation associated with the Hamiltonian H_I (see Example 2.1), we obtain the following generalization of Theorem 4.2.

Theorem 4.3. *Let M^{n+1} be the manifold associated with the family $\{\Lambda_t^n\}$, and suppose that there is a γ-atlas of M^{n+1} which consists of a single γ-patch (U_γ, Π_γ^I). Then the function*

$$S = (\Phi_I^\gamma + icD^{3/2}) \circ (\Pi_\gamma^I)^{-1},$$

where Φ^γ is the γ-phase of the germ in the zone Ω_I and c is the same as in the dissipativity inequality, satisfies the following Hamilton-Jacobi equation with dissipation:

$$\begin{aligned}
&\frac{\partial S}{\partial t} + \mathscr{H}\left(\frac{\partial S_1}{\partial x_I}, x_{\bar I}, x_I, -\frac{\partial S_1}{\partial x_{\bar I}}, t\right) + \\
&+ i\left\langle \mathscr{H}_{p_I}\left(\frac{\partial S_1}{\partial x_I}, x_{\bar I}, x_I, -\frac{\partial S_1}{\partial x_{\bar I}}, t\right), \frac{\partial S_2}{\partial x_I}\right\rangle - \\
&- i\left\langle \mathscr{H}_{q_{\bar I}}\left(\frac{\partial S_1}{\partial x_I}, x_{\bar I}, x_I, -\frac{\partial S}{\partial x_{\bar I}}\right), \frac{\partial S_2}{\partial x_{\bar I}}\right\rangle - \\
&- \frac{1}{2}\left\langle \frac{\partial S_2}{\partial x_I}, \mathscr{H}_{p_I p_I}\left(\frac{\partial S_1}{\partial x_I}, x_{\bar I}, x_I, -\frac{\partial S}{\partial x_{\bar I}}\right)\frac{\partial S_2}{\partial x_I}\right\rangle - \\
&- \frac{1}{2}\left\langle \frac{\partial S_2}{\partial x_{\bar I}}, \mathscr{H}_{q_{\bar I} q_{\bar I}}\left(\frac{\partial S_1}{\partial x_I}, x_{\bar I}, x_I, -\frac{\partial S}{\partial x_{\bar I}}\right)\frac{\partial S_2}{\partial x_{\bar I}}\right\rangle + \\
&+ \left\langle \frac{\partial S_2}{\partial x_I}, \mathscr{H}_{p_I q_{\bar I}}\left(\frac{\partial S_1}{\partial x_I}, x_{\bar I}, x_I, -\frac{\partial S}{\partial x_{\bar I}}\right)\frac{\partial S_2}{\partial x_{\bar I}}\right\rangle = o_{S_2}(h).
\end{aligned}$$

Sec. 5. Preservation of the Dissipativity Inequality. Bypassing Focuses Operation

In this section we shall prove Theorems 3.1 and 4.1.

Lemma 5.1. *Theorem 4.1 is valid under the conditions of Theorem 4.2.*

Proof. Let the germ $\{\Lambda_0^n, r_0^n\}$ be dissipative, that is

$$\varepsilon_0(\alpha) D(\alpha, 0) \leqslant \operatorname{Im} \Phi^\gamma(\alpha, 0) + c_0(\alpha)[D(\alpha, 0)]^{3/2},$$

where $\varepsilon_0 > 0$ and c_0 are smooth functions. We wish to show that there exist such smooth functions $\varepsilon(\alpha, t) > 0$ and $c(\alpha, t)$ that

$$\varepsilon(\alpha, t) D(\alpha, t) \leqslant \operatorname{Im} \Phi^\gamma(\alpha, t) = c(\alpha, t)[D(\alpha, t)]^{3/2}$$

for $0 \leqslant t \leqslant T$.

Set as above

$$\xi_1 = \operatorname{Re}(w - \mathscr{E}(z-\gamma)),$$
$$\xi_2 = \operatorname{Im}(w - \mathscr{E}(z-\gamma)),$$
$$\Phi^\gamma = \Phi_1^\gamma + i\Phi_2^\gamma.$$

By virtue of (3.3), we have

$$\frac{\partial \Phi_2^\gamma}{\partial \alpha} = \left[\frac{\partial(q+\gamma)}{\partial \alpha}\right]\xi_2 + O_D(h), \tag{5.1}$$

so

$$\xi_2 = A\frac{\partial \Phi_2^\gamma}{\partial \alpha} + O_D(h) \text{ with } A = {}^t\left[\frac{\partial(q+\gamma)}{\partial \alpha}\right]^{-1}.$$

Taking the imaginary part of (4.3) we obtain

$$\begin{aligned}\dot{\Phi}_2^\gamma = & -\widetilde{H}(p, q+\gamma, t) - \langle H_p(p, q+\gamma, t)\xi_2\rangle - \\ & -\langle \widehat{H}_p(p, q+\gamma, t), \xi_1\rangle - \langle \xi_1, H_{pp}(p, q+\gamma, t)\xi_2\rangle - \\ & -\frac{1}{2}\langle \xi_1, \widetilde{H}_{pp}(p, q+\gamma, t)\xi_1\rangle + \frac{1}{2}\langle \xi_2, \widetilde{H}_{pp}(p, q+\gamma, t)\xi_2\rangle + \\ & +\langle H_p(p, q, t) + \dot{\gamma}\xi_2\rangle + \langle \widetilde{H}_p, O_D(h)\rangle + O_D(h^{3/2}).\end{aligned} \tag{5.2}$$

It follows from (5.1) and (5.2) that Φ_2^γ satisfies the following differential equation:

$$\dot{\Phi}_2^\gamma = G + \langle {}^tA\Phi_{2\alpha}^\gamma, g + \dot{\gamma}\rangle, \tag{5.3}$$

where

$$G = -\widetilde{H}(p + \xi_1 + f, q + \gamma, t) + O_D(h^{3/2}),$$
$$f = O_D(h),$$
$$\begin{aligned}g = & H_p(p, q, t) - H_p(p, q+\gamma, t) - H_{pp}(p, q+\gamma, t)\xi_1 - \\ & -\frac{1}{2}\widetilde{H}_{pp}(p, q+\gamma, t)\xi_2 = O_D(h^{1/2}).\end{aligned}$$

Here we have used (4.2).

Denote by d/dl the vector field

$$\frac{d}{dt} + \left\langle A(g+\dot{\gamma}), \frac{\partial}{\partial \alpha}\right\rangle.$$

With this notation we can rewrite (5.3) in the form

$$\frac{d\Phi_2^\gamma}{dl} = G.$$

Set

$$G_1 = -\widetilde{H}(p + \xi_1 + f, q + \gamma, t),$$

and let ψ be the solution of the following Cauchy problem:

$$\frac{d\psi}{dl} = G_1, \quad \psi(\alpha, 0) = \varepsilon_0(\alpha) D(\alpha, 0).$$

Thus, ψ is defined on each trajectory of the vector field d/dl starting from the initial manifold Λ_0^n. Note that $dt/dl = 1$, so ψ is non-negative for $t \geqslant 0$. Furthermore, ψ does not decrease along every trajectory of the field d/dl. We shall see that ψ may be regarded as a "new dissipation", so the following estimates are valid:

$$\psi(l, t) \leqslant c_1(\alpha, t) D(\alpha, t), \tag{5.4}$$
$$D(\alpha, t) \leqslant c_2(\alpha, t) \psi(\alpha, t).$$

Here we encounter, however, some difficulty, because, in general, ψ is not defined everywhere (the trajectories of the field d/dl need not be continuable). Therefore, we localize the situation as follows: by Note 3.3 it is sufficient to prove the dissipativity inequality in a neighbourhood of an arbitrary point (α_0, t_0). Moreover, only the case $D(\alpha_0, t_0) = 0$ is to be considered, as otherwise the dissipativity inequality holds obviously.

The trajectory of the field d/dl, starting from a point (α_0, t_0) with $D(\alpha_0, t_0) = 0$ coincides for $0 \leqslant t \leqslant t_0$ with that of the field d/dt. Furthermore, we note that each point of a sufficiently small neighbourhood u of (α_0, t_0) lies on a trajectory of the field d/dl starting from Λ_0^n. In fact, let $\alpha = \approx{\alpha}(t)$ be the trajectory of d/dl passing through the point (α_1, t_1), let $\alpha = \widetilde{\alpha}(t)$ be the trajectory of d/dl starting from $(\alpha_0, 0)$, and let δ_0 be a positive number such that $\widetilde{\alpha}(t)$ is defined for $0 \leqslant t \leqslant t_0 + \delta_0$ (note that $\widetilde{\alpha}(t) = \alpha_0$ for $0 \leqslant t \leqslant t_0$). Denote by V the closed domain

$$0 \leqslant t \leqslant t_0 + \delta_0, \qquad |\alpha - \widetilde{\alpha}(t)| \leqslant \delta,$$

and let

$$c = \max_V |F_\alpha(\alpha, t)|, \text{ where } F(\alpha, t) = \frac{d\alpha}{dl}.$$

Since $dt/dl = 1$, we have

$$\frac{d\widetilde{\alpha}}{dt} = F(\widetilde{\alpha}(t), t), \quad \frac{d\overset{\approx}{\alpha}}{dt} = F(\overset{\approx}{\alpha}(t), t),$$

so

$$\left|\frac{d}{dt}(\tilde{\tilde{\alpha}}(t)-\tilde{\alpha}(t))\right|\leqslant c\,|\tilde{\tilde{\alpha}}(t)-\tilde{\alpha}(t)|$$

for t with $(\tilde{\tilde{\alpha}}(t), t)\in V$. It follows that

$$|\tilde{\tilde{\alpha}}(t)-\tilde{\alpha}(t)|\leqslant|\alpha_1-\tilde{\alpha}(t_1)|\,e^{c|t-t_1|}.$$

This estimate implies that the trajectory $\alpha=\tilde{\tilde{\alpha}}(t)$ does not meet the boundary of V for $0<t<t_0+\delta$ if u is sufficiently small. The same argument shows that all segments of trajectories of d/dt (that is, the lines $\alpha=$ const) starting from Λ_0^n and finishing at u lie in the range of definition of ψ.

Now we prove that (5.4) holds in U. We have

$$\begin{aligned}-\tilde{H}(p, q, t)\leqslant G_1+|\langle\tilde{H}_p(p+\xi_1+f, q+\gamma, t), \xi_1\rangle|+\\+|\langle\tilde{H}_q(p+\xi_1+f, q+\gamma, t), \gamma\rangle|+c_3D\leqslant\\\leqslant G_1+c_4|\tilde{H}(p+\xi_1+f, q+\gamma, t)|^{1/2}(|\xi_1|+|\gamma|)+\\+c_3D\leqslant c_5(G_1+D),\end{aligned}\tag{5.5}$$

where c_3, c_4, c_5 are smooth functions of α and t. Since

$$g=O_D(h^{1/2}),\quad \frac{\partial D}{\partial\alpha}=O_D(h^{1/2}),$$

$$\dot{\gamma}=M\tilde{H}_p+O_D(h^{1/2}),$$

where M is a smooth matrix function, and

$$|\tilde{H}_p|\times\left|\frac{\partial D}{\partial\alpha}\right|\leqslant|\tilde{H}|^{1/2}D^{1/2}\leqslant c_6(|\tilde{H}|+D)\leqslant c_7(G_1+D),$$

it follows that

$$\left|\frac{dD}{dl}\right|\leqslant|\tilde{H}|+\left|\left\langle\frac{\partial D}{\partial\alpha},\ A(g+\dot{\gamma})\right\rangle\right|\leqslant c_8(G_1+D).$$

As is seen from Lemma 2.1,

$$D(\alpha_1, t_1)\leqslant c_9(\alpha_1, t_1)\Big(D(\alpha, 0)+\int\limits_{L(\alpha_1, t_1)}G_1dt\Big)\leqslant$$

$$\leqslant c_{10}(\alpha, t)\Big(\varepsilon_0(\alpha_1)D(\alpha_1, 0)+\int\limits_{L(\alpha_1, t_1)}G_1dt\Big)=c_{11}(\alpha_1, t_1)\,\psi(\alpha_1, t_1),$$

where $L(\alpha_1, t_1)$ is the segment of a trajectory of d/dl starting from Λ_0^n and finishing with (α_1, t_1). This proves the second estimate in (5.4).

On the other hand,

$$\left|\frac{d\psi}{dt}\right| = \left|G_1 - \left\langle \frac{\partial\psi}{\partial\alpha}, \quad A(g+\dot{\gamma})\right\rangle\right| \leqslant$$
$$\leqslant G_1 + c_{12}\psi^{1/2}(D^{1/2} + |\widetilde{H}|^{1/2}) \leqslant c_{13}(G_1 + \psi + D). \tag{5.6}$$

The calculation similar to that in (5.5) shows that

$$G_1 = -\widetilde{H}(p+\xi_1+f, q+\gamma, t) \leqslant c_{14}(-\widetilde{H}+D). \tag{5.7}$$

By Lemma 2.1 we obtain from (5.6) and (5.7):

$$\psi(\alpha, t) \leqslant c_{15}\left(D(\alpha, 0) - \int_0^t \widetilde{H}\, dt\right) = c_{15}D(\alpha, t),$$

and (5.4) is proved.

Now we are able to obtain the required inequalities for Φ_2^γ. We have

$$\frac{d(\psi - \Phi_2^\gamma)}{dl} = O_D(h^{3/2}),$$

so

$$\psi(\alpha_1, t_1) - \Phi_2^\gamma(\alpha_1, t_1) \leqslant |\psi(\alpha_1, 0) - \Phi_2^\gamma(\alpha_1, 0)| +$$
$$+ M_1(\alpha_1, t_1) \int_{L(\alpha, t)} D^{3/2}\, dt \leqslant |\varepsilon_0(\alpha) D(\alpha_1, 0) - \Phi_2^\gamma(\alpha_1, 0)| +$$
$$+ M_2(\alpha_1, t_1) \int_{L(\alpha, t)} \psi^{3/2}\, dt \leqslant$$
$$\leqslant |\varepsilon_0(\alpha_1) D(\alpha_1, 0) - \Phi_2^\gamma(\alpha_1, 0)| + M_3(\alpha_1, t_1) \times$$
$$\times [\psi(\alpha_1, t_1)] \leqslant c_0(\alpha_1)[D(\alpha_1, 0)]^{3/2} +$$
$$+ M_4(\alpha_1, t_1)[D(\alpha_1, t_1)]^{3/2} \leqslant c(\alpha_1, t_1)[D(\alpha_1, t_1)]^{3/2}.$$

Thus,

$$\frac{1}{c_2} D \leqslant \psi \leqslant \Phi_2^\gamma + cD^{3/2}.$$

The lemma is proved.

Note 5.1. In the same way it can be proved that Theorem 4.1 is valid under the condition of Theorem 4.3.

Corollary. *Theorem 4.1 is valid if $\Lambda_t^n \subset \Omega_I$ for any t (with I being independent of t).*

Proof. Let $L: \{\alpha = \alpha_0, 0 \leqslant t \leqslant T\}$ be a segment of the trajectory of d/dt starting with $\alpha_0 \in \Lambda_0^n$. Then there exist such numbers $t_0, t_1, \ldots, t_N$ with $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$ that the

segment L_j of L corresponding to $[t_{j-1}, t_j]$ lies in U_j, where (U_j, Π_γ^I) is a γ-patch of M^{n+1}, Π_γ^I, $(\alpha, t)^j = (q(\alpha, t) + \gamma_j(\alpha, t), t)$. By Note 5.1 we can choose smooth functions $\varepsilon_1 > 0$ and c_1 in such a way that the dissipativity inequality

$$\varepsilon_1 D \leqslant \Phi_I^{\gamma_1} + c_1 D^{3/2}$$

holds in a neighbourhood of L_1 in M^{n+1}.

Since the property of a germ to be dissipative does not depend on the choice of a γ-atlas (we have proved this in the case of a γ-atlas consisting of patches of the same zone), we conclude that the dissipativity inequality

$$\varepsilon' D \leqslant \Phi_{I_2}^{\gamma_2} + c_1' D^{3/2}$$

with some ε' and c_1' holds in a neighbourhood of (α_0, t_1). It follows from Note 5.1 that there exist smooth functions $\varepsilon_2 > 0$ and c_2 such that

$$\varepsilon_2 D \leqslant \Phi_{I_2}^{\gamma_2} + c_2 D^{3/2}$$

in a neighbourhood of L_2 in M^{n+1}, and so on.

Now let us return to Theorem 3.1. Suppose, for the sake of simplicity, that $\alpha_0 \in \Gamma$ belongs to the intersection of a non-singular patch of the non-singular zone with a non-singular patch of Ω_I, and let the dissipativity inequality for the non-singular zone

$$\varepsilon D \leqslant \Phi_2 + c D^{3/2}$$

with some ε and c hold in a neighbourhood of α_0. We have to show that the dissipativity inequality for the zone Ω_I

$$\varepsilon' D \leqslant \Phi_{I_2} + c' D^{3/2} \tag{5.8}$$

with some ε' and c' holds in a neighbourhood of α_0.

Let $g_{H_I}^t$ be the canonical transformation considered in Example 2.1, and let $\Phi(\alpha, t)$ be the phase on $g_{H_I}^t \Lambda^n$. Then $\Phi_I(\alpha) = \Phi(\alpha, \pi/2)$. If the matrix $\partial(q(\alpha, t) + z(\alpha, t))/\partial\alpha$ is non-degenerate for $\alpha = \alpha_0$ and $0 \leqslant t \leqslant \frac{\pi}{2}$, then the validity of (5.1) follows from the corollary to Lemma 5.1.

If, however, for $t \in (0, \pi/2)$ the image of α_0 by the canonical transformation $g_{H_I}^t$ is a zero of the Jacobian $D(q + z)/D\alpha$ (such a point (α_0, t) is said to be focal), then the above argument fails. Note that even if the transformation $g_{H_I}^{\pi/2}$ (which is the rotation through 90° of some axes in the phase space) were embedded in the family of canonical transformations g_H^t associated with another Hamiltonian function H, then, nevertheless, the trajectory of the

Hamiltonian vector field d/dt starting with α_0 would be sure to have focal points.

It turns out that this difficulty can be avoided by using some other rotations of axes. This method will be called "bypassing focuses operation".

Definition 5.1. *Let $\alpha_0 \in \Gamma \subset \Lambda^n$ belong to the intersection of a non-singular patch of the zone Ω_K with a non-singular patch of the zone Ω_I. A real Hamiltonian function H will be called a bypassing focuses Hamiltonian associated with the point α_0 and the pair (K, I) if the following conditions are satisfied:*

(i) $\det C_K(\alpha, t) \neq 0 \quad \text{for } 0 \leqslant t \leqslant \frac{\pi}{2}$,

(ii) $\Phi_K\left(\alpha, \frac{\pi}{2}\right) = \Phi_I(\alpha)$ *in a neighborhood*

of α_0, where $C_K(\alpha, t)$, and $\Phi_K(\alpha, t)$ are the matrix introduced in the definition of μ-action and the phase in the zone Ω_I of $\overset{t}{g}_H\Lambda^n$, respectively.

Theorem 5.1. *Let $\alpha_0 \in \Gamma$ belong to the intersection of a non-singular patch of the non-singular zone with a non-singular patch of the zone Ω_I, and let Φ satisfy a dissipativity inequality in a neighborhood of α_0. Then there exist an orthogonal matrix $u = (u_{ij})_{i, j\in\bar{I}}$ and a diagonal matrix $\varepsilon = (\varepsilon_{ij})_{i, j\in\bar{I}}$ with $\varepsilon_{ii} \overset{\text{def}}{=\!=} \varepsilon_i = \pm 1$ such that the function*

$$H = \frac{1}{2}\{\langle p_{\bar{I}}, u^{-1}\varepsilon u p_{\bar{I}}\rangle + \langle q_{\bar{I}}, u^{-1}\varepsilon u q_{\bar{I}}\rangle\} \tag{5.9}$$

is a bypassing focuses Hamiltonian associated with α_0 and $(\bar{\Phi}, I)$, the image of α_0 by $\overset{t}{g}_H$ belonging to a non-singular patch of the non-singular zone of $\overset{t}{g}_H\Lambda^n$ for $t \in \left[0, \frac{\pi}{2}\right]$.

Proof. To simplify the notation we shall renumber the coordinates in the phase space so that $\bar{I}$ should be equal to $\{1, \ldots, k\}$. Set

$$A(\alpha) = p_\alpha q_\alpha^{-1} = p_q,$$

then $A(\alpha)$ is symmetric. Denote by $A^{(k)}$ the left upper block of A. Choose an orthogonal matrix u so that the matrix $uA^{(k)}u^{-1}$ is diagonal, and a diagonal matrix with $\varepsilon_{ii} = \pm 1$ so that the condition

$$\varepsilon u A^{(k)} u^{-1} \geqslant 0$$

is satisfied. Define the $n \times n$ matrices $\hat{u}$ and $\hat{\varepsilon}$ by

$$\hat{u} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\varepsilon} = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us consider the canonical transformation g generated by the orthogonal operator $\hat{u}$ in $\mathbf{R}_q^n$. This transformation is given by

$$p'(\alpha) = \hat{u}p(\alpha), \qquad q'(\alpha) = \hat{u}q(\alpha),$$
$$w'(\alpha) = \hat{u}w(\alpha), \qquad z'(\alpha) = \hat{u}z(\alpha),$$
$$D'(\alpha) = D(\alpha), \qquad E'(\alpha) = E(\alpha)$$

(we mark off functions on $g\Lambda^n$ by $'$). We have:

$$\Phi'(\alpha) = \Phi(\alpha), \quad C_I' = \hat{u}C_I, \quad B_I' = \hat{u}B_I, \quad \varepsilon_I' = \hat{u}\varepsilon_I\hat{u}^{-1},$$

in particular, the image of α_0 by g belongs to $\Omega \cap \Omega_I$. It is obvious that the function (5.9) is a bypassing focuses Hamiltonian associated with α_0 and (Ω, Ω_I) if and only if the function

$$H' = \frac{1}{2}\left(\langle p_{\bar{I}}, \ \varepsilon p_{\bar{I}}\rangle + \langle q_{\bar{I}}, \ \varepsilon q_{\bar{I}}\rangle\right) \tag{5.10}$$

is a bypassing Hamiltonian associated with $g\alpha_0$ and (Ω, Ω_I). Therefore, we shall omit strokes, assuming from the outset that A^k is diagonal and putting $u = 1$ in (5.9).

On solving the Hamilton-Jacobi system and the system (2.4) correspoding to the Hamiltonian function (5.10) we obtain

$$p_{\bar{I}}(\alpha, t) = p_{\bar{I}}(\alpha)\cos t - \varepsilon q_{\bar{I}}(\alpha)\sin t, \ \ p_I(\alpha, t) = p_I(\alpha),$$
$$q_{\bar{I}}(\alpha, t) = q_{\bar{I}}(\alpha)\cos t + \varepsilon p_{\bar{I}}(\alpha)\sin t, \ \ q_I(\alpha, t) = q_I(\alpha),$$
$$w_{\bar{I}}(\alpha, t) = w_{\bar{I}}(\alpha)\cos t - \varepsilon z_{\bar{I}}(\alpha)\sin t, \ \ w_I(\alpha, t) = w_I(\alpha),$$
$$z_{\bar{I}}(\alpha, t) = z_{\bar{I}}(\alpha)\cos t + \varepsilon w_{\bar{I}}(\alpha)\sin t, \ \ z_I(\alpha, t) = z_I(\alpha).$$

It follows that

$$p_{\bar{I}}(\alpha, \pi/2) = -\varepsilon q_{\bar{I}}(\alpha), \qquad w_{\bar{I}}(\alpha, \pi/2) = -\varepsilon z_{\bar{I}}(\alpha),$$
$$q_{\bar{I}}(\alpha, \pi/2) = \varepsilon p_{\bar{I}}(\alpha), \qquad z_{\bar{I}}(\alpha, \pi/2) = \varepsilon w_{\bar{I}}(\alpha),$$
$$\langle w(\alpha, \pi/2), \ z(\alpha, \pi/2)\rangle = \langle -\varepsilon z_{\bar{I}}(\alpha), \ \varepsilon w_{\bar{I}}(\alpha)\rangle + \langle w_I(\alpha), \ z_I(\alpha)\rangle =$$
$$= -\langle z_{\bar{I}}(\alpha), \ w_{\bar{I}}(\alpha)\rangle + \langle w_I(\alpha), \ z_I(\alpha)\rangle = \langle w(\alpha), z(\alpha)\rangle_I. \tag{5.11}$$

Set $P = p + w$, $Q = q + z$. Then

$$C(\alpha, t) = \begin{pmatrix} \dfrac{\partial Q_{\bar{I}}(\alpha)}{\partial \alpha}\cos t + \varepsilon \dfrac{\partial P_{\bar{I}}(\alpha)}{\partial \alpha}\sin t \\ \dfrac{\partial Q_I(\alpha)}{\partial \alpha} \end{pmatrix}.$$

So

$$C(\alpha, \pi/2) = \begin{pmatrix} \dfrac{\varepsilon \partial P_{\bar{I}}(\alpha)}{\partial \alpha} \\ \dfrac{\partial Q_I(\alpha)}{\partial \alpha} \end{pmatrix} = \hat{\varepsilon}\,\frac{\partial (P_{\bar{I}}(\alpha), Q_I(\alpha))}{\partial \alpha} = \hat{\varepsilon} C_I(\alpha).$$

In the same way we obtain

$$B(\alpha, \pi/2) = \hat{\varepsilon} B_I(\alpha).$$

Therefore

$$\mathscr{E}(\alpha, \pi/2) = \hat{\varepsilon}\mathscr{E}_I(\alpha)\,\hat{\varepsilon}.$$

By noting that $\hat{\varepsilon}^2 = 1$, we obtain

$$\begin{aligned}\mu(\alpha, \pi/2) &= \frac{1}{2}\langle z(\alpha, \pi/2),\ \mathscr{E}(\alpha, \pi/2)\, z(\alpha, \pi/2)\rangle = \\ &= \frac{1}{2}\left\langle \hat{\varepsilon}\begin{pmatrix} w_{\bar{I}}(\alpha) \\ z_I(\alpha)\end{pmatrix},\ \hat{\varepsilon}\mathscr{E}_I(\alpha)\,\hat{\varepsilon}\,\hat{\varepsilon}\begin{pmatrix} w_{\bar{I}}(\alpha) \\ z_I(\alpha)\end{pmatrix}\right\rangle = \\ &= \frac{1}{2}\left\langle \begin{pmatrix} w_{\bar{I}}(\alpha) \\ z_I(\alpha)\end{pmatrix},\ \mathscr{E}_I(\alpha)\begin{pmatrix} w_{\bar{I}}(\alpha) \\ z_I(\alpha)\end{pmatrix}\right\rangle = \mu_I(\alpha).\end{aligned} \tag{5.12}$$

We also have

$$\begin{aligned}s(\alpha, \pi/2) &= s(\alpha) + \frac{1}{2}\int_0^{\pi/2} \{\langle p_{\bar{I}}(\alpha, \tau),\ \varepsilon p_{\bar{I}}(\alpha, \tau)\rangle - \\ &\quad - \langle q_{\bar{I}}(\alpha, \tau),\ \varepsilon q_{\bar{I}}(\alpha, \tau)\rangle\}\, d\tau = \\ &= s(\alpha) + \frac{1}{2}\int_0^{\pi/2} \{\langle p_{\bar{I}}(\alpha) - q_{\bar{I}}(\alpha),\ \varepsilon(p_{\bar{I}}(\alpha) - q_{\bar{I}}(\alpha))\rangle \cos 2\tau - \\ &\quad - 2\langle p_{\bar{I}}(\alpha),\ q_{\bar{I}}(\alpha)\sin 2\tau\rangle\}\, d\tau = \\ &= s(\alpha) - \langle p_{\bar{I}}(\alpha),\ q_{\bar{I}}(\alpha)\rangle = s_I(\alpha).\end{aligned}$$

Since the potential is independent of t, the first assertion of the proposition is proved.

Now it is easily seen that

$$C(\alpha, t) = \begin{pmatrix} 1_{\bar{I}} & 0 \\ 0 & \dfrac{1}{\sin t + \cos t}\, 1_I \end{pmatrix}(C\cos t + \hat{\varepsilon} C_I \sin t),$$

where $1_{\bar{I}}$ is the identity $k \times k$ matrix and 1_I is the identity $(n-k)\times(n-k)$ matrix. Hence for any $t \in \left[0, \frac{\pi}{2}\right]$, the matrix $C(\alpha, t)$ is non-degenerate if and only if $C(\alpha)\cos t + \hat{\varepsilon} C_I(\alpha)\sin t$ is non-degenerate.

We are thus led to the criterion: H is a bypassing focused Hamiltonian if and only if the matrix $\hat{\varepsilon} C_I(\alpha_0)\, C^{-1}(\alpha_0)$ has no negative eigenvalue.

We have

$$\hat{\varepsilon C}_I C^{-1} = \begin{pmatrix} \dfrac{\partial P_{\bar{I}}}{\partial Q_{\bar{I}}} & \varepsilon \dfrac{\partial P_{\bar{I}}}{\partial Q_I} \\ 0 & 1 \end{pmatrix}. \tag{5.13}$$

Denote by $\mathscr{E}^{(k)}$ the upper left $k \times k$ block of $\mathscr{E}(\alpha_0)$:

$$\mathscr{E}^{(k)} = \frac{\partial P_{\bar{I}}}{\partial Q_{\bar{I}}}(\alpha_0).$$

By (5.13), it is sufficient for non-existence of negative eigenvalues of $\hat{\varepsilon C}_I C^{-1}|_{\alpha=\alpha_0}$ that $\varepsilon\mathscr{E}^{(k)}$ have no negative eigenvalue because the spectrum of the latter matrix contains that of $\hat{\varepsilon C}_I(\alpha_0) C_I^{-1}(\alpha_0)$. In fact, let x be the left eigenvector of $\hat{\varepsilon C}_I(\alpha_0) C^{-1}(\alpha_0)$ belonging to an eigenvalue λ, then

$$x_{\bar{I}}\varepsilon\mathscr{E}^{(k)} = \lambda x_{\bar{I}},$$

$$x_{\bar{I}}\varepsilon \frac{\partial P_{\bar{I}}}{\partial Q_{\bar{I}}}(\alpha_0)\, x_I = \lambda x_I.$$

Note that $x_{\bar{I}} \neq 0$, for otherwise the latter equation implies that $x_I = 0$, so $x = 0$. Thus, x_I is the left eigenvector of $\varepsilon\mathscr{E}^{(k)}$ belonging to the eigenvalue λ.

Similarly, for the matrix $q_\alpha(\alpha, t)$ to be non-degenerate when $t \in \left[0, \frac{\pi}{2}\right]$, it suffices for the matrix $\varepsilon A^{(k)}$ to have no negative eigenvalue. The last is ensured by the choice of ε. We shall show that the same choice of ε ensures that $\varepsilon\mathscr{E}^{(k)}$ has no negative eigenvalue either.

Let us accept the following agreement: for the rest of the proof all differentials in question are taken at α_0. Let E_0 be the subspace of $\mathbf{R}^n$ consisting of all vectors annihilating d^2D. Then the relations $z = O_D(h^{1/2})$ and $w = O_D(h^{1/2})$ imply that $dz|_{E_0} = 0$ and $dw|_{E_0} = 0$. In fact, let $f = O_D(h^{1/2})$ be a real function, then

$$d^2f^2(u) = 2\,[df(u)]^2$$

for each u. Since $f^2 \leqslant cD$ and $f^2 - cD$ has a maximum at α_0 it follows that

$$d^2f^2 - cd^2D \leqslant 0,$$

hence

$$[df(u)]^2 \leqslant \text{const}\; d^2D(u),$$

which implies immediately that

$$(u \in E_0) \Rightarrow (df(u) = 0).$$

Now suppose that $\varepsilon\mathscr{E}^{(k)}$ has a negative eigenvalue:

$$\varepsilon\mathscr{E}^{(k)}(u+iv) = -\lambda(u+iv), \tag{5.14}$$

where $\lambda > 0$. Rewrite (5.14) in the form of a system of two equations:

$$\left.\begin{aligned} \mathscr{E}_1^{(k)}u - \mathscr{E}_2^{(k)}v &= -\lambda\varepsilon u, \\ \mathscr{E}_1^{(k)}v + \mathscr{E}_2^{(k)}u &= -\lambda\varepsilon v, \end{aligned}\right\}$$

where $\mathscr{E}_1^{(k)}$ and $\mathscr{E}_2^{(k)}$ are the real symmetric matrices defined by $\mathscr{E}^{(k)} = \mathscr{E}_1^{(k)} + i\mathscr{E}_2^{(k)}$. Multiplying the first equation by v and subtracting it from the second one multiplied by u we obtain

$$\langle\mathscr{E}_2^{(k)}u; u\rangle + \langle\mathscr{E}_2^{(k)}v, v\rangle = 0. \tag{5.15}$$

Let us choose q as local coordinates at α_0. Then $d^2\Phi_2(u) = \langle\mathscr{E}_2 u, u\rangle$. By the dissipativity inequality,

$$d^2\Phi_2 \geqslant cd^2D \geqslant 0,$$

where c is a positive number. If $u' \in \mathbf{R}^n$ and $v' \in \mathbf{R}^n$ are defined by

$$u'_{\bar{I}} = u, \ v'_{\bar{I}} = v, \ u'_I = 0, \ v'_I = 0,$$

then

$$\langle\mathscr{E}_2^{(k)}u, u\rangle = d^2\Phi_2(u') \geqslant cd^2D(u') \geqslant 0,$$

$$\langle\mathscr{E}_2^{(k)}v, v\rangle = d^2\Phi_2(v') \geqslant cd^2D(v') \geqslant 0.$$

Hence (5.15) implies that

$$\langle\mathscr{E}_2^{(k)}u, u\rangle = \langle\mathscr{E}_2^{(k)}v, v\rangle = 0, \tag{5.16}$$

$$u' \in E_0, \qquad v' \in E_0.$$

Since $\mathscr{E}_2^{(k)} \geqslant 0$, it follows from (5.16) that $\mathscr{E}_2^{(k)}u = \mathscr{E}_2^{(k)}v = 0$; hence $u + iv$ is an eigenvector of the real matrix $\varepsilon\mathscr{E}_1^{(k)}$ belonging to the eigenvalue $-\lambda$. At least one of the two real vectors u and v must be non-zero. Let, for example, $u \neq 0$. Then

$$\mathscr{E}^{(k)}u = -\varepsilon\lambda u, \qquad u' \in E_0. \tag{5.17}$$

Let us show that

$$\mathscr{E}^{(k)}u = A^{(k)}u.$$

We have

$$C(\alpha_0)u' = [1 + z_q(\alpha_0)]u' = u' + dz(u') = u',$$

hence $C^{-1}(\alpha_0)u' = u'$. Next,

$$\mathscr{E}(\alpha_0)u' = B(\alpha_0)C^{-1}(\alpha_0)u' = B(\alpha_0)u' =$$
$$= p_q(\alpha_0)u' + dw(u') = p_q(\alpha_0)u'.$$

Since $u'_I = 0$, we obtain

$$[p_q(\alpha_0)\,u']_{\bar I} = A^{(k)}u$$

and

$$[\mathscr{E}(\alpha_0)\,u']_{\bar I} = \mathscr{E}^{(k)}u.$$

It follows from (5.17) and (5.18) that

$$A^{(k)}u = -\varepsilon\lambda u,$$

contrary to the condition $\varepsilon A^{(k)} \geqslant 0$. The theorem is proved.

Corollary. *Let $\alpha_0 \in \Gamma$ belong to the intersection of a non-singular patch of Ω_K with that of Ω_I, the phase Φ_K satisfying at α_0 the dissipativity inequality*

$$\varepsilon D \leqslant \operatorname{Im}\Phi_K + cD^{3/2}.$$

Then there exist smooth functions $\varepsilon_1 > 0$ and c_1 such that the dissipativity inequality

$$\varepsilon_1 D \leqslant \operatorname{Im}\Phi_I + c_1 D^{3/2} \tag{5.19}$$

holds in a neighborhood of α_0.

In fact, in the case with $\bar K = \varnothing$ this statement is a direct corollary of Lemma 5.1 because a bypassing focuses Hamiltonian exists. The general case can be reduced to that with $\bar K = \varnothing$ by applying the canonical transformation $g^{\pi/2}_{H_K}$, where H_K is the same as in Example 2.1.

Now let us turn to the singular patches case. Our treatment depends on the following observation: the image of $\alpha_0 \in \Lambda^n$ by an appropriate "small" canonical transformation is in a non-singular patch of the zone Ω_I for every I. To state this result more precisely we introduce the following definition.

Definition 5.2. *A point α_0 of a Lagrangean manifold*

$$\Lambda^n = \{q = q(\alpha),\ p = p(\alpha)\}$$

is said to be non-singular with respect to the canonical projection onto the coordinate plane $p_I = 0$, $q_{\bar I} = 0$ of the phase space if

$$\left.\frac{D(q_I, p_{\bar I})}{D\alpha}\right|_{\alpha=\alpha_0} \neq 0,$$

i.e., if the mapping

$$\alpha \to (q_I(\alpha),\ p_{\bar I}(\alpha))$$

is a diffeomorphism of a neighborhood of α_0.

Proposition 5.1. *Let Λ be a Lagrangean manifold. Then for any $\alpha_0 \in \Lambda$ and sufficiently small positive t, the point*

$$g_H^t \alpha_0 \in g_H^t \Lambda, \quad \text{where } H = \frac{1}{2}(p^2 + q^2),$$

is non-singular with respect to the canonical projection onto the coordinate plane $q_{\bar{I}} = 0$, $p_I = 0$ of th' phase space.

Note 5.2. Proposition 5.1 remains valid if H is any Hamiltonian function with $\frac{\partial^2 H}{\partial q_{\bar{I}} \partial p_I}(p(\alpha_0), q(\alpha_0)) > 0$.

This result is essentially the same as the known lemma of M. Morse which states that focal points of a submanifold of a Riemannian space corresponding to a geodesical are isolated in **R**.

Proof of Proposition 5.1. It suffices to check the result for $\bar{I} = \varnothing$. In fact we have the following commutative diagram

$$\begin{tikzcd} \Lambda \arrow[r, "g_{H_I}^{\pi/2}"] \arrow[d, "g_H^t"'] & \Lambda' \arrow[d, "g_H^t"] \\ \Lambda_t \arrow[r, "g_{H_I}^{\pi/2}"] & \Lambda_t' \end{tikzcd}$$

where H_I is the Hamiltonian function introduced in Example 2.1. Since $g_{H_I}^{\pi/2}$ sends $(p, q) \in \mathbf{R}^{2n}$ into (p', q') with $p' = (p_I, -q_{\bar{I}})$, $q' = (q_I, p_{\bar{I}})$, the point $g_H^t \alpha_0 \in g_H^t \Lambda$ is non-singular with respect to the canonical projection onto the plane $p_I = 0$, $q_{\bar{I}} = 0$ if, and only if, the point $(g_H^t \circ g_{H_I}^{\pi/2})\, \alpha_0 \in \Lambda_t'$ is non-singular with respect to the canonical projection onto the plane $p = 0$.

So let $\bar{I} = \varnothing$ and let Λ_t be given by $p = p(\alpha, t)$, $q = q(\alpha, t)$. We have to show that the matrix $\partial q(\alpha, t)/\partial \alpha$ is non-degenerate for $\alpha = \alpha_0$ and sufficiently small positive t. We have

$$\frac{\partial q(\alpha, t)}{\partial \alpha} = \frac{\partial q(\alpha)}{\partial \alpha} \cos t + \frac{\partial p(\alpha)}{\partial \alpha} \sin t = \cos t \left(\frac{\partial q(\alpha)}{\partial \alpha} + \tan t \frac{\partial p(\alpha)}{\partial \alpha} \right).$$

Set

$$\left.\frac{\partial q(\alpha)}{\partial \alpha}\right|_{\alpha=\alpha_0} = C, \quad \left.\frac{\partial p(\alpha)}{\partial \alpha}\right|_{\alpha=\alpha_0} = B,$$
$$\tan t = \lambda.$$

Thus, what we have to show is reduced to

$$\det(C + \lambda B) \neq 0$$

for small positive λ. Let rank $C = k$. We may assume without loss of generality that the first k rows of C are linearly independent. Then the matrix

$$\begin{pmatrix} \partial q_I/\partial\alpha \\ \partial p_{\bar{I}}/\partial\alpha \end{pmatrix}_{\alpha=\alpha_0}$$

with $I = \{1, 2, \ldots, k\}$ is non-degenerate (cf. the proof of Lemma 2.4). Choose q_I, $p_{\bar{I}}$ as local coordinates in Λ at α_0. Then we can express B and C in the form

$$B = \begin{pmatrix} \dfrac{\partial p_I}{\partial q_I} & \dfrac{\partial p_I}{\partial p_{\bar{I}}} \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ \dfrac{\partial q_{\bar{I}}}{\partial q_I} & 0 \end{pmatrix}$$

(for the rest of the proof the argument α_0 will be omitted everywhere), so non-degeneracy of $C + \lambda B$ is equivalent to that of

$$M(\lambda) = \begin{pmatrix} 1 + \lambda^{\partial p_I/\partial q_I} & \dfrac{\partial p_I}{\partial p_{\bar{I}}} \\ \dfrac{\partial q_{\bar{I}}}{\partial q_I} & 1 \end{pmatrix}.$$

Since $M(\lambda)$ is continuous at 0, it is sufficient to check that $M(0)$ is non-degenerate. Let $s_I(q_I, p_{\bar{I}})$ be a generating function of the Lagrangean manifold Λ (an s-action):

$$q_{\bar{I}} = -\frac{\partial s_I}{\partial p_{\bar{I}}}, \qquad p_I = \frac{\partial s}{\partial q_I}.$$

Then

$$\frac{\partial p_I}{\partial p_{\bar{I}}} = \frac{\partial^2 s_I}{\partial q_I\, \partial p_{\bar{I}}}, \qquad \frac{\partial q_{\bar{I}}}{\partial q_I} = -\frac{\partial^2 s_I}{\partial p_{\bar{I}}\, \partial q_I},$$

so to complete the proof, it suffices to apply the following result:

Lemma 5.2. *Let A be an $m \times n$ matrix. Then the matrix*

$$\mathcal{A} = \begin{pmatrix} 1 & A \\ -{}^tA & 1 \end{pmatrix}$$

is non-degenerate.

Proof. Suppose that this is not the case and let u be a non-zero vector such that $\mathcal{A}u = 0$. Write u in the form

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with $u_1 \in \mathbf{R}^n$, $u_2 \in \mathbf{R}^m$. Then

$$u_1 + Au_2 = 0, \quad -{}^tAu_1 + u_2 = 0,$$

which implies that

$$\langle u_1, u_2 \rangle = -\langle A\,{}^tAu_1, u_1 \rangle = -\langle {}^tAu_1, {}^tAu_1 \rangle.$$

Thus $u_1 = 0$, contradicting the assumption $u \neq 0$.

Proof of Theorem 3.1. We can now give the promised proof of Theorem 3.1. We have to show that if $\alpha_0 \in \Gamma$ is in the intersection of two patches (u_γ, π_γ^K) and $(u_{\gamma'}, \pi_{\gamma'}^I)$ of types (K, F) and (I, F'), respectively, Φ_K^γ satisfying a dissipativity inequality in a neighborhood of α_0, then there are $\varepsilon > 0$ and a such that

$$\varepsilon D \leqslant \operatorname{Im} \Phi_I^{\gamma'} + cD^{3/2}$$

near α_0.

Let

$$\varepsilon_0 (\alpha) D (\alpha) \leqslant \operatorname{Im} \Phi_K^\gamma (\alpha) + c_0 (\alpha) [D (\alpha)]^{3/2}$$

in a neighborhood of α_0. Set $H = \frac{1}{2}(p^2 + q^2)$ and consider the canonical transformation g_H^t, where t is so small that the image of α_0 under g_H^t is in the intersection of a non-singular patch of Ω_K with that of Ω_I (such a t exists by Proposition 5.1), being yet in the intersection of a patch of type (K, F) with that of (I, F'). Then for α near α_0 we have

$$\varepsilon_1 (\alpha, t) D (\alpha) \leqslant \operatorname{Im} \Phi_K^\gamma (\alpha, t) + c_1 (\alpha, t) [D (\alpha)]^{3/2}$$

by Lemma 5.1;

$$\varepsilon_2 (\alpha, t) D (\alpha) \leqslant \operatorname{Im} \Phi_K (\alpha, t) + c_2 (\alpha, t) [D (\alpha)]^{3/2}$$

by Lemma 3.4;

$$\varepsilon_3 (\alpha, t) D (\alpha) \leqslant \operatorname{Im} \Phi_I (\alpha, t) + c_3 (\alpha, t) [D (\alpha)]^{3/2}$$

by the corollary to Theorem 5.1;

$$\varepsilon_4 (\alpha, t) D (\alpha) \leqslant \operatorname{Im} \Phi_I^{\gamma'} (\alpha, t) + c_4 (\alpha, t) [D (\alpha)]^{3/2}$$

again by Lemma 3.4; and finally,

$$\varepsilon (\alpha) D (\alpha) \leqslant \operatorname{Im} \Phi_I^{\gamma'} (\alpha) + c (\alpha) [D (\alpha)]^{3/2}$$

by Lemma 5.1 with g_H^t replaced by g_{-H}^t which coincides with g_H^{-t}. Theorem 3.1 is proved.

Proof of Theorem 4.1 is similar to that of the corollary to Lemma 5.1, we need only replace Proposition 5.1 used there by this corollary, Lemma 3.4 by Theorem 3.1 and consider zones instead of patches. Details may be left to the reader.

Sec. 6. Solution of Transfer Equation with Dissipation

We have before by numerous examples illustrated the role that Jacobians and transfer equations play in asymptotic expansions and geometric constructions. Here we shall solve the transfer equation in the general situation corresponding to a Lagrangean manifold with a complex germ.

First of all we introduce the notation $g = O_f(h^\alpha)$ for functions dependent on the parameter h.

Definition 6.1. *Let $f(x)$ be a smooth non-negative function and let*

$$g(x, h) = \sum_{j=0}^{j_0} g_j(x) h^{j/2}.$$

Then we write $g = O_f(h^\nu)$ if $g_j = O_f(h^{\nu - j/2})$ for $j \leqslant 2\nu$.

It is clear that

$$O_f(h^{\nu_1}) O_f(h^{\nu_2}) = O_f(h^{\nu_1+\nu_2}),$$

$$O_f(h^{\nu_1}) + O_f(h^{\nu_2}) = O_f(h^{\min\{\nu_1, \nu_2\}})$$

and that any polynomial g in $h^{1/2}$ with coefficients smoothly dependent on x is at least $O_f(h^\alpha)$.

Definition 6.2. *We write $g = \Theta_f(h^\nu)$ if*

$$g = h^{-k/2} O_D\left(h^{\nu + \frac{k}{2}}\right)$$

for a non-negative integer k. This notation makes sense even if ν is negative.

Let

$s(x, t) = s_1(x, t) + i s_2(x, t)$

be the solution of the Hamilton-Jacobi equation with dissipation given in Theorem 4.2. Thus we consider the following object to be given: a family $\{\Lambda_t^n, r_t^n\}$ of Lagrangean manifolds with complex germ and the corresponding to this family diffeomorphism

$$\Pi_\gamma : (\alpha, t) \to (x, t)$$

which allows us to regard x, t as coordinates in the $(n + 1)$-dimensional manifold $\{\Lambda_t^n\}$. As before, differentiation with respect to t with fixed α will be denoted by $\frac{d}{dt}$ or by the point over the symbol of a function, while that with fixed x will be denoted by $\frac{\partial}{\partial t}$ or by the subscript t.

Now we introduce a class $\mathfrak{P}$ of differential operators which will be regarded as "perturbations".

Definition 6.3. *We say that a differential operator*

$$\mathcal{A}=\sum_{|j|=0}^{j_0} a_j(\alpha,\ t,\ h)\left(\frac{\partial}{\partial\alpha}\right)^j$$

dependent on the parameter h *belongs to the class* $\mathfrak{P}$ *if*

$$a_j=O_D\left(h^{\frac{|j|-1}{2}}\right)+\left\langle\widetilde{H}_p(p(\alpha,\ t),\ q(\alpha,\ t),\ t),\ \mathcal{O}_D\left(h^{\frac{|j|}{2}}\right)\right\rangle+$$
$$+h^{1/2}O_D\left(h^{\frac{|j|-1}{2}}\right)\ \textit{for}\ |j|\geqslant 1$$

and

$$a_0=\mathcal{O}_D(h^{1/2})+\langle\widetilde{H}_p,\ \mathcal{O}_D(h^0)\rangle.$$

Set $\int\limits_0^t f(\alpha,\ \tau)\,d\tau = If(\alpha,\ t)$. The following shows why it is natural to regard operators belonging to $\mathfrak{P}$ as "perturbations".

Lemma 6.1. *For any* $\mathcal{A}\in\mathfrak{P}$ *there exists an integer-valued function* $\pi(r,\ s)$ *of two non-negative integer variables such that the following conditions are satisfied:*

(i) $\mathcal{A}(I\mathcal{A})^r\,\dfrac{O_D(h^{s/2})}{h^{s/2}}=\mathcal{O}_D\left(h^{\frac{\pi(r,\ s)}{2}}\right)$;

(ii) $\lim\limits_{r\to\infty}\pi(r,\ s)=+\infty$.

Proof. Let $\mathcal{A}\in\mathfrak{P}$. Then there is an $l\geqslant 0$ such that

$$\mathcal{A}O_D(h^{s/2})=h^{-l/2}O_D\left(h^{\frac{s+l}{2}}\right)$$

and the operator $I\mathcal{A}$ can be represented as $A+B$, where

$$AO_D(h^{s/2})=h^{-l/2}O_D\left(h^{\frac{s+l+1}{2}}\right),$$

$$BO_D(h^{s/2})=\begin{cases} h^{1/2}O_D\left(h^{\frac{s-1}{2}}\right) & \text{for } s\geqslant 1\\ h^{1/2}O_D(h^0) & \text{for } s=0.\end{cases}$$

This partition of $I\mathcal{A}$ corresponds to the natural partition of a_j:

$$a_j=a_j'+a_j'',$$

$$a_j'=O_D\left(h^{\frac{|j|+1}{2}}\right)+\langle\widetilde{H}_p,\ \mathcal{O}_D(h^{|j|/2})\rangle,$$

$$a_j''=h^{1/2}O_D\left(h^{\frac{|j|-1}{2}}\right),$$

the estimate of A being obtained by noting that

$$\left|\int_0^t \langle \widetilde{H}_p, \chi\rangle\, dt\right| \leqslant c(\alpha, t)\,[D(\alpha, t)]^{\nu}\int_0^t |\widetilde{H}_p|\, dt \leqslant$$

$$\leqslant c_1(\alpha, t)\,[D(\alpha, t)]^{\nu}\left[\int_0^t |\widetilde{H}|\, dt\right]^{1/2} \leqslant c_1(\alpha, t)\,[D(\alpha, t)]^{\nu+\frac{1}{2}}$$

holds for each vector-function $\chi(\alpha, t) = O_D(h^{\nu})$.

Consider the operator $(A+B)^r$. We have

$$(A+B)^r = \sum_{k=0}^{r}\sum_{p\in G} \overset{p(1)}{B} \ldots \overset{p(k)}{B}\;\overset{p(k+1)}{A} \ldots \overset{p(r)}{A} \overset{\text{def}}{=} \sum_{k,\,p} C_{k,\,p},$$

where G is the group of all permutations of $\{1, \ldots, r\}$. It is easily seen that

$$C_{k,\,p}\frac{O_D(h^{s/2})}{h^{s/2}} = h^{\frac{-l(r-k)+s-k}{2}}\,O_D\left(h^{\max\left\{\frac{(1+l)(r-k)+s-k}{2},\,0\right\}}\right).$$

It follows that

$$C_{k,\,p}\frac{O_D(h^{s/2})}{h^{s/2}} = h^{-\frac{p(r-k)+s-k}{2}}\,O_D\left(h^{\frac{(1+l)(r-k)+s-k}{2}}\right) = \mathcal{O}_D\left(h^{\frac{r-s}{2(l+2)}}\right)$$

for $k \leqslant \dfrac{(1+l)\,r+s}{l+2}$, $r > s$, and

$$C_{k,\,p}\frac{O_D(h^{s/2})}{h^{s/2}} = h^{-\frac{l(r-k)+s-k}{2}}\,O_D(h^0) = \mathcal{O}_D\left(h^{\frac{r-s}{2(l+2)}}\right)$$

for $k \geqslant \dfrac{(1+l)\,r+s}{l+2}$, $r > s$. At last for $r \leqslant s$, we have

$$C_{k,\,p}\frac{O_D(h^{s/2})}{h^{s/2}} = \mathcal{O}_D(h^0).$$

Thus, one may put

$$\pi(r, s) = \begin{cases} 0 & \text{for } r \leqslant s, \\ \left[\dfrac{r-s}{l+2}\right] & \text{for } r > s, \end{cases} \qquad Q.E.D.$$

Definition 6.4. *Any function* $\pi(r, s)$ *satisfying the conditions* (i) *and* (ii) *of Lemma* 6.1 *will be called a* $\mathcal{P}$*-function of* $\mathcal{A}$. *The smallest non-negative integer* l *with the property*

$$\mathcal{A}O_D(h^{s/2}) = h^{-l/2}O_D\left(h^{\frac{s+l}{2}}\right)$$

will be called the type of $\mathcal{A}$.

Problem 6.1. Let

$$\mathcal{A} = \sum_{|j|=0}^{J} h^{-1} O_D \left(h^{\frac{|j|+3}{2}}\right) \left(\frac{\partial}{\partial \alpha}\right)^j + \sum_{|j|=2}^{J} h O_D \left(h^{\frac{|j|-2}{2}}\right) \left(\frac{\partial}{\partial \alpha}\right)^j .$$

Then

$$\pi(r, s) = \begin{cases} \left[\frac{2r-s}{5}\right]+1 & \text{for } \left\{\frac{2r-s}{5}\right\} \geqslant \frac{1}{5} \\ 2r-s-4\left[\frac{2r-s}{5}\right] & \text{for } \left\{\frac{2r-s}{5}\right\} \leqslant \frac{1}{5} \end{cases}$$

is a $\mathcal{P}$-function of $\mathcal{A}$; here $[x]$ denotes the greatest integer not exceeding x, and $\{x\} = x - [x]$. In particular, $\pi(r, 0) \geqslant 1$.

Given a Hamiltonian function $\mathcal{H}(p, q, t)$, a solution $s(x, t)$ of the Hamilton-Jacobi equation with dissipation corresponding to $\mathcal{H}$, a smooth function $g(p, q, t)$ and an operator $\mathcal{A} \in \mathcal{P}$, the following relation will be called the *transfer equation with dissipation in the* $\frac{N}{2}$*- order approximation*:

$$\varphi_t + \langle \mathcal{H}_p(S_{1x}, x, t) + i\mathcal{H}_{pp}(S_{1x}, x, t) S_{2x}, \varphi_x \rangle +$$
$$+ \left\{ \frac{1}{2} \operatorname{tr} [\mathcal{H}_{pp}(S_{1x}, x, t) S_{xx}] + g(S_{1x}, x, t) + \mathcal{A} \right\} \varphi =$$
$$= \mathcal{O}_{S_2}(h^{N/2}). \tag{6.1}$$

We shall consider only those solutions of the transfer equation with dissipation which have the form $\varphi = h^{-m/2} O_{S_2}(h^{m/2})$, where m is a non-negative integer.

The relation (6.1) with the right-hand side replaced by

$$\mathcal{O}_{S_2}(h^{1/2}) + \langle \mathcal{O}_{S_2}(h^0), \quad \widetilde{H}_p(S_{1x}, x, t) \rangle$$

will be called the transfer equation with dissipation in the zero approximation. Recall that

$$\widetilde{H}_p(S_{1x}, x, t) = \mathcal{O}_{S_2}(h^{1/3})$$

for $t > 0$.

For convenience we shall identify $\varphi(\alpha, t, h)$ by the diffeomorphism Π_γ^I with a function (dependent on the parameter h) on the $(n+1)$-dimensional manifold $\{\Lambda_t^n\}$. Since

$$\varphi_t = \dot{\varphi} - \langle \dot{q} + \dot{\gamma}, \varphi_x \rangle = \dot{\varphi} - \langle (q_\alpha + \gamma_\alpha)^{-1} (H_p + \dot{\gamma}), \varphi_\alpha \rangle$$

and

$$S_x = p + w - \mathscr{E}(z-\gamma) + u_1,$$
$$S_{xx} = \mathscr{E} + u_2,$$
$$\mathscr{H}_p(S_{1x}, x, t) = \mathscr{H}_p(p, q, t) + \mathscr{H}_{pp}(p, q, t) \times$$
$$\times (\omega - \varepsilon(z-\gamma)) + \mathscr{H}_{pq}(p, q, t)\gamma + u_3,$$
$$\mathscr{H}_{pp}(S_{1x}, x, t) = \mathscr{H}_{pp}(p, q, t) + u_4,$$
$$g(S_{1x}, x, t) = g(p, q, t) + u_5,$$

where

$$u_1 = O_D(h), \quad u_2 = O_D(h^{1/2}), \quad u_3 = O_D(h),$$
$$u_4 = O_D(h^{1/2}), \quad u_5 = O_D(h^{1/2}),$$

then the transfer equation with dissipation can be rewritten in the coordinates α, t as follows:

$$\dot\varphi + \langle \varphi_\alpha, (q_\alpha + \gamma_\alpha)^{-1}[i\widetilde{H}_p - \dot\gamma + \mathscr{H}_{pp}(w - \mathscr{E}(z-\gamma)) + \mathscr{H}_{pq}\gamma]\rangle +$$
$$+ \left\{\frac{1}{2}\operatorname{tr}(\mathscr{H}_{pp}\mathscr{E}) + g(p, q, t) + \mathscr{B}\right\}\varphi = \mathcal{O}_D(h^{N/2}), \tag{6.2}$$

where $\mathscr{B} \in \mathscr{F}$ is related to $\mathscr{A}$ by the formula

$$\mathscr{B} = \mathscr{A} + \left\langle (q_\alpha + \gamma_\alpha)^{-1}[u_3 + iu_4 S_{2x} + i\mathscr{H}_{pp}\operatorname{Im} u_1], \frac{\partial}{\partial\alpha}\right\rangle +$$
$$+ \frac{1}{2}\operatorname{tr}(\mathscr{H}_{pp}u_2 + u_4 S_{xx}) + u_5.$$

Define the transfer operator P^γ by

$$P^\gamma = \frac{d}{dt} + \left\langle f, \frac{\partial}{\partial\alpha}\right\rangle, \tag{6.3}$$

where

$$f = (q_\alpha + \gamma_\alpha)^{-1}[i\widetilde{H}_p - \dot\gamma + \mathscr{H}_{pp}(w - \varepsilon(z-\gamma)) + \mathscr{H}_{pq}\gamma].$$

We begin by solving the transfer equation with dissipation in the zero approximation which has the form

$$P^\gamma\varphi + \left[\frac{1}{2}\operatorname{tr}(\mathscr{H}_{pp}\varepsilon) + g + \mathscr{B}\right]\varphi \sim 0, \tag{6.4}$$

where the sign $\sim$ means equality modulo

$$\mathcal{O}_D(h^{1/2}) + \langle \mathcal{O}_D(h^0), \widetilde{H}_p\rangle.$$

By analogy with the real Hamiltonian function case, we introduce a new function ψ to be found, related with φ by

$$\varphi(\alpha, t, h) = \frac{1}{\sqrt{J(\alpha, t)}}\psi(\alpha, t, h),$$

where $J(\alpha, t) = \det C(\alpha, t)$. In order to rewrite (6.4) in terms of ψ, we shall use the following result.

Lemma 6.2. *The Jacobian J satisfies the relation*

$$\dot{J} = J \operatorname{tr} [\mathscr{H}_{pp}\mathscr{E} + \mathscr{H}_{pq}] + O_D(h^{1/2}). \tag{6.5}$$

Proof. By the rule of differentiation of a determinant we have

$$\dot{J} = \det \begin{pmatrix} \frac{\partial(\dot{q}_1 + \dot{z}_1)}{\partial \alpha_1} & \cdots & \frac{\partial(\dot{q}_1 + \dot{z}_1)}{\partial \alpha_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial(q_n + z_n)}{\partial \alpha_1} & \cdots & \frac{\partial(q_n + z_n)}{\partial \alpha_n} \end{pmatrix} + \ldots +$$

$$+ \det \begin{pmatrix} \frac{\partial(q_1 + z_1)}{\partial \alpha_1} & \cdots & \frac{\partial(q_1 + z_1)}{\partial \alpha_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial(\dot{q}_n + \dot{z}_n)}{\partial \alpha_1} & \cdots & \frac{\partial(\dot{q}_n + \dot{z}_n)}{\partial \alpha_n} \end{pmatrix}. \tag{6.6}$$

In virtue of (2.6)

$$\frac{\partial(\dot{q} + \dot{z})}{\partial \alpha} = \mathscr{H}_{pq}C + \mathscr{H}_{pp}B + O_D(h^{1/2}) =$$

$$= (\mathscr{H}_{pq} + \mathscr{H}_{pp}\mathscr{E})C + O_D(h^{1/2}).$$

Thus,

$$\frac{\partial(\dot{q}_b + \dot{z}_b)}{\partial \alpha_j} = \sum_{k=1}^{n} (\mathscr{H}_{pq} + \mathscr{H}_{pp}\mathscr{E})_{lk} \frac{\partial(q_k + z_k)}{\partial \alpha_j} + O_D(h^{1/2}),$$

so the l-th determinant in the right-hand side of (6.6) equals

$$(\mathscr{H}_{pq} + \mathscr{H}_{pp}\mathscr{E})_{ll} J + O_D(h^{1/2}),$$

and it follows that (6.5) holds.

Corollary.

$$\frac{d}{dt}\frac{1}{\sqrt{J}} = -\frac{1}{2\sqrt{J}} \operatorname{tr}(\mathscr{H}_{pp}\mathscr{E} + \mathscr{H}_{pq}) + \chi, \tag{6.7}$$

where $\chi = O_D(h^{1/2})$.

It follows by noting the corollary that ψ must satisfy the relation

$$(P^{\gamma} + G(p, q, t) + \mathscr{B}')\psi \sim 0 \tag{6.8}$$

with

$$G = g - \frac{1}{2} \operatorname{tr} \mathscr{H}_{pq} \qquad \mathscr{B}' = \sqrt{J}\left(\mathscr{B}\frac{1}{\sqrt{J}} + \chi\right) \in \mathscr{P}.$$

Note that in spite of P^{γ} being equal to $\frac{d}{dt}+\left\langle f, \frac{\partial}{\partial\alpha}\right\rangle$ with $f\sim 0$, we may not ignore the term $\left\langle f, \frac{\partial\psi}{\partial\alpha}\right\rangle$ if we wish to find a solution of (6.8) of the form $\psi=h^{-\frac{m}{2}}O_D(h^{m/2})$ with $m>0$.

The idea of our method of solving (6.8) is as follows. If we could transform P^{γ} into $\frac{d}{dt}$ by an invertible operator L:

$$P^{\gamma}L=L\frac{d}{dt}, \tag{6.9}$$

then the problem would be trivial. In general it is impossible; nevertheless, it turns out that there is an "almost invertible" operator "almost satisfying" (6.9).

Namely, let

$$L_N^{\gamma}\overset{\text{def}}{=}\sum_{|j|=0}^{N}\frac{[\beta(\alpha, t)]^j}{j!}\left(\frac{\partial}{\partial\alpha}\right)^j, \tag{6.10}$$

where

$$\beta=-C^{-1}(z-\gamma). \tag{6.11}$$

Lemma 6.3. *L_N^{γ} satisfies the relation*

$$P^{\gamma}L_N^{\gamma}=L_N^{\gamma}\frac{d}{dt}+\hat{\varepsilon}+\sum_{|k|=N+1}g_k\left(\frac{\partial}{\partial\alpha}\right)^k$$

with $\hat{\varepsilon}\in\mathcal{P}$ *and*

$$g_k=O_D\left(h^{\frac{N+1}{2}}\right)+\left\langle O_D\left(h^{\frac{N}{2}}\right)\tilde{H}_p\right\rangle.$$

Proof. We have

$$P^{\gamma}L_N^{\gamma}-L_N^{\gamma}\frac{d}{dt}=\sum_{|k|=0}^{N}\frac{1}{k!}B_k, \tag{6.12}$$

where

$$B_k=P^{\gamma}\beta^k\left(\frac{\partial}{\partial\alpha}\right)^k-\beta^k\left(\frac{\partial}{\partial\alpha}\right)^k\frac{d}{dt}. \tag{6.13}$$

Let 1_j denote the multi-index $(\delta_{j1}, \ldots, \delta_{jn})$, where

$$\delta_{jk}=\begin{cases}1 & \text{if } k=j\\ 0 & \text{if } k\neq j.\end{cases}$$

Then

$$B_k = \sum_{j=1}^{n} k_j \beta^{k-1j} \left(\dot{\beta}_j + \left\langle f, \frac{\partial \beta_j}{\partial \alpha} \right\rangle\right) \left(\frac{\partial}{\partial \alpha}\right)^k +$$

$$+ \beta^k \sum_{j=1}^{n} f_j \left(\frac{\partial}{\partial \alpha}\right)^{k+1j}. \tag{6.14}$$

Substitution of (6.14) into (6.12) gives

$$P^\gamma L_N^\gamma - L_N^\gamma \frac{d}{dt} = \sum_{j=1}^{n} \left(\sum_{|k|=0}^{N-1} \frac{k_j}{k!} \beta^{k-1j} \left[P^\gamma (\alpha_j + \beta_j)\right] \left(\frac{\partial}{\partial \alpha}\right)^k + \right.$$

$$\left. + \sum_{|k|=N} \frac{\beta^k}{k!} f_j \left(\frac{\partial}{\partial \alpha}\right)^{k+1j} \right).$$

Since $\beta = O_D(h^{1/2})$ and $f_j = O_D(h^{1/2}) + \langle a, \widetilde{H}_p \rangle$, where a is a smooth function of α and t, it suffices only to check that

$$P^\gamma (\alpha_j + \beta_j) = O_D(h) + \langle O_D(h^{1/2}), \widetilde{H}_p \rangle \tag{6.15}$$

to complete the proof of the lemma.

We have

$$P^\gamma (\alpha_j + \beta_j) = f_j + \dot{\beta}_j + \left\langle f, \frac{\partial \beta_j}{\partial \alpha} \right\rangle \overset{\text{def}}{=} A_j.$$

Let A denote the vector $(A_1, \ldots, A_n)$, then

$$A = \dot{\beta} + (1 + \beta_\alpha) f. \tag{6.16}$$

Using the formula

$$\dot{C} = \mathscr{H}_{pq} C + \mathscr{H}_{pp} B + O_D(h^{1/2}),$$

we obtain

$$\dot{\beta} = C^{-1} (\mathscr{H}_{pq} C + \mathscr{H}_{pp} B) C^{-1} (z - \gamma) - C^{-1} (\dot{z} - \dot{\gamma}) + O_D(h) =$$

$$= C^{-1} [\mathscr{H}_{pq} (z - \gamma) + \mathscr{H}_{pp} \mathscr{E} (z - \gamma) - i\widetilde{H}_p -$$

$$- \mathscr{H}_{pp} w - \mathscr{H}_{pq} z + \dot{\gamma}] + O_D(h) =$$

$$= - C^{-1} [\mathscr{H}_{pp} (w - \mathscr{E} (z - \gamma)) + i\widetilde{H}_p + \mathscr{H}_{pq} \gamma - \dot{\gamma}] + O_D(h) =$$

$$= - C^{-1} (q_\alpha + \gamma_\alpha) f + O_D(h). \tag{6.17}$$

Further,

$$1 + \beta_\alpha = 1 - C^{-1} (z_\alpha - \gamma_\alpha) + O_D(h^{1/2}) =$$

$$= C^{-1} (q_\alpha + \gamma_\alpha) + O_D(h^{1/2}). \tag{6.18}$$

Equations (6.17) and (6.18) imply that

$$A = O_D(h) + O_D(h^{1/2})\,\tilde{H}_p,$$

Q.E.D.

The following two lemmas give the construction of an "almost-inverse" of L_N^γ.

Lemma 6.4. *Let* $\rho(\alpha, t)$ *be a smooth vector function with* $\det(1 + \rho_\alpha) \neq 0$. *Then there exist two sequences* $\{\mathcal{P}_k(\rho)\}$ *and* $\{Q_k(\rho)\}$ *of homogeneous polynomials in* ρ *with vector coefficients smoothly dependent on* α *and* t, $\mathcal{P}_k$ *and* Q_k *being of degree* k, *such that*

$$\rho + \sum_{|j|=0}^{R-1} \frac{\rho^j}{j!}\left(\frac{\partial}{\partial\alpha}\right)^j \sum_{k=1}^{R-1} \mathcal{P}_k(\rho) = Q_R(\rho) \tag{6.19}$$

holds for every $R > 1$.

Proof. We shall construct $\{\mathcal{P}_k\}$ and $\{Q_k\}$ by induction. Set

$$\mathcal{P}_1(\rho) = -(1 + \rho_\alpha)^{-1}\rho.$$

Then

$$\rho + \sum_{|j|=0}^{1} \frac{\rho^j}{j!}\frac{\partial^{|j|}\mathcal{P}_1(\rho)}{\partial\alpha^j} =$$

$$= \rho - (1+\rho_\alpha)^{-1}\rho - (1+\rho_\alpha)^{-1}\rho_\alpha\rho + Q_2(\rho) = Q_2(\rho),$$

so (6.19) holds for $R = 2$.

Suppose we have constructed $\mathcal{P}_1, \ldots, \mathcal{P}_m$ and $Q_2, \ldots, Q_{m+1}$ so that (6.19) holds for $R = 2, \ldots, m + 1$, and let $\mathcal{P}_{m+1}$ be any homogeneous polynomial in ρ of degree $m + 1$ with vector coefficients smoothly dependent on α and t. Then

$$\rho + \sum_{|j|=0}^{m+1} \frac{\rho^j}{j!}\left(\frac{\partial}{\partial\alpha}\right)^j \sum_{k=1}^{m+1} \mathcal{P}_k(\rho) = \sum_{|j|=0}^{m+1} \frac{\rho^j}{j!}\frac{\partial^{|j|}\mathcal{P}_{m+1}(\rho)}{\partial\alpha^j} +$$

$$+ Q_{m+1}(\rho) + \sum_{|j|=m+1} \frac{\rho^j}{j!}\left(\frac{\partial}{\partial\alpha}\right)^j \sum_{k=1}^{m} \mathcal{P}_k(\rho) = \bar{Q}_{m+1}(\rho) +$$

$$+ \sum_{|j|=0}^{m+1} \frac{\rho^j}{j!}\frac{\partial^{|j|}\mathcal{P}_{m+1}(\rho)}{\partial\alpha^j}, \tag{6.20}$$

where $\bar{Q}_{m+1}(\rho)$ is a homogeneous polynomial of degree $m + 1$ with smooth coefficient which does not depend on the choice of $\mathcal{P}_{m+1}$. Since

$$\frac{\partial^{|j|}\mathcal{P}_{m+1}(\rho)}{\partial\alpha^j} = \left({}^t\rho_\alpha \frac{\partial}{\partial\rho}\right)^j \mathcal{P}_{m+1}(\rho) + \bar{\bar{Q}}_{j,\,m+2-j}(\rho),$$

where $\overline{\overline{Q}}_{j,m+2-j}(\rho)$ is a homogeneous polynomial of degree $m+2-j$ with smooth coefficients, (6.20) implies that if $\mathcal{P}_{m+1}$ satisfies the equation

$$\sum_{|j|=0}^{m+1} \frac{\rho^j}{j!}\left({}^t\rho_\alpha \frac{\partial}{\partial\rho}\right)^j \mathcal{P}_{m+1}(\rho) = -\bar{Q}_{m+1}(\rho), \tag{6.21}$$

then there is a Q_{m+2} such that (6.19) is satisfied for $R = m+2$. Thus, the lemma will be proved if we show that (6.21) has a solution of the form

$$\mathcal{P}_{m+1}(\rho) = \sum_{|j|=m+1} b_j \rho^j,$$

where b_j is a smooth vector function of α and t. Let

$$\tilde{Q}_{m+1}(\rho) = -\sum_{|j|=m+1} a_j \rho^j.$$

Then $\mathcal{P}_{m+1}$ satisfies (6.21) if $b_1, \ldots, b_n$ satisfies the equations

$$\left[\sum_{k\leqslant j} \frac{j!}{k!\,(j-k)!}({}^t\rho_\alpha)^{|k|}\right] b_j = a_j, \qquad j = 1, \ldots, n. \tag{6.22}$$

So it suffices to show that for any j the matrix

$$A_j \stackrel{\text{def}}{=} \sum_{k\leqslant j} \frac{j!}{k!\,(j-k)!}({}^t\beta_\alpha)^{|k|}$$

is non-degenerate. But note that

$$\sum_{k\leqslant j} \frac{j!\,x^{|k|}}{k!\,(j-k)!} = (1+x)^{|j|},^*$$

hence

$$\det A_j = \det\,[(1+{}^t\beta_\alpha)^{|j|}] = [\det(1+\beta_\alpha)]^{|j|} \neq 0$$

as required.

Lemma 6.5. *Let* $\{\mathcal{P}_k\}$ *be the sequence of Lemma* 6.4, *and let*

$$\beta_N = \sum_{k=1}^{N} \mathcal{P}_k(\beta).$$

---

* This follows from the generalized binomial formula

$$(x+y)^j = \sum_{k\leqslant j} \frac{j!}{k!\,(j-k)!} x^k y^{j-k};$$

here j and k are multi-indices of length n, and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

Then the operator

$$R_N^\gamma = \sum_{|j|=0}^{N} \frac{1}{j!} \beta_N^j \left(\frac{\partial}{\partial\alpha}\right)^j \tag{6.23}$$

satisfies the conditions

$$L_N^\gamma R_N^\gamma = 1 + \hat{\varepsilon}_1, \tag{6.24}$$

$$R_N^\gamma L_N^\gamma = 1 - \hat{\varepsilon}_2, \tag{6.25}$$

where

$$\hat{\varepsilon}_1 = \sum_{|r|=1}^{2N} a_r(\alpha, t) \left(\frac{\partial}{\partial\alpha}\right)^r,$$

$$\hat{\varepsilon}_2 = \sum_{|r|=1}^{2N} b_r(\alpha, t) \left(\frac{\partial}{\partial\alpha}\right)^r,$$

$$a_r(\alpha, t) = O_D\left(h^{\frac{N+1}{2}}\right), \quad b_r(\alpha, t) = O_D\left(h^{\frac{N+1}{2}}\right).$$

Proof. First we check that R_N^γ is a right "almost inverse" of L_N^γ, i.e., (6.24) holds. We have

$$L_N^\gamma R_N^\gamma = \sum_{|k|=0}^{N} \frac{1}{k!} \beta^k \left(\frac{\partial}{\partial\alpha}\right)^k \sum_{|l|=0}^{N} \frac{1}{l!} \beta_N^l \left(\frac{\partial}{\partial\alpha}\right)^l =$$

$$= \sum_{|k|=0}^{N} \sum_{|l|=0}^{N} \sum_{0\leqslant j\leqslant k} \frac{1}{j!\,(k-j)!\,l!} \beta^k \frac{\partial^{|j|}\beta_N^l}{\partial\alpha^j} \left(\frac{\partial}{\partial\alpha}\right)^{l+k-j}.$$

Introducing the index $r=l+k-j$ instead of j, by a change of order of summation we obtain

$$L_N^\gamma R_N^\gamma = \sum_{|k|=0}^{N} \sum_{|l|=0}^{N} \sum_{l\leqslant r\leqslant l+k} \frac{\beta^k}{(l+k-r)!\,(r-l)!\,l!} \times$$

$$\times \frac{\partial^{|l+k-r|}\beta_N^l}{\partial\alpha^{l+k-r}} \left(\frac{\partial}{\partial\alpha}\right)^r = 1 + \sum_{|r|=1}^{2N} a_r \left(\frac{\partial}{\partial\alpha}\right)^r,$$

where

$$a_r = \sum_{\substack{0\leqslant l\leqslant r \\ |l|\geqslant N}} \sum_{\substack{k\geqslant r-l \\ |k|\leqslant N}} \frac{\beta^k}{(l+k-r)!\,(r-l)!\,l!} \frac{\partial^{|l+k-r|}\beta_N^l}{\partial\alpha^{l+k-r}}. \tag{6.26}$$

Now introduce the index $j = l + k - r$ instead of k, then (6.26) becomes

$$a_r = \sum_{\substack{0 \leq l \leq r \\ |l| \leq N}} \sum_{|j|=0}^{N+|l|-|r|} \frac{\beta^{j+r-l}}{j!\,(r-l)!\,l!} \frac{\partial^{|j|}\beta_N^l}{\partial \alpha^j} =$$

$$= \sum_{\substack{0 \leq l \leq r \\ |l| \leq N}} \frac{\beta^{r-l}}{l!\,(r-l)!} \sum_{|j|=0}^{N+|l|-|r|} \frac{\beta^j}{j!} \frac{\partial^{|j|}\beta_N^l}{\partial \alpha^j}. \tag{6.27}$$

Let us show that

$$\sum_{|j|=0}^{N} \frac{\beta^j}{j!} \frac{\partial^{|j|}\beta_N^l}{\partial \alpha^j} = (-\beta)^l + O_D\left(h^{\frac{N+1}{2}}\right). \tag{6.28}$$

To do this, we observe that since $\beta = O_D(h^{1/2})$ and the relation

$$1 + \beta_\alpha = 1 - (q_\alpha + z_\alpha)^{-1}(z_\alpha - \gamma_\alpha) + O_D(h^{1/2}) =$$
$$= (q_\alpha + z_\alpha)^{-1}(q_\alpha + \gamma_\alpha) + O_D(h^{1/2})$$

implies that the matrix $1 + \beta_\alpha$ is non-degenerate on the zero-set of the dissipation, it follows from Lemma 6.4 and the definition of β_N that

$$\sum_{|j|=0}^{N} \frac{\beta^j}{j!} \frac{\partial^{|j|}\beta_N}{\partial \alpha^j} = -\beta + O_D\left(h^{\frac{N+1}{2}}\right).$$

Thus, to prove (6.28) it is sufficient to show that

$$\sum_{|j|=0}^{N} \frac{\beta^j}{j!} \frac{\partial^{|j|}\beta_N^l}{\partial \alpha^j} - \left[\sum_{|j|=0}^{N} \frac{\beta^j}{j!} \frac{\partial^{|j|}\beta_N}{\partial \alpha^j}\right]^l = O_D\left(h^{\frac{N+1}{2}}\right). \tag{6.29}$$

Let $f(\alpha, t)$ be a smooth vector function. Consider the following polynomial $F_f(\beta)$ with coefficients smoothly dependent on α and t:

$$F_f(\beta) = \sum_{|j|=0}^{N} \frac{\beta^j}{j!} \frac{\partial^{|j|} f^l}{\partial \alpha^j} - \left[\sum_{|j|=0}^{N} \frac{\beta^j}{j!} \frac{\partial^{|j|} f}{\partial \alpha^j}\right]^l = \sum_k c_k(\alpha, t)\,\beta^k.$$

If x runs over a small neighborhood of 0 in $\mathbf{R}^n$, then

$$F_f(x) = f^l(\alpha + x, t) + O(|x|^{N+1}) -$$
$$-[f(\alpha + x, t) + O(|x|^{N+1})]^l = O(|x|^{N+1}).$$

It follows that $c_k(\alpha, t) = 0$ for $|k| \leq N$, so

$$F_f(\beta) = O_D\left(h^{\frac{N+1}{2}}\right).$$

Putting $f = \beta_N$ in this relation gives (6.29). Equation (6.28) implies that

$$\sum_{|j|=0}^{N+|l|-|r|} \frac{\beta^j}{j!}\frac{\partial^{|j|}\beta_N^l}{\partial\alpha^j} = (-\beta)^l + O_D\left(h^{\frac{N+|l|-|r|+1}{2}}\right) \tag{6.30}$$

for $l \leqslant r$. Putting (6.30) into (6.27), we see that

$$a_r = \sum_{l\leqslant r} \frac{\beta^{r-l}}{l!\,(r-l)!}(-\beta)^l + O_D\left(h^{\frac{N+1}{2}}\right) =$$

$$= (\beta-\beta)^r + O_D\left(h^{\frac{N+1}{2}}\right) = O_D\left(h^{\frac{N+1}{2}}\right)$$

for $l \leqslant r$. To complete the proof of (6.24) it remains to note that in the case $|r| > N$ the estimate $a_r = O_D(h^{\frac{N+1}{2}})$ obviously holds.

Turning to the proof of (6.25) we note that only the following two properties of $\beta(\alpha, t)$ have been used for constructing the right "almost-inverse" of L_N^γ: (1) $\beta = O_D(h^{1/2})$ and (2) $\det(1+\beta_\alpha)_{\alpha\in\Gamma_t} \neq 0$. The function $\beta_N(\alpha, t)$ has these properties too, for

$$\beta_N = -(1+\beta_\alpha)^{-1}\beta + O_D(h),$$

$$\left(1+\frac{\partial\beta_N}{\partial\alpha}\right) = 1-(1+\beta_\alpha)^{-1}\beta_\alpha + O_D(h^{1/2}) = (1+\beta_\alpha)^{-1} + O_D(h^{1/2}),$$

hence there is a function $\overline{\beta} = O_D(h^{1/2})$ such that the operator

$$S_N^\gamma = \sum_{|j|=0}^{N} \frac{\overline{\beta}^j}{j!}\left(\frac{\partial}{\partial\alpha}\right)^j$$

is a right "almost-inverse" of R_N^γ, that is,

$$R_N^\gamma S_N^\gamma = 1 + \hat{\varepsilon}_3, \tag{6.31}$$

where

$$\hat{\varepsilon}_3 = \sum_{|r|=1}^{2N} d_r(\alpha, t)\left(\frac{\partial}{\partial\alpha}\right)^r,$$

$$d_r = O_D\left(h^{\frac{N+1}{2}}\right).$$

The multiplication of (6.24) from the left by R_N^γ and from the right by S_N^γ gives

$$R_N^\gamma L_N^\gamma R_N^\gamma S_N^\gamma = R_N^\gamma S_N^\gamma + R_N^\gamma \hat{\varepsilon}_1 S_N^\gamma = 1 + \hat{\varepsilon}_3 + R_N^\gamma \hat{\varepsilon}_1 S_N^\gamma. \tag{6.32}$$

It follows from (6.31) and (6.32) that

$$R_N^\gamma L_N^\gamma = 1 + \hat{\varepsilon}_3 + R_N^\gamma \hat{\varepsilon}_1 S_N^\gamma - R_N^\gamma L_N^\gamma \hat{\varepsilon}_3. \tag{6.33}$$

Set

$$\hat{\varepsilon}_2 \overset{\text{def}}{=} \hat{\varepsilon}_3 + R_N^\gamma \hat{\varepsilon}_1 S_N^\gamma - R_N^\gamma L_N^\gamma \hat{\varepsilon}_3. \tag{6.34}$$

Because of (6.33), $\hat{\varepsilon}_2$ is a differential operator of order $2N$:

$$\hat{\varepsilon}_2 = \sum_{|j|=0}^{2N} b_j \left(\frac{\partial}{\partial\alpha}\right)^j.$$

It is easily seen from (6.34) that $b_j = O_D\left(h^{\frac{N+1}{2}}\right)$, which proves the lemma.

Theorem 6.1. *If* $\mathcal{A}\dfrac{O_D\left(h^{\frac{m}{2}}\right)}{h^{\frac{m}{2}}} \sim 0$, *then, for any function* $v_0(\alpha, h)$ *of the form*

$$v_0(\alpha, h) = \frac{O_D(h^{m/2})}{h^{m/2}},$$

the function

$$\varphi(\alpha, t, h) = \frac{1}{\sqrt{J(\alpha, t)}} L_m^\gamma v_0(\alpha, h)\, e^{-\int_0^t G(p(\alpha, \tau), q(\alpha, \tau), \tau)\, d\tau}$$

satisfies the transfer equation with dissipation in the zero approximation.

Note 6.1. If the condition $\mathcal{A}\dfrac{O_D(h^{m/2})}{h^{m/2}} \sim 0$ is not satisfied, then the function $\varphi(\alpha, t, h)$ introduced in the theorem can be used for solving the transfer equation by the theory of perturbations (see Theorem 6.2 below).

Proof of Theorem 6.1. Set

$$v(\alpha, t, h) = v_0(\alpha, h)\, e^{-\int_0^t G(p(\alpha, \tau), q(\alpha, \tau), \tau)\, \alpha\tau}.$$

Then v satisfies the equation $\dot{v} + Gv = 0$, and

$$v = h^{-\frac{m}{2}} O_D\left(h^{\frac{m}{2}}\right).$$

By Lemma 6.3 we have

$$(P^{\gamma}+G)\,L_m^{\gamma}v=\Big\{L_m^{\gamma}\left(\frac{d}{dt}+G\right)+\hat{\varepsilon}+$$
$$+\sum_{|k|=m+1} g_k\left(\frac{\partial}{\partial\alpha}\right)^k+[G,\ L_m^{\gamma}]\Big\}\,v=$$
$$=\Big\{\hat{\varepsilon}+\sum_{|k|=m+1} g_k\left(\frac{\partial}{\partial\alpha}\right)^k+[G,\ L_m^{\gamma}]\Big\}\,v.$$

It is easy to see that

$$\Big[\hat{\varepsilon}+\sum_{|k|=m+1} g_k\left(\frac{\partial}{\partial\alpha}\right)^k\Big]\,v\sim 0.$$

Moreover,

$$[G,\ L_m^{\gamma}]=\sum_{|j|=0}^{m} G\frac{\beta^j}{j!}\left(\frac{\partial}{\partial\alpha}\right)^j-$$
$$-\sum_{|j|=0}^{m}\sum_{k\leqslant j}\frac{\beta^j}{k!\,(j-k)!}\,\frac{\partial^{|j-k|}G}{\partial\alpha^{j-k}}\left(\frac{\partial}{\partial\alpha}\right)^k=$$
$$=-\sum_{|j|=0}^{m}\sum_{k<j}\frac{\beta^j}{k!\,(j-k)!}\,\frac{\partial^{|j-k|}G}{\partial\alpha^{j-k}}\left(\frac{\partial}{\partial\alpha}\right)^k\in\mathcal{P},\tag{6.35}$$

hence $[G,\ L_m^{\gamma}]\,v\sim 0$. So $\psi=L_m^{\gamma}v$ satisfies (6.8). The theorem is proved.

Now let us tackle the transfer equation with dissipation in the approximation of order N. As before, we set $\varphi=\frac{1}{\sqrt{J}}\psi$. Then (6.2) becomes

$$(P^{\gamma}+G+\mathcal{B}')\,\psi=\mathcal{O}_D(h^{N/2}),\tag{6.36}$$

(note (6.7), (6.8)).

We shall look for ψ in the form

$$\psi=L_M^{\gamma}v\tag{6.37}$$

with m to be determined later. Putting (6.37) into (6.36), we obtain by Lemma 6.3

$$\Big[L_m^{\gamma}\left(\frac{d}{dt}+G\right)+\hat{\varepsilon}'+\sum_{|k|=m+1} g_k\left(\frac{\partial}{\partial\alpha}\right)^k\Big]\,v=\mathcal{O}_D(h^{N/2}),\tag{6.38}$$

where

$$\hat{\varepsilon}'=\hat{\varepsilon}+[G,\ L_m^{\gamma}]+\mathcal{B}'L_m^{\gamma}\in\mathcal{P},\tag{6.39}$$

the operator $\hat{\varepsilon}$ and the functions g_k being those of Lemma 6.3. If $v=\frac{O_D(h^{s/2})}{h^{s/2}}$, then it follows, because of $g_k=O_D(h^{m/2})$, that we may drop the term $\sum_{|k|=m+1} g_k\left(\frac{\partial}{\partial\alpha}\right)^k v$ in (6.38) provided that $m\geqslant N+s$.

Now, by virtue of Lemma 6.5, we obtain

$$\hat{\varepsilon}' = (L_m^{\gamma}R_m^{\gamma} - \hat{\varepsilon}_1)\,\hat{\varepsilon}' = L_m^{\gamma}\hat{\varepsilon}'' - \hat{\varepsilon}_1\hat{\varepsilon}',$$

where

$$\hat{\varepsilon}'' = R_m^{\gamma}\hat{\varepsilon}' \in \mathcal{P} \tag{6.40}$$

the operator $\hat{\varepsilon}_1$ being as in Lemma 6.5, in particular,

$$\hat{\varepsilon}_1\hat{\varepsilon}'\,\frac{O_D(h^{s/2})}{h^{s/2}} = \Theta_D(h^{N/2}), \tag{6.41}$$

whenever $m \geqslant N+s+k-1$, where k is the type of $\mathcal{A}$. Noting that

$$L_m^{\gamma}\Theta_D(h^{N/2}) = \Theta_D(h^{N/2}),$$

we arrive at

$$\left(\frac{d}{dt}+G+\hat{\varepsilon}''\right)v=\Theta_D(h^{N/2}). \tag{6.42}$$

We can construct a function v satisfying (6.42) by the perturbation theory, $\hat{\varepsilon}''$ being considered as a perturbation. We have the following result.

Lemma 6.6. *Let I be the operator of Lemma 6.1, let $\hat{\varepsilon}\in\mathcal{P}$, and set*

$$\hat{\delta} = -e^{\int_0^t G\,dt}\circ\hat{\varepsilon}\circ e^{-\int_0^t G\,dt}.$$

Then for any

$$v_0(\alpha,\ h)=\frac{O_D(h^{s_0/2})}{h^{s_0/2}}$$

the function

$$v = e^{-\int_0^t G\,dt}\sum_{j=0}^{N}(I\hat{\delta})^j v_0$$

satisfies the relation

$$\left(\frac{d}{dt}+G+\hat{\varepsilon}\right)v=\Theta_D\left(h^{\frac{\pi(N,\ s)}{2}}\right),$$

where π is a $\mathcal{P}$-function of $\hat{\delta}$.

Proof. We have

$$\left(\frac{d}{dt}+G+\hat{\varepsilon}\right)v=e^{-\int_0^t G\,dt}\left(\frac{d}{dt}-\hat{\delta}\right)\sum_{j=0}^{N}(I\hat{\delta})^j v_0=$$

$$=-e^{-\int_0^t G\,dt}\,\hat{\delta}\,(I\hat{\delta})^N v_0,$$

and the lemma is proved.

Now it is easy to formulate the main result of this section.

Theorem 6.2. *Let $\mathcal{A}$ in* (6.1) *be of type k. Then for any function $v_0(\alpha, h)$ of the form $v_0=h^{-s_0/2}O_D(h^{s_0/2})$, the following function satisfies* (6.2):

$$\varphi(\alpha, t, h)=\frac{1}{\sqrt{J(\alpha, t)}}L_m^{\gamma}e^{-\int_0^t G(p(\alpha,\tau),\, q(\alpha,\tau),\, \tau)\,d\tau}\times$$

$$\times\sum_{j=0}^{N(k+2)+s_0}(I\hat{\delta})^j v_0(\alpha, h),$$

where

$$\hat{\delta}=-e^{\int_0^t G\,dt}\circ\hat{\varepsilon}''\circ e^{-\int_0^t G\,dt}$$

(with $\hat{\varepsilon}''$ given by (6.40)) *and*

$$m=N(k+1)^2+s_0(k+1)+\max\{k-1,\ 0\}. \tag{6.43}$$

Proof. Essentially, the theorem has already been proved above. There are only two points to be noted.

(1) By Lemma 6.6 the function

$$v=e^{-\int_0^t G\,dt}\sum_{j=0}^{N(k+2)+s_0}(I\hat{\delta})^j v_0 \tag{6.44}$$

satisfies (6.42). In fact, if $\mathcal{A}$ is of type k, then it is easy to see that the same is true of $\hat{\delta}$. It follows from the proof of Lemma 6.1 that

$$\pi(r, s)=\begin{cases}\dfrac{r-s}{k+2} & \text{if } r\geqslant s\\ 0 & \text{if } r<s\end{cases}$$

is a $\mathcal{P}$-function of $\hat{\delta}$, so

$$\pi(N(k+2)+s_0,\ s_0)=N.$$

(2) It is easy to see that the function (6.44) has the form

$$v=h^{-s/2}O_D(h^{s/2})$$

with $s=k[N(k+2)+s_0]+s_0$.

For this s, (6.41) is satisfied whenever $m\geqslant N(k+1)^2+$ $+s_0(k+1)+k-1$, and the condition

$$\sum_{|k|=m+1} g_k\left(\frac{\partial}{\partial\alpha}\right)^k\frac{O_D(h^{s/2})}{h^{s/2}}$$

is satisfied whenever $m\geqslant N(k+1)^2+s_0(k+1)$.

Note 6.2. Let $\pi(r, s)$ be a $\mathcal{P}$-function of $\hat{\delta}$, and let $r=r(s_0, N)$ satisfy the equation $\pi(r, s_0)=N$. Then Theorem 6.2 remains valid if we replace $\sum\limits_{j=0}^{N(k+2)+s_0}$ by $\sum\limits_{j=0}^{r(s_0,\,N)}$ and (6.43) by $m=\max\{r+s,\ r+s+k-1\}$.

Note 6.3. Let $\frac{\partial}{\partial l}=\left(\frac{\partial}{\partial l_1},\ \ldots,\ \frac{\partial}{\partial l_n}\right)$ be a set of commuting complex vector fields on u_γ which are linearly independent at every point (a complex vector field on a real manifold M is an element of the complexification of C^∞ module of all vector fields on M). Theorem 6.2 remains valid if one replaces $\partial/\partial\alpha$ by $\partial/\partial l$ in the definitions of $J(\alpha, t)$, L_m^γ and $\hat{\delta}$, for instance:

$$L_m^\gamma=\sum_{|j|=0}^{m}\frac{(\beta_\gamma)^j}{j!}\left(\frac{\partial}{\partial l}\right)^j,$$

where $\beta_\gamma=-\left[\frac{\partial(q+z)}{\partial l}\right]^{-1}(z-\gamma)$.

V. CANONICAL OPERATOR ON A LAGRANGEAN MANIFOLD WITH A COMPLEX GERM AND PROOF OF THE MAIN THEOREM

Notation. In this chapter we shall use the spaces H_k^l introduced in Sec. 11 of Chapter II:

$$\| f(x) \|_{H_k^l} = \left\| (1+|x|^2)^{k/2} \left(1+\left|\frac{\partial}{\partial x}\right|^2\right)^{l/2} f(x) \right\|_{L_2}.$$

We shall write $H_0^k = H^k$, $H_l^0 = H_l$. Thus H^l is the Sobolev space W_2^l. The symbol $H_{l,m}^{I,h}$ will denote the completion of the Schwartz space S with respect to the norm

$$\| f(x) \|_{H_{l,m}^{I,h}}^2 = \sum_{\substack{|\alpha| \leq l \\ |\beta| \leq m}} \left\| \left(\frac{x_I}{h}\right)^{\alpha} \left(\frac{\partial}{\partial x}\right)^{\beta} f(x) \right\|_{L_2}^2,$$

where h is a (small) positive parameter.

Sec. 1. Quantum Bypassing Focuses Operation

In this section we shall give an asymptotic expansion for $h \to 0$ of the Fourier transform of a function of the form

$$\left[\frac{1}{\sqrt{J(\alpha)}} e^{\frac{i}{h}\Phi^{\gamma}(\alpha)} \varphi(\alpha)\right]_{\alpha=\pi_{\gamma}^{-1}(x)}, \quad \operatorname{Im}\Phi^{\gamma} \geqslant 0, \quad x \in \mathbf{R}^n.$$

We have to do this to define the canonical operator as we have already done in the case $\operatorname{Im}\Phi^{\gamma} = 0$, $n = 1$ (see Chapter III).

In the case under consideration, a method which is similar to the saddle-point method is likely to be applied to obtain an asymptotic expansion of the integral, for, in general, $e^{\frac{i}{h}S}$ decreases rapidly as $h \to +0$. However, a difficulty arises because it would be desirable to obtain as the result an expression similar to the integrand with the phase and the Jacobian replaced by those corresponding to the zone Ω_I, if the Fourier transformation $F_{x_{\bar{I}} \to p_{\bar{I}}}$ is considered (it is

such a geometric interpretation of the asymptotic expansion of the integral (3.1) that has led in Chapter II to the definition of the canonical operator and thus to the global construction of solutions of (pseudo)differential equations). It appears to be very difficult to do this by the saddle-point method because a complex-valued function is not uniquely expressible as the sum of an s-action, a μ-action and a potential. This difficulty will be avoided by a method similar to that of Sec. 8 of Introduction. The method is based on a special transformation which is connected with the bypassing focuses operation and will be called the quantum bypassing focuses operation.

We shall first introduce some terminology.

Definition 1.1. *Let $\{\varphi_j(\alpha, h)\}$ be a family of smooth functions of α dependent on the standard small positive parameter h, with the index j running over a certain set J. Given a smooth non-negative function $D(\alpha)$, the series $\sum_{j\in J}\varphi_j(\alpha, h)$ will be called D-asymptotic if*

(i) $\varphi_j = \mathcal{O}_D(h^{\nu_j})$, supp φ_j *lying in a compact set independent of j and h;*

(ii) *for any natural number m, there is such a finite set $J_0 \subset J$ that $\varphi_j = \mathcal{O}_D(h^m)$ for $j \notin J_0$.*

Definition 1.2. *Let $\{\varphi_j(x, h)\}$ be a family of smooth functions of $x \in \mathbf{R}^n$ dependent on a small positive parameter h, with the index j running over a certain set J. The series $\sum_{j\in J}\varphi_j(x, h)$ will be called h-asymptotic if*

(i) *for any j, k and l there is such a number m_{jkl} that*

$$\|\varphi_j\|_{C_l^k(\mathbf{R}^n)} = O(h^{m_{jkl}})$$

(*for the definition of C_l^k see* Sec. 11 of Chapter II);

(ii) *for any N, k and l, there is such a finite set $J_0 \subset J$ that*

$$\|\varphi_j\|_{C_l^k(\mathbf{R}^n)} = O(h^N) \text{ for } j \notin J_0.$$

In the following we shall mean, by an asymptotic series, either D-asymptotic series, or an h-asymptotic series.

Example 1.1. Let $\varphi(\alpha) \in C_0^\infty$ and set $\beta_I = -C_I^{-1}(w_{\bar{I}}, z_I)$, then the series

$$L_I\varphi \stackrel{\text{def}}{=} \sum_{|j|=0}^{\infty} \frac{\beta_I^j}{j!}\left(\frac{\partial}{\partial\alpha}\right)^j \varphi$$

is D-asymptotic.

Example 1.2. Let $\mathscr{A} \in \mathscr{P}$, $\varphi(\alpha) \in C_0^\infty$, and let I be the same operator as in Sec. 6 of Chapter IV. Then the series

$$\sum_{j=0}^{\infty} (I\mathscr{A})^j \varphi$$

is D-asymptotic by Lemma 6.1 of Chapter IV.

We now define addition and multiplication of asymptotic series.

Definition 1.3. *The sum $\varphi \dot{+} \psi$ of two asymptotic series, $\varphi = \sum_{j\in J} \varphi_j$ and $\psi = \sum_{k\in K} \psi_k$, is the following asymptotic series:*

$$\varphi \dot{+} \psi = \sum_{l\in(J \vee K)} \chi_l,$$

where the sign $\vee$ stands for disjoint union and

$$\chi_l = \begin{cases} \varphi_l \text{ for } l \in J \\ \psi_l \text{ for } l \in K. \end{cases}$$

Let $-\psi$ be the asymptotic series $\sum_{k\in K} (-\psi_k)$. The difference $\varphi \dot{-} \psi$ is the asymptotic series $\varphi \dot{+} (-\psi)$.

The product $\varphi\psi$ is the following asymptotic series:

$$\varphi\psi = \sum_{(j,\,k)\in(J\times K)} \varphi_j \psi_k.$$

The reader is invited to check and see that this definition is correct, i.e., to verify that both $\varphi \dot{+} \psi$ and $\varphi\psi$ are asymptotic.

Definition 1.4. *Let $\varphi \in \sum_j \varphi_j$ and $\psi = \sum_j \psi_j$ be two asymptotic series, with the index j running over one and the same set J. Then the term-by-term sum $\varphi + \psi$ is the following asymptotic series:*

$$\varphi + \psi = \sum_{j\in J} (\varphi_j + \psi_j).$$

Definition 1.5. *We write:*

$$\varphi(\alpha) = \hat{O}_D(h^s), \textit{ if } \left| e^{-\frac{D(\alpha)}{h}} \varphi(\alpha) \right| \leqslant ch^s;$$

$$\varphi(\alpha,\ h) = \hat{O}_D(h^s), \textit{ if } \varphi(\alpha,\ h) = \sum_{j=0}^{j_0} \varphi_j(\alpha) h^{j/2},$$

where $\varphi_j(\alpha) = \hat{O}_D\left(^{s-\frac{j}{2}}\right)$ *for* $j \leqslant 2s$;

$$\varphi(\alpha,\ h) = O_D(h^s) \textit{ if } \varphi = h^{-\frac{k}{2}} \hat{O}_D(h^{s+k/2}),$$

for some integer $k > 0$. Here all the estimates are assumed to be locally uniform in α.

Definition 1.6. *We say that a D-asymptotic series $\sum_{j\in J} \varphi_j$ is weakly equivalent to zero if for any natural number N, there is such a finite set that*

$$\sum_{j\in J_1} \varphi_j = \mathcal{O}_D(h^N),$$

whenever $J_1 \subset J$ is any finite set containing J_0.

An h-asymptotic series $\sum_{j\in J} \varphi_j$ will be called weakly equivalent to zero if, for any natural numbers N and k, there is such a finite set $J_0 \subset J$ that

$$\Big\| \sum_{j\in J_1} \varphi_j \Big\|_{C_k^0} = O(h^N),$$

whenever $J_1 \subset J$ is any finite set containing J_0.

We write $\varphi \sim 0$ to denote that φ is weakly equivalent to zero.

Definition 1.7. *We say that an asymptotic series $\sum_{j\in J} \varphi_j$ is equivalent to zero, $\varphi \approx 0$ in symbol, if for any multi-index k,*

$$\sum_{j\in J} \left(\frac{\partial}{\partial\alpha}\right)^k \varphi_j \sim 0.$$

Two asymptotic series, φ and ψ, are said to be weakly equivalent, $\varphi \sim \psi$ in symbol, if $\varphi \dot{-} \psi \sim 0$, and equivalent, $\varphi \approx \psi$ in symbol, if $\varphi \dot{-} \psi \approx 0$.

The equivalence relation $\approx$ is compatible with the arithmetical operations defined above:

$$((\varphi_1 \approx \varphi_2) \text{ and } (\psi_1 \approx \psi_2)) \Rightarrow ((\varphi_1 \dot{+} \psi_1) \approx \varphi_2 \dot{+} \psi_2) \quad \text{and} \quad (\varphi_1\psi_1 \approx \varphi_2\psi_2).$$

Moreover, the term-by-term sum of two asymptotic series, $\sum_{j\in J} \varphi_j$ and $\sum_{j\in J} \psi_j$, is equivalent to their sum. Because of this, we shall, as a rule, write $+$ in place of $\dot{+}$ and use the standard symbol $\sum$ to denote the summation operator for asymptotic series.

Definition 1.8. *An asymptotic series $\varphi = \sum_{j\in J} \varphi_j$ is called an asymptotic solution of the equation $A\varphi = 0$ if the series $\sum_{j\in J} A\varphi_j$ is asymptotic and equivalent to zero.*

Example 1.3. The series

$$J^{-1/2} L \sum_{j=0}^{\infty} (I\hat{\delta})^j v(\alpha)$$

is a D-asymptotic solution of the transfer equation with dissipation for a non-singular patch (see Theorem 6.2 of Chapter IV).

Definition 1.9. *A D-asymptotic differential operator L is such an operator series* $L = \sum_{j\in J} L_j$ *that if* $\varphi = \sum_{k=K} \varphi_k$ *is a D-asymptotic series, then the series*

$$L\varphi = \sum_{j\in J} \sum_{k\in K} L_j \varphi_k$$

is D-asymptotic, and $L\varphi \approx 0$ *whenever* $\varphi = 0$.

In the case where L_j *is differential for every* j, *L will be called a differential D-asymptotic operator.*

Definition 1.10. *We shall say that the series* $\sum_{j\in J} \varphi_j(\alpha, h)$ *is asymptotic at* α_0 *if there is a function* $e(\alpha) \in C_0^\infty$ *equal to* 1 *near* α_0 *such that the series*

$$\sum_{j\in J} e(\alpha)\varphi_j(\alpha, h)$$

is asymptotic.

Definition 1.11. *Let both* φ *and* ψ *be asymptotic at* α_0. *We say that* φ *and* ψ *are equivalent (respectively weakly equivalent) at* α_0, $\varphi \overset{\alpha_0}{\approx} \psi$ (respectively $\varphi \overset{\alpha_0}{\sim} \psi$) *in symbol, if there is a function* $e(\alpha) \in C_0^\infty$ *equal to* 1 *near* α_0 *such that* $e\varphi \approx e\psi$ *(respectively* $e\varphi \sim e\psi$*).*

Definition 1.12. *Let* $\varphi \in \sum_{j\in J} \varphi_j$ *be a D-asymptotic series. If* φ *is not equivalent to zero, then the order of* φ, ord φ *in symbol, is the smallest half-integer k with the following property: there is such a finite subset* $J' \subset J$ *that*

$$\sum_{j\in J'} \varphi_j \neq \mathcal{O}_D(h^{k+1/2}), \quad \varphi_j = \mathcal{O}_D(h^{k+1/2}) \text{ for } j \notin J'.$$

If $\varphi \approx 0$, *then we set* ord $\varphi = \infty$.

Definition 1.13. *Set* $p(\varphi) = \sum_{j\in J_0} \varphi_j$, *where* J_0 *is the set of all j such that* ord $\varphi_j \leqslant$ ord φ *(it is easy to see that* ord $p(\varphi)$ = ord φ*). Let*

$$p(\varphi) = \sum_{k=k_1}^{k_2} h^{k/2} c_k(\alpha).$$

The smallest k with the property ord $(h^{k/2}c_k(\alpha))$ = ord φ *is called the type of* φ. *The principal monomial of* φ, $m[\varphi]$ *in symbol, is* $h^{k/2}c_k(\alpha)$, *where k is the type of* φ.

If ord $\varphi = \infty$, *then we set* $m[\varphi] = 0$.

Problem 1.1. Let A and I be the operators of Example 1.2. Then $\sum\limits_{j=0}^{\infty}(IA)^j$ is a D-asymptotic differential operator preserving the principal monomial.

Definition 1.14. *A D-asymptotic operator A is called a quasi-identity operator if*

$$m[A\varphi] = m[\varphi] + h^k O_D(h^{s+1/2})$$

for any D-asymptotic series φ, where k is the type of φ and $s + k =$ ord φ.

Given a Lagrangean manifold with a complex germ, let α_0 be in $\Omega \cap \Omega_I \cap \Gamma$, and suppose that the assumptions of Theorem 5.1 of Chapter IV are satisfied. Let $\alpha(x)$ be the solution of the equation $q(\alpha) = x$. For the integral

$$I(x_I, \xi_{\bar I}, h) = F_{x_{\bar I}\to\xi_{\bar I}}\left\{e^{\frac{i}{h}\Phi(\alpha)}[J(\alpha)]^{-1/2}L\varphi(\alpha)\right\}_{\alpha=\alpha(x)} =$$
$$= (2\pi h)^{-\frac{m}{2}}e^{-\frac{i\pi m}{4}}\times$$
$$\times\int_{\mathbf{R}^m} e^{-\frac{i}{h}\langle x_{\bar I},\, \xi_{\bar I}\rangle}\left\{e^{\frac{i}{h}\Phi(\alpha)}[J(\alpha)]^{-1/2}L\varphi(\alpha)\right\}_{\alpha=\alpha(x)}dx_{\bar I},$$

where L is the D-asymptotic differential operator of Example 1.1,

$$[J(\alpha)]^{1/2} = |J(\alpha)|^{1/2}\exp\left(\frac{i}{2}\operatorname{Arg} J(\alpha)\right),$$

Arg $J(\alpha)$ is continuous near α_0, and φ is supported near α_0, we shall obtain an h-asymptotic expansion, i.e., an h-asymptotic series which is equivalent to $I(x_I, \xi_{\bar I}, h)$ and does not involve integration (that series will be defined via a D-asymptotic differential operator applied to φ).

Let H be a bypassing focuses Hamiltonian such as that of Theorem 5.1 of Chapter IV:

$$H = \frac{1}{2}\{\langle p_{\bar I}, u^{-1}\varepsilon u p_{\bar I}\rangle + \langle q_{\bar I}u^{-1}\varepsilon u q_I\rangle\}, \tag{1.1}$$

and let $J(\alpha, t)$ be the complex Jacobian corresponding to the bypassing focuses operation associated with H. Further let Arg $J(\alpha, t)$ be the function determined by the following conditions:

(i) Arg $J(\alpha, t)$ is continuous in t,

(ii) $J(\alpha,\ t) = |J(\alpha,\ t)|\, e^{i \operatorname{Arg} J(\alpha, t)}$,
(iii) $\operatorname{Arg} J(\alpha,\ 0) = \operatorname{Arg} J(\alpha)$.

Proposition 1.1. *Under the above assumptions, for any* $\varphi(\alpha) \in C_0^\infty$ *supported near* α_0, *we have*

$$I(x_I,\ \xi_{\bar I},\ h) \approx e^{\frac{i}{h}\Phi(\alpha) - \frac{i}{2}\operatorname{Arg} J\left(\alpha,\ \frac{\pi}{2}\right) - \frac{i\pi\sigma}{2}} \times \\ \times |J_I(\alpha)|^{-1/2} L_I v_I \varphi(\alpha) \Big|_{\alpha = \alpha^I(x_I,\ \xi_{\bar I})}, \tag{1.2}$$

where $\alpha^I(x_I,\ \xi_{\bar I})$ *is the solution of the system*

$$\begin{cases} q_I(\alpha) = x_I, \\ p_{\bar I}(\alpha) = \xi_{\bar I}, \end{cases}$$

σ *is the number of negative eigenvalues of* ε, $J_I = \det C_I$, *and* v_I *is a D-asymptotic differential quasi-identity operator.*

Note. Since $J(\alpha,\ \pi/2) = (-1)^\sigma J_I(\alpha)$, the function $e^{\frac{i}{2}[\operatorname{Arg} J(\alpha,\ \pi/2) + \pi\sigma]}\, |J_I(\alpha)|^{1/2}$ is a branch of $\sqrt{J_I(\alpha)}$.

The proof of the proposition will be by the technique which we have used in Sec. 8 of Introduction to obtain an asymptotic expansion for $h \to 0$ of the integral $I(h)$.

Consider the Cauchy problem

$$\begin{cases} -ih\dfrac{\partial\psi}{\partial t} + H\left(-ih\dfrac{\partial}{\partial x},\ x\right)\psi = 0, \\ \psi(x,\ 0) = \psi_0(x), \end{cases} \tag{1.3}$$

where H is the bypassing focuses Hamiltonian. The solution of (1.2) is given by

$$\psi(x,\ t) = \frac{e^{-\frac{i\pi m}{4} + \frac{i\pi\sigma}{2}}}{(2\pi h \sin t)^{m/2}} \times \\ \times \int\limits_{\mathbf{R}^m} e^{\frac{i}{h}(\sin t)^{-1}[H(y,\ x)\cos t - \langle x_{\bar I},\ u^{-1}\varepsilon u y_{\bar I}\rangle]}\, \psi_0(x_I,\ y_{\bar I})\, dy_{\bar I} \tag{1.4}$$

where m is the number of elements of $\bar I$. Note that if ψ_0 is an asymptotic series, then the last formula defines an asymptotic series which satisfies (1.3) (both the equation and the initial condition) term by term. It is natural to regard such a series as a precise solution of the Cauchy problem in contrast to asymptotic solutions introduced in Definition 1.8.

Putting $t = \pi/2$ in (1.4) we obtain

$$\psi\left(x, \frac{\pi}{2}\right) = \frac{e^{-\frac{i\pi m}{4} + \frac{i\pi\sigma}{2}}}{(2\pi h)^{\frac{m}{2}}} \int\limits_{\mathbf{R}^m} e^{-\frac{i}{h}\langle x_{\overline{I}},\, u^{-1}\varepsilon u y_{\overline{I}}\rangle} \psi_0(x_I, y_{\overline{I}})\, dy_{\overline{I}}.$$

Thus,

$$I(x_I, \xi_{\overline{I}}, h) = \psi\left(x_I, u^{-1}\varepsilon u \xi_{\overline{I}}, \frac{\pi}{2}\right) \tag{1.5}$$

if

$$\psi_0(x) = \left[e^{-\frac{i\pi\sigma}{2} + \frac{i}{h}\Phi(\alpha) - \frac{i}{2}\operatorname{Arg} J(\alpha)} |J(\alpha)|^{-1/2} L\varphi(\alpha)\right]_{\alpha=\alpha(x)}. \tag{1.6}$$

So the problem of constructing an asymptotic expansion for $I(x_I, \xi_{\overline{I}}, h)$ is reduced to that of the solution of the Cauchy problem (1.3). The standard way for obtaining an asymptotic expansion of the solution of a differential equation is as follows:

(a) one constructs an asymptotic solution of this equation;

(b) by an appropriate estimate of the solution of the corresponding non-homogeneous Cauchy problem one proves that the asymptotic solution is equivalent to the precise one.

Therefore, we give some estimates of solutions of the Schrödinger equation corresponding to the bypassing focuses Hamiltonian.

Lemma 1.1. 1°. *Let* $\psi_0(x)$, $f(x, t)$, $x \in \mathbf{R}^m$, $t \in \left[0, \frac{\pi}{2}\right]$ *be smooth functions vanishing for* $|x| > R$, *and let* $\psi(x, t)$ *be the solution of the Cauchy problem*

$$\begin{cases} -ih\dfrac{\partial\psi(x, t)}{\partial t} + \\ \qquad + \dfrac{1}{2}\sum\limits_{j=1}^{m} \varepsilon_j\left[\left(-ih\dfrac{\partial}{\partial x}\right)^2 + x^2\right]\psi(x, t) = f(x, t) \\ \psi(x, 0) = \psi_0(x), \end{cases} \tag{1.7}$$

where $\varepsilon_j = \pm 1$.

Then the following estimate holds for sufficiently large l:

$$\|\psi(x, t)\|_{C_l^k(\mathbf{R}_x^m)} \leqslant c_{k,l}(R)\, h^{-k-1-\left[\frac{m}{2}\right]} \times$$
$$\times \left(\|\psi_0\|_{C^{k+l}} + h^{-1}\max_{0 \leqslant t \leqslant \frac{\pi}{2}} \|x\|_{C^{k+l}(\mathbf{R}_x^m)}\right).$$

2°. *Let* $\psi_0 \in S$, $f = 0$. *Then*

$$\|\psi(x, t)\|_{C_l^k(\mathbf{R}_x^m)} \leqslant c_{k,l}(t)\, h^{-\frac{m}{2}-k} \|\psi_0\|_{C_{l+k+m}^{l+k}}, \quad t \in \left(0, \frac{\pi}{2}\right],$$

where $c_{k,l}(t)$ *is a continuous function.*

Proof. (1) Let $f=0$. Then

$$\psi(x,\ t)=\frac{e^{-\frac{i\pi m}{4}+\frac{i\pi\sigma}{2}}}{(2\pi h\sin t)^{m/2}}\times$$

$$\times\int\limits_{\mathbf{R}^m}\exp\left\{\frac{i}{h\sin t}\left[\sum_{j=1}^{m}\varepsilon_j\left(\frac{x_j^2+y_j^2}{2}\cos t-x_jy_j\right)\right]\right\}\psi_0(y)\,dy=$$

$$\stackrel{\text{def}}{=}\frac{1}{(2\pi)^{m/2}}\exp\left(-\frac{i\pi m}{4}+\frac{i\pi\sigma}{2}\right)(h\sin t)^{-\frac{m}{2}}J[\psi_0].\tag{1.8}$$

Set

$$\Phi(x,\ y,\ t)=\sum_{j=1}^{m}\varepsilon_j\left(\frac{x_j^2+y_j^2}{2}\cos t-x_jy_j\right).$$

Then we have

$$\frac{\partial\Phi}{\partial y_j}=\varepsilon_j(y_j\cos t-x_j),\tag{1.9}$$

so $\dfrac{\partial\Phi}{\partial y}$ vanishes only if $y=x/\cos t$.

Let $\Omega(x,\ t)$ denote the ball

$$\left\{y\in\mathbf{R}^m\,\middle|\,\left|y-\frac{x}{\cos t}\right|<\frac{|x|+1}{2\cos t}\right\}.$$

Then $\partial\Phi/\partial y\neq 0$ for $y\in\mathbf{R}^m\setminus\Omega(x,\ t)$. Therefore the integral

$$I_1[\psi_0]=\int\limits_{\mathbf{R}^m\times\Omega(x,\ t)}\exp\left\{\frac{i}{h\sin t}\Phi(x,\ y,\ t)\right\}\psi_0(y)\,dy$$

can be transformed by partial integration as follows:

$$I_1[\psi_0]=-i\int\limits_{\partial\Omega(x,\ t)}\exp\left(\frac{i}{h\sin t}\Phi(x,\ y,\ t)\right)\psi_0(y)\frac{\partial g}{\partial n}\,d\sigma+$$

$$+h\sin t\int\limits_{\mathbf{R}^m\times\Omega(x,\ t)}\exp\left(\frac{i}{h\sin t}\Phi(x,\ y,\ t)\right)L\psi_0(y)\,dy,$$

where

$$Lf(y)=i\operatorname{div}_y(g(x,\ y,\ t)f(y)),\quad g=\Phi_y\cdot|\Phi_y|^{-2},\tag{1.10}$$

$d\sigma$ is the area element on the sphere $\partial\Omega(x,\ t)$, and $\partial/\partial n$ is the exterior normal derivation. Repeating integration by parts M times

yields

$$I_1[\psi_0] = -i \int_{\partial\Omega(x,\,t)} \exp\left(\frac{i}{h\sin t}\Phi(x,\,y,\,t)\right) \times$$

$$\times \sum_{l=0}^{M-1} [(h\sin t)^l L^l \psi_0(y)] \frac{\partial g}{\partial n} d\sigma +$$

$$+ (h\sin t)^M \int_{\mathbf{R}^m\setminus\Omega(x,\,t)} \exp\left(\frac{i}{h\sin t}\Phi(x,\,y,\,t)\right) L^M\psi_0(y)\,dy.$$

It follows from (1.9) and (1.10) by a simple calculation that

$$|L^M\psi_0| \leqslant \text{const}\,|y\cos t - x|^{-M} \sup_{|j|\leqslant M}\left|\left(\frac{\partial}{\partial y}\right)^j \psi_0(y)\right|$$

for x, y and t such that $|y\cos t - x| \geqslant a > 0$, const depending on a. Hence

$$|I_1(\psi_0)| \leqslant \text{const}\left[\sum_{l=0}^{M-1} (h\sin t)^l (|x|+1)^{-l-1+\frac{m}{2}} \|\psi_0\|_{C^l(\Omega(x,\,t))} +\right.$$

$$\left. + (h\sin t)^M (|x|+1)^{-M} \int_{\mathbf{R}^m} \sup_{|j|\leqslant M}\left|\left(\frac{\partial}{\partial y}\right)^j \psi_0(y)\right| dy\right]. \quad (1.11)$$

As to the integral

$$I_2[\psi_0] = I[\psi_0] - I_1[\psi_0] = \int_{\Omega(x,\,t)} \exp\left(\frac{i}{h}\Phi(x,\,y,\,t)\right)\psi_0(y)\,dy,$$

it is easy to obtain the following estimate:

$$|I_2[\psi_0]| \leqslant \int_{\Omega(x,\,t)} |\psi_0(y)|\,dy \leqslant \text{const}\cdot|x|^m\cdot \sup_{y\in\Omega(x,\,t)} |\psi_0(y)| \leqslant$$

$$\leqslant \text{const}\,(|x|+1)^{m-l}.$$

Combining the estimates for $I_1[\psi_0]$ and $I_2[\psi_0]$ gives

$$|I[\psi_0]| \leqslant \text{const}\,(|x|+1)^{-M} \|\psi_0\|^M_{C_{M+m}}. \quad (1.12)$$

Now derivating (1.8), we see that $\left(\frac{\partial}{\partial x}\right)^j \psi(x,\,t)$ can be expressed as the sum of terms of the form

$$c(x,\,t)(h\sin t)^{-\frac{m}{2}-l} x^k I[y^n\psi_0],$$

where

$$\frac{1}{2}|j| \leqslant l \leqslant |j|, \quad |n| \leqslant 2l - |j|, \quad |k| = 2l - |n| - |j|,$$

and $c(x,\,t)$ is a smooth function. Hence 2° follows from (1.12).

(2) Now let the assumptions of 1° be satisfied. First we estimate ψ for large x. We have

$$\psi = \bar{\psi} + \bar{\bar{\psi}},$$

where $\bar{\psi}$ and $\bar{\bar{\psi}}$ are the solutions of (1.7) with f replaced by 0 and ψ replaced by 0, respectively.

If $|x| \geqslant 2R+1$, then ψ_0 vanishes on $\Omega(x, t)$, so $I_2[\psi_0] = 0$ and

$$|I_1[\psi_0]| \leqslant \text{const}\,(h \sin t)^M\, |x|^{-M} \|\psi_0\|_{C^M}.$$

Therefore, we can estimate $\bar{\psi}$ for $|x| \geqslant 2R+1$ as follows:

$$\left|\left(\frac{\partial}{\partial x}\right)^j \bar{\psi}(x, t)\right| \leqslant$$
$$\leqslant \text{const}\,(h \sin t)^{M-\frac{m}{2}-|j|}\, |x|^{|j|-M} \|\psi_0\|_{C^M}. \tag{1.13}$$

We further have

$$\bar{\bar{\psi}}(x, t) = \frac{i}{h}\int_0^t \psi(x, t, \tau)\, d\tau,$$

where $\psi(x, t, \tau)$ is the solution of the Cauchy problem

$$\begin{cases} -ih\dfrac{\partial\psi}{\partial t} + \displaystyle\sum_{j=1}^{m} \frac{\varepsilon_j}{2}\left[\left(-ih\frac{\partial}{\partial x}\right)^2 + x^2\right]\psi = 0, \\ \psi(x, t, \tau) = f(x, \tau). \end{cases}$$

Estimating $\psi(x, t, \tau)$ by (1.13) and integrating with respect to τ on $[0, t]$ we obtain for $|x| \geqslant 2R+1$:

$$\left|\left(\frac{\partial}{\partial x}\right)^j \psi(x, t)\right| \leqslant C_{M,j}(R)\, h^{M-\frac{m}{2}-|j|-1}\, t^{M-\frac{m}{2}-|j|+1} \times$$
$$\times |x|^{|j|-M} \max_{0 \leqslant t \leqslant \frac{\pi}{2}} \|f(x, t)\|_{C^M(\mathbf{R}_x^m)},$$

$$\forall M > \frac{m}{2} + |j| - 1.$$

Combining the last estimate with (1.13) gives for $|x| \geqslant 2R+1$

$$\left|\left(\frac{\partial}{\partial x}\right)^j \psi(x, t)\right| \leqslant C_{M,j}(R)\, h^{M-\frac{m}{2}-|j|}\, |x|^{|j|-M} \times$$
$$\times \left(\|\psi_0\|_{C^M} + \max_t \|f\|_{C^M(\mathbf{R}_x^m)}\right). \tag{1.14}$$

Now let us give estimates on the whole of $\mathbf{R}^n$. The following is well known:

$$\|\psi\|_{L_2(\mathbf{R}_x^m)} \leqslant \|\psi_0\|_{L_2} + \frac{1}{h}\int_0^t \|f(x, \tau)\|_{L_2(\mathbf{R}_x^m)}\, d\tau. \tag{1.15}$$

Set $\psi^{(j)} = \left(\frac{\partial}{\partial x}\right)^j \psi$. Then $\psi^{(j)}$ satisfies (1.7) with ψ_0 replaced by $\left(\frac{\partial}{\partial x}\right)^j \psi_0$ and f replaced by a linear combination of the functions $\left(\frac{\partial}{\partial x}\right)^j f$, $\psi^{(l)}$ and $x_s\psi^{(p)}$ with $|l| = |j| - 2$, $|p| = |j| - 1$, $s = 1, 2, \ldots, n$. Using this fact we prove by induction that

$$\|\psi^{(j)}\|_{L_2(\mathbf{R}_x^m)} \leqslant \operatorname{const} h^{-|j|}\Big(\|\psi_0\|_{C^{|j|+\left[\frac{m}{2}\right]}} + h^{-1}\max_{0\leqslant t\leqslant\frac{\pi}{2}}\|f\|_{C^{|j|+\left[\frac{m}{2}\right]}(\mathbf{R}_x^m)}\Big), \tag{1.16}$$

with const depending on j and R. For $j = 0$, (1.16) follows from (1.15) by noting that any function φ with support in a ball $\overline{\Omega}_R \subset \mathbf{R}^n$ of radius R satisfies the inequality

$$\|\varphi\|_{L_2(\mathbf{R}^m)} \leqslant \|\varphi\|_C \cdot V_R^{1/2},$$

where V_R is the volume of $\overline{\Omega}_R$. Suppose (1.16) holds for $|j| \leqslant N$ and estimate $\psi^{(k)}$ with $|k| = N + 1$. We have

$$\left\|\left(\frac{\partial}{\partial x}\right)^k \psi_0\right\|_{L_2} \leqslant \operatorname{const}\|\psi_0\|_{C^{N+1}},$$

$$\left\|\left(\frac{\partial}{\partial x}\right)^k f\right\|_{L_2(\mathbf{R}_x^m)} \leqslant \operatorname{const}\|f\|_{C^{N+1}(\mathbf{R}_x^m)}.$$

If $|l| = N - 1$, $|p| = N$, then

$$\|\psi^{(l)}\|_{L_2(\mathbf{R}_x^m)} \leqslant \operatorname{const} h^{1-N}\Big(\|\psi_0\|_{C^{N+\left[\frac{m}{2}\right]}} + h^{-1}\max_t\|f\|_{C^{N+\left[\frac{m}{2}\right]-1}(\mathbf{R}_x^m)}\Big)$$

by the induction hypothesis, and

$$\|x_s\psi^{(p)}\|_{L_2(\mathbf{R}_x^m)} \leqslant \operatorname{const} h^{-N}\Big(\|\psi_0\|_{C^{N+\left[\frac{m}{2}\right]}} + h^{-1}\max_t\|f\|_{C^{N+\left[\frac{m}{2}\right]}(\mathbf{R}_x^m)}\Big)$$

by (1.14) $\left(\text{with } M=N+\left[\frac{m}{2}\right]+1\right)$ and the induction hypothesis. These estimates imply that

$$\|\psi^{(h)}\|_{L_2(\mathbf{R}_x^m)} \leqslant \text{const}\, h^{-N-1}\left(\|\psi_0\|_{C^{N+\left[\frac{m}{2}\right]+1}}+\right.$$
$$\left.+h^{-1}\max_t\|f\|_{C^{|j|+2\left[\frac{m}{2}\right]+1}(\mathbf{R}_x^m)}\right),$$

which proves (1.16).

It follows from (1.16) by the Sobolev embedding theorem that

$$|\psi^{(j)}(x,\ t)| \leqslant \text{const}\, h^{-\left(|j|+\left[\frac{m}{2}\right]+1\right)}\left(\|\psi_0\|_{C^{|j|+2\left[\frac{m}{2}\right]+1}}+\right.$$
$$\left.+h^{-1}\max_t\|f\|_{C^{|j|+2\left[\frac{m}{2}\right]+1}(\mathbf{R}_x^m)}\right).$$

This estimate together with (1.14) prove 1°.

Corollary 1.1. *Let $\psi_0(x,\ h)$ be an h-asymptotic series equivalent to zero. Then $F_{x_{\bar I}\to\xi_{\bar I}}\psi_0\approx 0$.*

Proof. Note that

$$F_{x_{\bar I}\to\xi_{\bar I}}\psi_0(x,\ h)=\psi\left(x_I,\ \xi_{\bar I},\ \frac{\pi}{2},\ h\right),$$

where $\psi(x,\ t,\ h)$ is the solution of (1.3) with $\varepsilon=1$, $u=1$, and use 2° of Lemma 1.1.

Proof of Proposition 1.1. Parallel with the precise solution of (1.2) consider an asymptotic solution of the following form:

$$\psi_1(x,\ t,\ h)=e^{\frac{i}{h}S(x,\ t)}v(x,\ t,\ h). \tag{1.17}$$

We have

$$\left[-ih\frac{\partial}{\partial t}+H\left(-ih\frac{\partial}{\partial x},\ x\right)\right]\psi_1=e^{\frac{i}{h}S}\{\hat H v+S_t v-ihv_t\},$$

where

$$\hat H=H(S_x,\ x)-ih\left[\frac{1}{2}\ \mathrm{tr}\ (H_{pp}S_{xx})+\right.$$
$$\left.+\left\langle H_p(S_x,\ x),\ \frac{\partial}{\partial x}\right\rangle\right]-\frac{h^2}{2}\ \mathrm{tr}\ \left(H_{pp}\frac{\partial^2}{\partial x^2}\right).$$

Set

$$(\Lambda_t^n,\ r_t^n)=(g_H^t\Lambda^n,\ (dg_H^t)\,r^n).$$

Let $\Phi(\alpha,\ t)$ be the phase on Λ_t^n, and let Π: $(\alpha,\ t)\to(x,\ t)$ be the diffeomorphism corresponding to a non-singular patch of the non-

singular zone. Put $S=\Phi\circ\Pi^{-1}$. We have already shown that $S(x, t)$ satisfies the Hamilton-Jacobi equation with dissipation

$$\frac{\partial S}{\partial t}+H(S_x, x)\overset{\text{def}}{=\!=}g(x, t)=O_{S_2}(h^{3/2}).$$

Substitution of (1.17) into (1.2) leads by noting the last equation for S to the following *transfer equation* to be satisfied by v:

$$v_t+\langle H_p(S_x, x),\ v_x\rangle+\left[\frac{1}{2}\operatorname{tr}(H_{pp}S_{xx})+\mathcal{A}\right]v\approx 0, \tag{1.18}$$

where

$$\mathcal{A}=\frac{i}{h}g(x, t)-\frac{ih}{2}\operatorname{tr}\left(H_{pp}\frac{\partial^2}{\partial x^2}\right)\in\mathcal{P}.$$

By Example 1.3 this equation has (in the coordinates α, t) a D-asymptotic solution of the form

$$v=|J(\alpha, t)|^{-1/2}e^{-\frac{i}{2}\operatorname{Arg}J(\alpha, t)}L\sum_{j=0}^{\infty}(I\hat{\delta})^j v_0(\alpha) \tag{1.19}$$

with $\hat{\delta}\in\mathcal{P}$ determined by the Lagrangean manifold with the complex germ and the bypassing focuses Hamiltonian. (Note that if $\rho(\alpha, h)$ is a D-asymptotic series equivalent to zero, then $\exp\left(\frac{i}{h}\Phi(\alpha)\right)\rho(\alpha, h)$ is an h-asymptotic series equivalent to zero.) In order to satisfy the initial condition $\psi_1|_{t=0}\approx\psi_0$, where ψ_0 is defined by (1.6), we put

$$v_0(\alpha)=e^{-\frac{i\pi}{2}\sigma}\varphi(\alpha). \tag{1.20}$$

Now we observe that

$$\beta\left(\alpha, \frac{\pi}{2}\right)=\beta_I(\alpha),$$

so L turns for $t=\frac{\pi}{2}$ into L_I; moreover,

$$J\left(\alpha, \frac{\pi}{2}\right)=(-1)^{\sigma}J_I(\alpha).$$

Recall further that

$$\Phi\left(\alpha, \frac{\pi}{2}\right)=\Phi_I(\alpha)$$

and

$$q\left(\alpha, \frac{\pi}{2}\right)=\{q_I(\alpha),\ u^{-1}\varepsilon up_{\bar{I}}(\alpha)\}.$$

Therefore, we obtain the following expression of $\psi_1(x, t, h)$ for $t=\pi/2$ and $x=\{q_I(\alpha),\ u^{-1}\varepsilon up_{\bar{I}}(\alpha)\}$:

$$\psi_1\left(\{q_I(\alpha),\ u^{-1}\varepsilon p_{\bar{I}}(\alpha)\}, \frac{\pi}{2}, h\right)=e^{\frac{i}{h}\Phi(\alpha)}J_I^{-\frac{1}{2}}(\alpha)L_I v_I\varphi(\alpha),$$

where

$$J_I^{-\frac{1}{2}}(\alpha) \overset{\text{def}}{=\!=} e^{-\frac{i}{2}[\operatorname{Arg} J(\alpha,\, \pi/2) - \pi\sigma]} \left| J_I(\alpha) \right|^{-\frac{1}{2}},$$

$$v_I = \sum_{j=0}^{\infty} (I\hat{\delta})^j \Big|_{t=\frac{\pi}{2}}.$$

Note here that v_I satisfies the requirement of the proposition (see Problem 1.1).

To complete the proof, we shall show that $\psi_1(x, t, h) \approx \approx \psi(x, t, h)$. Let

$$\psi = \sum_{k \in K} \psi_k,$$

$$\psi_1 = \sum_{k \in K_1} \psi_{1k},$$

and set

$$\psi^{(R)} = \sum_{k \in R} \psi_k, \qquad \psi_1^{(R_1)} = \sum_{k \in R_1} \psi_{1k},$$

$$\chi_{RR_1} = \psi^{(R)} - \psi_1^{(R_1)},$$

where both R and R_1 are finite. Fix a natural N. Since $\psi|_{t=0} \approx \approx \psi_1|_{t=0}$ and

$$\left[-ih\frac{\partial}{\partial t} + H\left(-ih\frac{\partial}{\partial x},\, x\right)\right](\psi - \psi_1) = f,$$

where f has the form

$$e^{\frac{i}{h}\Phi(\alpha,\, t)} \rho(\alpha,\, t,\, h)\,|_{(\alpha,\, t) = \Pi^{-1}(x,\, t)}$$

with some D-asymptotic $\rho \approx 0$, it follows that for any k and l, there are finite sets $R \subset K$ and $R_1 \subset K_1$ such that for any finite $R' \supset R$ and $R_1' \supset R_1$,

$$\| \chi_{R',\, R_1'}(x, 0, h) \|_{C^{k+l}(\mathbf{R}_x^n)} = O\left(h^{N + [\frac{m}{2}] + k + 1}\right),$$

$$\left[-ih\frac{\partial}{\partial t} + H\left(-ih\frac{\partial}{\partial x},\, x\right)\right]\chi_{R',\, R_1'} = f_{R'\, R_1'}(x, t, h)$$

with

$$\| f_{R',\, R_1'} \|_{C^{k+l}(\mathbf{R}_x^n)} = O\left(h^{N + [\frac{m}{2}] + k + 2}\right).$$

Furthermore, $\chi_{R_1',\, R_1'}(x, 0, h)$ and $f_{R',\, R_1'}(x, t, h)$ vanish for x being outside a ball independent of h and t. Hence

$$\| \chi_{R',\, R_1'} \|_{C_l^k(\mathbf{R}_x^n)} = O(h^N)$$

by lemma 1.1, which means that $\psi \approx \psi_1$.

The proposition is proved.

Now we give a generalization of Proposition 1.1 to the case where α_0 need not be in the intersection of non-singular patches of the zones Ω and Ω_I. To do this we use the fact that for the canonical transformation $g^\tau_{H_0}$ associated with the harmonic oscillator Hamiltonian function $H_0 = \frac{1}{2}(p^2 + q^2)$, a neighborhood in $g^\tau_{H_0}\Lambda^n$ of the point $g^\tau_{H_0}\alpha_0$ satisfies the condition of Theorem 5.1 of Chapter IV provided that τ is sufficiently small.

For the rest of this section we shall fix the following notation. Let $J(\alpha, \tau)$ be the complex Jacobian corresponding to the family $\{g^\tau_{H_0}\Lambda^n, dg^\tau_{H_0}r^n\}$. Given a continuous branch $\operatorname{Arg} J(\alpha)$ of the phase argument of $J(\alpha)$, we define $\operatorname{Arg} J(\alpha, \tau)$ as the continuous function satisfying the conditions

$$J(\alpha, \tau) = |J(\alpha, \tau)| \exp(i \operatorname{Arg} J(\alpha, \tau)),$$

$$\operatorname{Arg} J(\alpha, 0) = \operatorname{Arg} J(\alpha).$$

For every small $\tau > 0$ choose a bypassing focuses Hamiltonian H^τ associated with $g^\tau_{H_0}\alpha_0$ and the pair $(\overline{\Phi}, I)$ and having the form (5.9) of Chapter IV. Let $J(\alpha, \tau, t)$ be the complex Jacobian corresponding to the bypassing focuses operation associated with H^τ. We define $\operatorname{Arg} J(\alpha, \tau, t)$ as the continuous in t function satisfying the conditions

$$J(\alpha, \tau, t) = |J(\alpha, \tau, t)| \exp(i \operatorname{Arg} J(\alpha, \tau, t)),$$

$$\operatorname{Arg} J(\alpha, \tau, 0) = \operatorname{Arg} J(\alpha, \tau).$$

Set

$$\operatorname{Arg} J_I(\alpha, \tau) = \operatorname{Arg} J\left(\alpha, \tau, \frac{\pi}{2}\right) + \pi\sigma(\tau),$$

where $\sigma(\tau)$ is the number of negative eigenvalues of the matrix $\varepsilon(\tau)$ which occurs in the Hamiltonian H^τ. In Sec. 3 (see Lemma 3.5) we shall show that the limit

$$\lim_{\tau\to+0} \operatorname{Arg} J_I(\alpha, \tau) \overset{\text{def}}{=\!=} \operatorname{Arg} J_I(\alpha) \tag{1.22}$$

exists and does not depend on the choice of H^τ.

Proposition 1.2. *Let $\alpha_0 \in \Gamma$ be in the intersection of two patches (u_γ, π_γ) and $(u_{\gamma'}, \pi^I_{\gamma'})$. Then for any $\varphi(\alpha) \in C_0^\infty$ supported near α_0, we have*

$$F_{x_{\bar I}\to\xi_{\bar I}}\left[e^{\frac{i}{h}\Phi^\gamma(\alpha)-\frac{i}{2}\operatorname{Arg} J(\alpha)}|J(\alpha)|^{-\frac{1}{2}} L^\gamma\varphi(\alpha)\right]_{\alpha=\pi_\gamma^{-1}(x)} \approx$$
$$\approx \left[e^{\frac{i}{h}\Phi_I^{\gamma'}(\alpha)-\frac{i}{2}\operatorname{Arg} J_I(\alpha)}|J_I|^{-\frac{1}{2}} L_I^{\gamma'} v^{\gamma,\gamma'}_{\overline{\Phi},I}\,\varphi(\alpha)\right]_{\alpha=(\pi^I_{\gamma'})^{-1}(x_I,\xi_{\bar I})}, \tag{1.23}$$

where

$$L_I^{\gamma'} = \sum_{|j|=0}^{\infty} \frac{(\beta_{\gamma'}^I)^j}{j!} \left(\frac{\partial}{\partial \alpha}\right)^j,$$

$$\beta_{\gamma'}^I = -C_I^{-1}\left[\{z_I, w_{\bar{I}}\} - \gamma'\right]$$

and $v_{\Phi, I}^{\gamma, \gamma'}$ *is a D-asymptotic differential quasi-identity operator.*

To prove this proposition we need the following lemma.

Lemma 1.2. *Let* $\alpha_0 \in \Gamma$ *be in the intersection of two patches,* (u_γ, π_γ^I) *and* $(u_{\gamma'}, \pi_{\gamma'}^I)$, *of the zone* Ω_I. *Then for any* $\varphi(\alpha) \in C_0^\infty$ *supported near* α_0, *we have*

$$[J_I(\alpha)]^{-\frac{1}{2}} e^{\frac{i}{h}\Phi_I^\gamma(\alpha)} L_I^\gamma \varphi(\alpha)\,|_{\alpha=(\pi_\gamma^I)^{-1}(x)} \approx$$

$$\approx e^{\frac{i}{h}\Phi_I^{\gamma'}(\alpha)} [J_I(\alpha)]^{-\frac{1}{2}} L_I^{\gamma'} v_I^{\gamma, \gamma'} \varphi(\alpha)\,|_{\alpha=(\pi_{\gamma'}^I)^{-1}(x)}, \tag{1.24}$$

where $v_I^{\gamma, \gamma'}$ *is a D-asymptotic differential quasi-identity operator. Here* $[J_I(\alpha)]^{\frac{1}{2}}$ *is a smooth branch of the root of the Jacobian in the neighborhood of* α_0.

Proof. As before, it suffices to check the result for the special case where Ω_I is the non-singular zone.

First we prove that

$$\exp\left\{\frac{i}{h}\Phi^\gamma(\sigma(\alpha))\right\} \overset{\alpha_0}{\approx} \exp\left\{\frac{i}{h}\Phi^{\gamma'}(\alpha)\right\}\psi(\alpha),$$

where $\sigma = \pi_\gamma \circ \pi_{\gamma'}^{-1}$ and ψ is the D-asymptotic at α_0 series defined by

$$\psi = \sum_{j=0}^{\infty} \frac{1}{j!}\left[\frac{i}{h}(\Phi^\gamma(\sigma(\alpha)) - \Phi^{\gamma'}(\alpha))\right]^j.$$

Recall that

$$\sigma(\alpha) = \alpha + O_D(h^{1/2})$$

by Lemma 3.3 of Chapter IV, and

$$\Phi^\gamma(\sigma(\alpha)) = \Phi^{\gamma'}(\alpha) + O_D(h^{3/2})$$

by Lemma 3.4 of Chapter IV.

We have to show that

$$\left(\frac{\partial}{\partial\alpha}\right)^j \exp\left\{\frac{i}{h}\Phi^\gamma(\sigma(\alpha))\right\} \overset{\alpha_0}{\sim} \left(\frac{\partial}{\partial\alpha}\right)^j \exp\left\{\frac{i}{h}\Phi^{\gamma'}(\alpha)\right\}\psi$$

for any multi-index j. For $j=0$, this follows from the identity

$$\exp\left\{\frac{i}{h}\Phi^{\nu}(\sigma(\alpha))\right\}=\exp\left\{\frac{i}{h}\Phi^{\nu'}(\alpha)\right\}\times$$
$$\times\exp\left\{\frac{i}{h}[\Phi^{\nu}(\sigma(\alpha))-\Phi^{\nu'}(\alpha)]\right\}=$$
$$=\sum_{j=0}^{N}\exp\left\{\frac{i}{h}\Phi^{\nu'}(\alpha)\right\}\frac{1}{j!}\left\{\frac{i}{h}[\Phi^{\nu}(\sigma(\alpha))-\Phi^{\nu'}(\alpha)]\right\}^{j}+$$
$$+\frac{1}{(N+1)!}\left\{\frac{i}{h}[\Phi^{\nu}(\sigma(\alpha))-\Phi^{\nu'}(\alpha)]\right\}^{N+1}\times$$
$$\times\exp\left\{\frac{i}{h}[\Phi^{\nu'}(\alpha)+\theta(\alpha)(\Phi^{\nu}(\sigma(\alpha))-\Phi^{\nu'}(\alpha))]\right\},$$

where $0<\theta(\alpha)<1$. To prove it for all j, note that

$$\left(\frac{\partial}{\partial\alpha}\right)^{j}\exp\left\{\frac{i}{h}\Phi^{\nu'}(\alpha)\right\}\psi=\exp\left\{\frac{i}{h}\Phi^{\nu'}(\alpha)\right\}\left(\frac{\partial}{\partial\alpha}+\frac{i}{h}\frac{\partial\Phi^{\nu'}(\alpha)}{\partial\alpha}\right)^{j}\psi$$

and

$$\left(\frac{\partial}{\partial\alpha}\right)^{j}\exp\left\{\frac{i}{h}\Phi^{\nu}(\sigma(\alpha))\right\}=$$
$$=\exp\left\{\frac{i}{h}\Phi^{\nu}(\sigma(\alpha))\right\}\left(\frac{\partial}{\partial\alpha}+\frac{i}{h}\frac{\partial\Phi^{\nu}(\sigma(\alpha))}{\partial\alpha}\right)^{j}1\overset{\alpha_0}{\sim}$$
$$\overset{\alpha_0}{\sim}\exp\left\{\frac{i}{h}\Phi^{\nu'}(\alpha)\right\}\psi\left(\frac{\partial}{\partial\alpha}+\frac{i}{h}\frac{\partial\Phi^{\nu}(\sigma(\alpha))}{\partial\alpha}\right)^{j}1,$$

which shows that it suffices to check the relation

$$\left(\frac{\partial}{\partial\alpha}+\frac{i}{h}\frac{\partial\Phi^{\nu'}(\alpha)}{\partial\alpha}\right)^{j}\psi\overset{\alpha_0}{\approx}\psi\left(\frac{\partial}{\partial\alpha}+\frac{i}{h}\frac{\partial\Phi^{\nu}(\sigma(\alpha))}{\partial\alpha}\right)^{j}1,\quad |j|\geqslant 1.$$

The proof will be by induction on $|j|$. We have

$$\left(\frac{\partial}{\partial\alpha_s}+\frac{i}{h}\frac{\partial\Phi^{\nu'}(\alpha)}{\partial\alpha_s}\right)\psi=$$
$$=\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{i}{h}\right)^{r}\left\{r[\Phi^{\nu}(\sigma(\alpha))-\Phi^{\nu'}(\alpha)]^{r-1}\times\right.$$
$$\times\left(\frac{\partial\Phi^{\nu}(\sigma(\alpha))}{\partial\alpha_s}-\frac{\partial\Phi^{\nu'}(\alpha)}{\partial\alpha_s}\right)+$$
$$\left.+\frac{i}{h}\frac{\partial\Phi^{\nu'}(\alpha)}{\partial\alpha_s}\left(\Phi^{\nu}(\sigma(\alpha))-\Phi^{\nu'}(\alpha)\right)^{r}\right\}\overset{\alpha_0}{\approx}$$
$$\overset{\alpha_0}{\approx}\sum_{r=0}\frac{1}{r!}\left(\frac{i}{h}\right)^{r+1}\frac{\partial\Phi^{\nu'}(\alpha)}{\partial\alpha_s}(\Phi^{\nu}(\sigma(\alpha)-\Phi^{\nu'}(\alpha))^{r}=$$
$$=\psi\left(\frac{\partial}{\partial\alpha_s}+\frac{i}{h}\frac{\partial\Phi^{\nu'}(\alpha)}{\partial\alpha_s}\right)1.$$

So the relation in question holds for $|j|=1$. Assume that it holds for $|j|\leqslant N$ and let $|k|=N+1$, $k_s\neq 0$. Then

$$\left(\frac{\partial}{\partial\alpha}+\frac{i}{h}\frac{\partial\Phi^{\gamma'}(\alpha)}{\partial\alpha}\right)^k\psi=$$

$$=\left(\frac{\partial}{\partial\alpha_s}+\frac{i}{h}\frac{\partial\Phi^{\gamma'}(\alpha)}{\partial\alpha_s}\right)\left(\frac{\partial}{\partial\alpha}+\frac{i}{h}\frac{\partial\Phi^{\gamma'}(\alpha)}{\partial\alpha}\right)^{k-1_s}\psi\overset{\alpha_0}{\approx}$$

$$\overset{\alpha_0}{\approx}\left(\frac{\partial}{\partial\alpha_s}+\frac{i}{h}\frac{\partial\Phi^{\gamma'}(\alpha)}{\partial\alpha_s}\right)\left[\psi\left(\frac{\partial}{\partial\alpha}+\frac{i}{h}\frac{\partial\Phi^{\gamma}(\sigma(\alpha))}{\partial\alpha}\right)^{k-1_s}1\right]\overset{\alpha_0}{\approx}$$

$$\overset{\alpha_0}{\approx}\psi\left[\frac{i}{h}\frac{\partial\Phi^{\gamma}(\sigma(\alpha))}{\partial\alpha_s}\left(\frac{\partial}{\partial\alpha}+\frac{i}{h}\frac{\partial\Phi^{\gamma}(\sigma(\alpha))}{\partial\alpha}\right)^{k-1_s}1+\right.$$

$$\left.+\frac{\partial}{\partial\alpha_s}\left(\frac{\partial}{\partial\alpha}+\frac{i}{h}\frac{\partial\Phi^{\gamma}(\sigma(\alpha))}{\partial\alpha}\right)^{k-1_s}1\right]=\psi\left(\frac{\partial}{\partial\alpha}+\frac{i}{h}\frac{\partial\Phi^{\gamma}(\sigma(\alpha))}{\partial\alpha}\right)^k 1,$$

which proves the desired result.

Next we have

$$[J(\sigma(\alpha))]^{-1/2}=[J(\alpha)]^{-1/2}(1+\chi(\alpha))$$

with $\chi(\alpha)=O_D(h^{1/2})$, and

$$[L^{\gamma}\varphi](\sigma(\alpha))\approx\sum_{|j|=0}^{\infty}\sum_{|k|=0}^{\infty}\sum_{l\leqslant k}\frac{[\sigma(\alpha)-\alpha]^k}{l!\,j!\,(k-l)!}\frac{\partial^{|l|}\beta_{\gamma}^{j}}{\partial\alpha^l}\frac{\partial^{|j+k-l|}\varphi}{\partial\alpha^{j+k-l}}\approx$$

$$\approx\sum_{|r|=0}^{\infty}\sum_{s\leqslant r}\sum_{|l|=0}^{\infty}\frac{[\sigma(\alpha)-\alpha]^{l+s}}{l!\,s!\,(r-s)!}\frac{\partial^{|l|}\beta^{r-s}}{\partial\alpha^l}\frac{\partial^{|r|}\varphi}{\partial\alpha^r}\approx$$

$$\approx\sum_{|r|=0}^{\infty}\frac{1}{r!}[\sigma(\alpha)-\alpha+\beta_{\gamma}(\sigma(\alpha))]^r\frac{\partial^{|r|}\varphi}{\partial\alpha^r}\overset{\text{def}}{=\!=}L'\varphi.$$

Let $R^{\gamma'}$ be the D-asymptotic differential operator defined by

$$R^{\gamma'}=\sum_{|j|=0}^{\infty}\frac{1}{j!}\left(\sum_{|k|=1}^{\infty}\mathcal{P}_k(\beta_{\gamma'})\right)^j\left(\frac{\partial}{\partial\alpha}\right)^j,$$

where $\{\mathcal{P}_k\}$ is the sequence of polynomials of Lemma 6.4 in Chapter IV. According to Lemma 6.5 in Chapter IV,

$$L^{\gamma'}R^{\gamma'}\Psi\approx\Psi$$

for any D-asymptotic series Ψ supported in a small neighborhood of α_0. Put

$$v^{\gamma,\gamma'}=R^{\gamma'}\psi(1+\chi)L'.$$

It remains to be shown that $v^{\gamma,\gamma'}$ is a quasi-identity operator. To do this, we use the observation that $L'-L^{\gamma'}$ raises the order. In fact,

we have

$$(L-L^{\gamma'})\,\varphi(\alpha)=\sum_{|r|=0}^{\infty}\frac{1}{r!}\{[\sigma(\alpha)-\alpha+\beta_\gamma(\sigma(\alpha))]^r-$$
$$-[\beta_{\gamma'}(\alpha)]^r\}\frac{\partial^{|r|}\varphi(\alpha)}{\partial\alpha^r}$$

for any $\varphi(\alpha)\in C^\infty$ supported near α_0. It follows from Lemma 3.3 of Chapter IV that

$$\sigma(\alpha)-\alpha+\beta_\gamma(\sigma(\alpha))-\beta_{\gamma'}(\alpha)=O_D(h),$$

which implies

$$[\sigma(\alpha)-\alpha+\beta_\gamma(\sigma(\alpha))]^r-[\beta_{\gamma'}(\alpha)]^r=O_D\left(h^{\frac{r+1}{2}}\right),$$

so

$$\operatorname{ord}\,[(L'-L^{\gamma'})\,\varphi(\alpha)]\geqslant\frac{1}{2}+\operatorname{ord}\varphi(\alpha).$$

Now the required assertion follows from the identity

$$v^{\gamma,\gamma'}=R^{\gamma'}L^{\gamma'}+R^{\gamma'}(L'-L^{\gamma'})+R^{\gamma'}(\psi-1+\chi\psi)\,L'$$

by noting that χ and $\psi-1$ raise the order, $L^{\gamma'}$ and $R^{\gamma'}$ do not lower the order, and $R^{\gamma'}L^{\gamma'}$ is a quasi-identity operator. The lemma is proved.

Proof of Proposition 1.2. We need some further notation. For every h-asymptotic series ψ_0, let $R_t\psi_0(x,h)=\psi(x,t,h)$ be the solution of the Cauchy problem

$$-ih\frac{\partial\psi}{\partial t}+\left[\frac{x^2}{2}-h^2\left(\frac{\partial}{\partial x}\right)^2\right]\psi=0, \tag{1.25}$$

$$\psi|_{t=0}=\psi_0(x,h), \tag{1.26}$$

and let $R_t^{\mathrm{as}}\psi_0(x,h)=\psi_1(x,t,h)$ be the asymptotic solution of (1.25), (1.26) with the special Cauchy datum

$$\psi_0(x,h)=e^{\frac{i}{h}\Phi^\gamma(\alpha)}\,|J(\alpha)|^{-1/2}\,e^{-\frac{i}{2}\operatorname{Arg}J(\alpha)}\,L^\gamma\varphi(\alpha,h)\,|_{\alpha=\pi_\gamma^{-1}(x)}$$

corresponding to the path (u_γ,π_γ):

$$\psi_1(x,t,h)=e^{\frac{i}{h}\Phi^\gamma(\alpha,t)-\frac{i}{2}\operatorname{Arg}J(\alpha,t)}\times$$
$$\times|J(\alpha,t)|^{-1/2}L^\gamma\sum_{j=0}^{\infty}(I\hat{\delta})^j\varphi(\alpha,h)\,|_{\alpha=\alpha^\gamma(x,t)},$$

where $\alpha^\gamma(x,t)$ is the solution of the equation $q(\alpha,t)+\gamma(\alpha,t)=x$, $\hat{\delta}\in\mathcal{F}$ and Φ^γ, J, L^γ are determined via the $(n+1)$-dimensional manifold $\{g_{H_0}^t\Lambda^n\}$ (with the complex germ $dg_{H_0}^t r^n$). Quite similarly, we define the operator $\hat{R}_t^{\mathrm{as}}$ which sends any h-asymptotic series ψ_τ

of the form

$$\psi_\tau(x, h) = \left[e^{\frac{i}{h}\Phi_I^{\gamma'}(\alpha,\tau) - \frac{i}{2}\operatorname{Arg} J_I(\alpha,\tau)} \times \right.$$
$$\left. \times |J_I(\alpha, \tau)|^{-1/2} L_I^{\gamma'} |_{g_{H_0}^\tau \Lambda^n} \varphi(\alpha, h) \right]_{\alpha=\alpha_{\gamma'}^I(x,\tau)},$$

where $\alpha_{\gamma'}^I(x, \tau)$ is the solution of the system

$$q_I(\alpha, \tau) = \gamma_I'(\alpha, \tau) = x_I, \qquad p_{\bar I}(\alpha, \tau) + \gamma_{\bar I}'(\alpha, \tau) = x_{\bar I}$$

into the asymptotic solution of (1.25) with the initial condition $\psi|_{t=\tau} \approx \psi_\tau$. Let F be the h^{-1}-Fourier transformation with respect to the $\bar I$th group of variables. According to Proposition 2.1, for any D-asymptotic series χ supported near $g_{H_0}^\tau \alpha_0 \in g_{H_0}^\tau \Lambda^n$, we have

$$F\psi(x, \tau, h) \approx F_{\text{as}}\psi(x, \tau, h),$$

where

$$\psi(x, \tau, h) = e^{\frac{i}{h}\Phi(\alpha,\tau) - \frac{i}{2}\operatorname{Arg} J(\alpha,\tau)} |J(\alpha, \tau)|^{-1/2} L(\tau)\chi(\alpha)|_{\alpha=\alpha(x,\tau)}$$

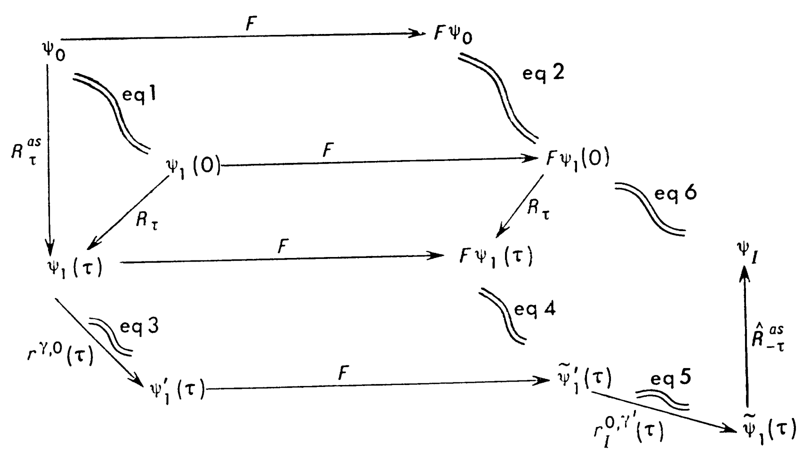
and

$$F_{\text{as}}\psi(x, \tau, h) \overset{\text{def}}{=\!=} e^{\frac{i}{h}\Phi(\alpha,\tau) - \frac{i}{2}\operatorname{Arg} J_I(\alpha,\tau)} \times$$
$$\times |J_I(\alpha, \tau)|^{-1/2} L_I(\tau) v_I(\tau) \chi(\alpha)|_{\alpha=\alpha^I(x,t)},$$

$L(\tau)$, $L_I(\tau)$ and $v_I(\tau)$ being determined via $(g_{H_0}^\tau \Lambda^n, dg_{H_0}^\tau r^n)$.

Finally, we let $r_I^{\gamma,\gamma'}(\tau)$ denote the operator corresponding to $(g_{H_0}^\tau \Lambda^n, dg_{H_0}^\tau r^n)$, which sends, for any φ supported near $g_{H_0}^\tau \alpha_0$, the asymptotic series on the left of (1.24) into that on the right.

Now the proposition is proved by considering the diagram:



where

$$\psi_0(x) = e^{\frac{i}{h}\Phi^{\gamma}(\alpha) - \frac{i}{2}\operatorname{Arg} J(\alpha)} |J(\alpha)|^{-1/2} L^{\gamma}\varphi(\alpha)|_{\alpha=\pi_{\gamma}^{-1}(x)}.$$

The desired result follows from the fact that $F\psi_0 \approx \psi_I$. Justification of the equivalences which occur in the diagram is as follows:

eq1 is by Lemma 1.1,
eq2 follows from eq 1 and Lemma 1.1,
eq3 is by Lemma 1.1,
eq4 follows from eq 3 and Corollary 1.1,
eq5 is by Lemma 1.2,
eq6 follows from eq 4, eq 5 and Lemma 1.1.
This completes the proof.

Note 1.1. Let $\partial/\partial l = (\partial/\partial l_1, \ldots, \partial/\partial l_n)$ be a set of commutating complex vector fields on u_γ which are linearly independent at every point. Lemma 1.2 and Propositions 1.1, 1.2 remain valid if we set

$$J_I(\alpha) = \frac{D(q_I + z_I,\ p_{\bar{I}} + w_{\bar{I}})}{Dl} = \det \frac{\partial\{q_I + z_I,\ p_{\bar{I}} + w_{\bar{I}}\}}{\partial l}, \tag{1.27}$$

$$L_I^{\gamma} = \sum_{|j|=0}^{\infty} \frac{(\beta_{\gamma}^I)^j}{j!} \left(\frac{\partial}{\partial l}\right)^j,$$

where

$$\beta_{\gamma}^I = -\left[\frac{\partial\{q_I + z_I,\ p_{\bar{I}} + w_{\bar{I}}\}}{\partial l}\right]^{-1} [\{z_I,\ w_{\bar{I}}\} - \gamma].$$

Sec. 2. Commutation Formulas for a Complex Exponential and a Hamiltonian

Let $x = (x_1, \ldots, x_n)$ be the generating set of multiplication operators by independent variables operating in the Sobolev scale $\{H^k(\mathbf{R}^n)\}$ with $D = C_0^{\infty}$ (see the definition of a Banach scale in Sec. 1 of Chapter II), and let $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_n)$ be the generating set of operators in the same scale defined by $p_j = -ih\,\partial/\partial x_j$, where h is a positive parameter. Further let

$$\mathcal{H}(x,\ p) \in \mathcal{B}_{\infty}(\mathbf{R}^{2n}) \overset{\text{def}}{=} \bigcap_{s=0}^{\infty} \mathcal{B}_s(\mathbf{R}^{2n}),$$

and let $S(x)$ be an infinitely differentiable finite function with $\operatorname{Im} S(x) \geqslant 0$.

In this section we shall obtain commutation formulas for the Hamiltonian $\mathcal{H}\left(\overset{2}{x}, \overset{1}{p}\right)$ and the multiplication operator by

$\exp\left\{\frac{i}{h}S\right\}$. To this end we first consider the following two special cases: (1) $\operatorname{Re} S = 0$ and (2) $\operatorname{Im} S = 0$.

Lemma 2.1. *Under the above assumptions suppose additionally that* $\operatorname{Re} S = 0$. *Then for any natural* k,

$$\mathscr{H}\left(\overset{3}{x}, \overset{2}{p}\right) e^{\frac{i}{h}S\left(\overset{1}{x}\right)} = \sum_{0\leqslant|\alpha|\leqslant k-1} \frac{(-ih)^{|\alpha|}\partial^{|\alpha|}e^{\frac{i}{h}S(x)}}{\alpha!\,\partial x^{\alpha}} \frac{\partial^{|\alpha|}\mathscr{H}}{\partial p^{\alpha}}\left(\overset{2}{x}, \overset{1}{p}\right) +$$

$$+(-ih)^k r_k(S), \tag{2.1}$$

where $r_k(S)$ *is an operator such that*

$$\|r_k(S)\|_{H^l\to H^l} \leqslant c_{k,l}h^{-\frac{k+l}{2}} \tag{2.2}$$

for all $l \geqslant 0$.

Proof. Using the commutation formula of Chapter II, it is easy to prove by induction that (2.1) holds with

$$r_k(S) = \sum_{j=1}^{n} \sum_{\substack{|\alpha|=k-1\\ \alpha_{j+1}=\ldots=\alpha_n=0}} \left\{\frac{1}{\alpha!}\overset{2}{\left[\!\left[\frac{\partial^k e^{\frac{i}{h}S}}{\partial x^{\alpha}\,\partial x_j}\right]\!\right]}\times\right.$$

$$\left.\times\left(\frac{\partial}{\partial p^{\alpha}}\frac{\delta}{\delta p_j}\mathscr{H}\right)\left(\overset{4}{x}, \overset{1}{p_1}, \ldots, \overset{1}{p_j}, \overset{3}{p_j}, \ldots, \overset{3}{p_n}\right)\right\}. \tag{2.3}$$

According to Corollary 1.1 of Chapter IV,

$$\left\|\frac{\partial^k e^{\frac{i}{h}S}}{\partial x^{\alpha}\,\partial x_j}\right\|_{C^l(\mathbf{R}^n)} \leqslant \operatorname{const} h^{-\frac{k+l}{2}},$$

hence, letting A denote the multiplication operator by $\frac{\partial^k e^{\frac{i}{h}S}}{\partial x^{\alpha}\,\partial x_j}$ acting on H^l to itself, we get

$$\|A\| \leqslant \operatorname{const} h^{-\frac{k+l}{2}}.$$

Since

$$\left\|\overset{2}{A}f\left(\overset{4}{x}, \overset{1}{p_1}, \ldots, \overset{1}{p_j}, \overset{3}{p_j}, \ldots, \overset{3}{p_n}\right)\right\|_{H^l\to H^l} \leqslant$$

$$\leqslant \operatorname{const}\|A\|_{H^l\to H^l}\|f\|_{\mathscr{B}_l},$$

this implies (2.2), Q.E.D.

As to the case of a real S, it is not difficult to obtain the expansion which generalizes (7.7) of Chapter II. To state the result we introduce non-linear differential operators $P^{(l)}_{\alpha,\beta}$ as follows: if $\frac{\partial S}{\partial x}(x_0)=0$, then $P^{(l)}_{\alpha,\beta}(S)\,|_{x=x_0}$ is defined by the identity

$$e^{-\frac{i}{h}S(x)}\left(-ih\frac{\partial}{\partial x}\right)^{\alpha}e^{\frac{i}{h}S(x)}(x-x_0)^{\beta}\,|_{x=x_0}=$$

$$=\sum_{l=0}^{|\alpha|}(-ih)^{l}\,\alpha!\,\beta!\,P^{(l)}_{\alpha,\beta}(S)\,|_{x=x_0}; \tag{2.4}$$

in the general case we set

$$P^{(l)}_{\alpha,\beta}(S)\,|_{x=x_0}=P^{(l)}_{\alpha,\beta}(S_0)\,|_{x=x_0},$$

where

$$S_0(x)=S(x_0)-\left\langle x-x_0,\ \frac{\partial S}{\partial x}(x_0)\right\rangle.$$

In particular,

$$P^{(0)}_{0,0}\,|\,(S)=1,$$

$$P^{(1)}_{\alpha,\beta}(S)=0 \text{ if } |\alpha|=1,\ |\beta|=0 \text{ or } |\alpha|=2,\ |\beta|=1,$$

$$P^{(1)}_{\alpha,\beta}(S)=\langle\alpha,\ \beta\rangle \text{ if } |\alpha|=|\beta|=1,$$

$$P^{(1)}_{\alpha,\beta}(S)=\frac{1}{\alpha!}\frac{\partial^{\alpha}S}{\partial x^{\alpha}} \text{ if } |\alpha|=2.$$

Lemma 2.2. *Under the assumptions of Lemma* 2.1 *with* $\operatorname{Re} S=0$ *replaced by* $\operatorname{Im} S=0$ *we have for any natural* k

$$\mathcal{H}\left(\overset{3}{x},\ \overset{2}{p}\right)e^{\frac{i}{h}S\left(\overset{1}{x}\right)}=e^{\frac{i}{h}S(x)}\left\{\sum_{l=0}^{k-1}(-ih)^{l}\,\Phi_l(S)+h^{k}\bar{r}_k(S)\right\}, \tag{2.5}$$

where

$$\bar{r}_k(S)\in \operatorname{Hom}(H^{s+k},\ H^{s}),\quad \textit{all}\quad s\in\mathbf{R},$$

and

$$\Phi_l(S)=\sum_{l\leq|\alpha|\leq 2l}\frac{\partial^{|\alpha|}\mathcal{H}}{\partial p^{\alpha}}\left(\frac{\partial S}{\partial x},\ x\right)P^{l}_{\alpha,\beta}(S)\frac{\partial^{|\beta|}}{\partial x^{\beta}} \tag{2.6}$$

In particular, $\Phi_0(S)=\mathcal{H}(x,\ S_x)$.

Proof. Obviously, (2.5) can be rewritten as

$$\mathcal{H}\left(\overset{2}{x},\ \frac{\partial S}{\partial x}+\overset{1}{\hat{p}}\right)\sum_{l=0}^{k-1}(-ih)^{l}\,\Phi_l(S)+h^{k}\bar{r}_k(S), \tag{2.5'}$$

but the left-hand member of (2.5′) has the form $f\left(\overset{2}{C}, \overset{1}{A+B}\right)$, so we use the following generalization of (6.8) of Introduction:

$$f(A+B)=f(A)+\sum_{l=1}^{k-1}\sum_{j_1=1}^{n}\cdots\sum_{j_l=j_{l-1}}^{n}\left\{\overset{2}{B}_{j_1}\cdots\overset{2l}{B}_{j_l}\times\right.$$
$$\left.\times\,\delta_{j_1}\cdots\delta_{j_l}f\left(\overset{1}{A}_1,\ldots,\overset{1}{A}_{j_1},\overset{3}{A}_{j_1},\ldots,\overset{2l-1}{A}_{j_l},\overset{2l+1}{A}_{j_l},\ldots,\overset{2l+1}{A}_n\right)\right\}+$$
$$+\sum_{1\leqslant j_1\leqslant\ldots\leqslant j_k\leqslant n}\left\{\overset{2}{B}_{j_1}\cdots\overset{2k}{B}_{j_k}\times\right.$$
$$\times\,\delta_{j_1}\cdots\delta_{j_k}f\left(\overset{1}{A}_1,\ldots,\overset{1}{A}_{j_1},\overset{3}{A}_{j_1},\ldots,\overset{2k-1}{A}_{j_k},\overset{2k+1}{A_{j_k}+}\right.$$
$$\left.+B_{j_k},\ldots,\overset{2k+1}{A_n+B_n}\right),\tag{2.7}$$

where, as well as in the one-dimensional case, some operators may be additionally included as parameters (cf. the notes following Theorem 6.4, 6.6 and 6.7 of Introduction, respectively). Thus we obtain (2.5′) with

$$\Phi_l(S)=\sum_{1\leqslant j_1\leqslant\ldots\leqslant j_l\leqslant n}\left\{\overset{2}{\frac{\partial}{\partial x_{j_1}}}\cdots\overset{2l}{\frac{\partial}{\partial x_{j_l}}}\times\right.$$
$$\left.[\delta_{j_1}\cdots\delta_{j_l}\mathscr{H}]\left(\overset{1}{S}_{x_1},\ldots,\overset{1}{S}_{x_{j_1}},\overset{3}{S}_{x_{j_1}},\ldots,\overset{2l-1}{S}_{x_j},\overset{2l+1}{S}_{x_{j_l}},\ldots,\overset{2l+1}{S}_{x_n}\right)\right\}$$

(here we have dropped x being the last to operate). In the expression of Φ_l, let us change the order of operating so that $\partial/\partial x$ would be the first to operate. Using the commutation formula of Chapter II, we arrive at

$$\Phi_l=\sum_{\substack{l\leqslant|\alpha|\leqslant 2l\\ 0\leqslant|\beta|\leqslant l}}\frac{\partial^\alpha\mathscr{H}}{\partial p^\alpha}\left(\frac{\partial S}{\partial x},x\right)Q^{(l)}_{\alpha,\beta}\left(\frac{\partial}{\partial x}\right)^\beta,$$

where $Q^{(l)}_{\alpha,\beta}(S)$ is an independent of $\mathscr{H}$ polynomial (with constant coefficients) in the derivatives of S of orders $2,\ldots,l+1$.

What remains to be proved now is that $Q^{(l)}_{\alpha,\beta}=P^{(l)}_{\alpha,\beta}$. It is easily seen that for this it suffices to show that

$$Q^{(l)}_{\alpha,\beta}(S)\,|_{x=0}=P^{(l)}_{\alpha,\beta}(S)\,|_{x=0}$$

for all S with $S_x(0)=0$. To do this, put $\mathscr{H}(x,p)=p^\alpha$. In spite of $p^\alpha\notin\mathscr{B}_\infty$, both (2.7) with $f=\mathscr{H}$, $A=S_x$, $B=\hat{p}$ and the commutation formula of Chapter II remain clearly valid in the case, the

remainder in (2.7) vanishing if k is large enough. Hence

$$\overset{2}{p}{}^{\alpha} \exp\left\{\frac{i}{h} S\left(\overset{1}{x}\right)\right\} = \exp\left\{\frac{i}{h} S(x)\right\} \times$$
$$\times \sum_{l} (-ih)^{l} \sum_{\gamma,\delta} \frac{\alpha!}{(\alpha-\gamma)!} S_x^{\alpha-\gamma} Q^{(l)}_{\gamma,\delta}(S) \frac{\partial^{|\delta|}}{\partial x^{\delta}}. \tag{2.8}$$

Applying this operator indentity to x^{β} and putting $x=0$, we obtain

$$e^{-\frac{i}{h}S(x)}\left(-ih\frac{\partial}{\partial x}\right)^{\alpha} e^{\frac{i}{h}S(x)} x^{\beta}\big|_{x=0} = \sum_{l} (-ih)^{l} \alpha!\, \beta!\, Q^{(l)}_{\alpha,\beta}(S)\big|_{x=0}.$$

Comparison of the last formula with (2.4) produces the desired result, Q.E.D.

We now generalize (2.5) to the case where $\operatorname{Im} S$ need not vanish.

Theorem 2.1. *Under the assumptions stated at the outset of the section we have*

$$\mathscr{H}\left(\overset{3}{x}, \overset{2}{p}\right) e^{\frac{i}{h}S\left(\overset{1}{x}\right)} = e^{\frac{i}{h}S(x)} \sum_{k=0}^{N-1} (-ih)^{k} \Phi_k^{(N)}(S) + R_N(S), \tag{2.9}$$

where

$$\Phi_k^{(N)}(S) = \sum_{\substack{0\le|\gamma|\le 2(N-k)-1\\ k\le|\alpha|\le 2k\\ 0\le|\beta|\le k}} \left\{\frac{1}{\gamma!}\left(i\frac{\partial \operatorname{Im} S}{\partial x}\right)^{\gamma} \times\right.$$
$$\left.\times \frac{\partial^{|\alpha|+|\gamma|}\mathscr{H}}{\partial p^{\alpha+\gamma}}\left(x, \frac{\partial \operatorname{Re} S}{\partial x}\right) P^{(k)}_{\alpha,\beta}(S) \frac{\partial^{|\beta|}}{\partial x^{\beta}}\right. \tag{2.10}$$

and $R_N(S)$ can be estimated as follows:

$$\|R_N(S)\|_{H^{N+l}\to H^{l}} \le c_{N,l} h^{N-\frac{3}{2}l}, \quad l=0, 1, 2, \ldots . \tag{2.11}$$

Proof. Set $S_1 = \operatorname{Re} S$, $S_2 = \operatorname{Im} S$. We begin by commuting the Hamiltonian $\mathscr{H}\left(\overset{2}{x}, \overset{1}{p}\right)$ with the multiplication operator by $\exp\left[-\frac{1}{h}S_2(x)\right]$; according to (2.1), we get

$$\mathscr{H}\left(\overset{3}{x}, \overset{2}{p}\right) e^{\frac{i}{h}S\left(\overset{1}{x}\right)} = \sum_{0\le|\alpha|\le 2N-1} \left\{\frac{(-ih)^{|\alpha|}}{\alpha!} \frac{\partial^{|\alpha|} e^{-\frac{1}{h}S_2}}{\partial x^{\alpha}} \times\right.$$
$$\left.\times \frac{\partial^{|\alpha|}\mathscr{H}}{\partial p^{\alpha}}\left(\overset{3}{x}, \overset{2}{p}\right) e^{\frac{i}{h}S\left(\overset{1}{x}\right)}\right\} + R_{N,1}(S), \tag{2.12}$$

where

$$R_{N,1}(S) = (-ih)^{2N} r_{2N}(S_2) \circ e^{\frac{i}{h}S_1(x)}.$$

By (2.5),

$$\frac{\partial^{|\alpha|}\mathscr{H}}{\partial p^{\alpha}}\left(\overset{3}{x}, \overset{2}{p}\right) e^{\frac{i}{h}S_1\left(\overset{1}{x}\right)} = e^{\frac{i}{h}S_1(x)} \times$$
$$\times\left\{\sum_{k=0}^{N-1} \sum_{k\leqslant|\gamma|\leqslant 2k} \sum_{0\leqslant|\beta|\leqslant k}\left[(-ih)^k \frac{\partial^{|\alpha+\gamma|}\mathscr{H}}{\partial p^{\alpha+\gamma}}\left(x, \frac{\partial S_1}{\partial x}\right)\times\right.\right.$$
$$\left.\left.\times P^{(k)}_{\gamma,\beta}(S_1)\frac{\partial^{|\beta|}}{\partial x^{\beta}}\right] + h^N \bar{r}_{N,\alpha}(S_1)\right\}, \tag{2.13}$$

where

$$\bar{r}_{N,\alpha}(S_1) \in \operatorname{Hom}(H^{s+N}, H^s)$$

for any $s \in \mathbf{R}$.

Thus (2.12), after substitution of (2.13), gives

$$\mathscr{B}\left(\overset{3}{x}, \overset{2}{p}\right) e^{\frac{i}{h}S\left(\overset{1}{x}\right)} =$$
$$= e^{\frac{i}{h}S}\sum_{k=0}^{N-1}(-ih)^k \sum_{\substack{0\leqslant|\gamma|\leqslant k\\ k\leqslant|\alpha|\leqslant 2k\\ 0\leqslant|\beta|\leqslant 2(N-k)-1}} \left\{\frac{\partial^{|\alpha+\beta|}\mathscr{H}}{\partial p^{\alpha+\beta}}(x, S_{1x})\times\right.$$
$$\left.\times(iS_{2x})^{\beta} Q_{k,\alpha,\beta,\gamma}(S)\left(\frac{\partial}{\partial x}\right)^{\gamma}\right\} +$$
$$+ R_{N,1}(S) + R_{N,2}(S) + R_{N,3}(S),$$

where $Q_{k,\alpha,\beta,\gamma}(S)$ is a polynomial in the derivatives of S_1 and S_2 of orders $2, \ldots, k+1$,

$$R_{N,2}(S) = \sum_{0\leqslant|\alpha|\leqslant 2N-1} \frac{(-ih)^{|\alpha|+N}}{\alpha!}\frac{\partial^{|\alpha|}e^{-\frac{1}{h}S_2}}{\partial x^{\alpha}} e^{\frac{i}{h}S_1}\bar{r}_{N,\alpha}(S_1) \tag{2.15}$$

and $R_{N,3}(S)$ is expressible as the sum of operators of the form

$$e^{\frac{i}{h}S(x)} h^k f(x)\left(\frac{\partial S_2}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial x}\right)^{\gamma}$$

with $f \in C_0^{\infty}$, $|\gamma| \leqslant N-1$, $k + \frac{|\beta|}{2} \geqslant N$.

To compute $Q_{k,\alpha,\beta,\gamma}$, we apply the method similar to that used in the proof of Lemma 2.2. Put $\mathscr{B}(x, p) = p^{\delta}$. The expansion (2.14) is, of course, valid in this case, the remainders $R_{N,1}$, $R_{N,2}$ and $R_{N,3}$ vanishing provided that N is large enough. Thus,

$$\overset{2}{p}{}^{\delta} e^{\frac{i}{h}S\left(\overset{1}{x}\right)} = e^{\frac{i}{h}S(x)}\sum_{k,\alpha,\beta,\gamma}\left[(-ih)^k \frac{\delta!}{(\delta-\alpha-\beta)!}\left(\frac{\partial S_1}{\partial x}\right)^{\delta-\alpha-\beta}\times\right.$$
$$\left.\times\left(i\frac{\partial S_2}{\partial x}\right)^{\beta} Q_{k,\alpha,\beta,\gamma}(S)\left(\frac{\partial}{\partial x}\right)^{\gamma}\right].$$

Let us apply this operator equality to x^{ε}; if $\frac{\partial S_1}{\partial x}(0)=0$, which we may assume without loss of generality, then

$$\hat{p}^{\delta} e^{\frac{i}{h}S(x)} x^{\varepsilon}\Big|_{x=0} = e^{\frac{i}{h}S(x)} \sum_{k,\beta} (-ih)^k \delta!\varepsilon! \left(i\frac{\partial S_2}{\partial x}\right)^{\beta} Q_{k,\delta-\beta,\beta,\varepsilon}(S)\Big|_{x=0}. \quad (2.16)$$

On the other hand, (2.8) holds for complex S too. Hence

$$\hat{p}^{\delta} e^{\frac{i}{h}S(x)} x^{\varepsilon}\Big|_{x=0} = e^{\frac{i}{h}S(x)} \sum_{k,\gamma} (-ih)^k \frac{\delta!\varepsilon!}{(\delta-\gamma)!} \left(i\frac{\partial S_2}{\partial x}\right)^{\delta-\gamma} P^{(k)}_{\gamma,\varepsilon}(S)\Big|_{x=0}.$$

Comparing the last formula with (2.16) we conclude that

$$Q_{k,\alpha,\beta,\gamma} = \frac{1}{\beta!} P^{(k)}_{\alpha,\gamma}.$$

To complete the proof, we have to check (2.11) for

$$R_N(S) = \sum_{j=1}^{3} R_{N,j}(S).$$

To estimate $R_{N,2}$, consider (up to a numerical factor) the general term in the right-hand member of (2.15):

$$A \overset{\text{def}}{=} h^{N+|\alpha|} \frac{\partial^{|\alpha|} e^{-\frac{1}{h}S_2}}{\partial x^{\alpha}} e^{\frac{i}{h}S_1} \bar{r}_{N,\alpha}(S_1).$$

We have

$$\|\bar{r}_{N,\alpha}(S_1)\|_{H^{l+N}\to H^l} \leqslant \text{const},$$

$$\left\| e^{\frac{i}{h}S_1} \right\|_{H^l\to H^l} \leqslant \text{const}\, h^{-l}$$

and

$$\left\| \frac{\partial^{|\alpha|} e^{-\frac{1}{h}S_2}}{\partial x^{\alpha}} \right\|_{H^l\to H^l} \leqslant \text{const}\, h^{-\frac{|\alpha|+l}{2}}$$

(for the last estimate see the proof of Lemma 2.1). Hence

$$\|A\|_{H^{N+l}\to H^l} \leqslant \text{const}\, h^{N-\frac{3}{2}l}$$

which implies the required estimate for $R_{N,2}(S)$. Estimation of $R_{N,1}$ and $R_{N,3}$ may be left to the reader.

Let $F^{-1}_{\xi_I \to x_I}$ be the inverse of the h^{-1}-Fourier transformation with respect to the Ith group of variables:

$$F^{-1}_{\xi_I \to x_I} \varphi(\xi_I, x_{\bar I}) = (2\pi h)^{-\frac{m}{2}} \int\limits_{\mathbf{R}^m} e^{\frac{i}{h}\langle \xi_I, x_I\rangle} \varphi(\xi_I, x_{\bar I})\, d\xi_I,$$

where m is the number of elements in I. Next, we must obtain a commutation formula for $\mathscr{H}\left(\overset{2}{x}, \overset{1}{p}\right)$ and the operator

$$F^{-1}_{\xi_I \to x_I} \circ e^{\frac{i}{h} S(\xi_I, x_{\bar I})},$$

generalizing (2.9).

Let r_I be the operator defined by

$$r_I \varphi(x) = \varphi(-x_I, x_{\bar I}),$$
$$x = (x_1, \ldots, x_n).$$

Set

$$P^{l, I}_{\alpha, \beta} = (-1)^{|\beta_I|} r_I \circ P^l_{\alpha, \beta} \circ r_I.$$

As before, we assume that $S \in C_0^\infty$, $\operatorname{Im} S \geqslant 0$ and $\mathscr{H} \in \mathscr{B}_\infty$, and use the notation $S_1 = \operatorname{Re} S$, $S_2 = \operatorname{Im} S$.

Theorem 2.2. *Under the above assumptions we have*

$$\mathscr{H}\left(\overset{2}{x}, \overset{1}{p}\right) F^{-1}_{\xi_I \to x_I} e^{\frac{i}{h} S(\xi_I, x_{\bar I})} \varphi(\xi_I, x_{\bar I}) =$$
$$= F^{-1}_{\xi_I \to x_I} e^{\frac{i}{h} S(\xi_I, x_{\bar I})} \sum_{k=0}^{N-1} (-ih)^k \Phi^N_{k, I}(S) \varphi(\xi_I, x_{\bar I}) +$$
$$+ R_N(S) \varphi(\xi_I, x_{\bar I}),$$

where

$$\Phi^N_{k, I}(S) = \sum_{k \leqslant |\alpha| \leqslant 2k} \sum_{0 \leqslant |\beta| \leqslant k} \sum_{\gamma \leqslant \beta_I} \sum_{0 \leqslant |\delta| \leqslant 2N-2k-1} \left\{ \frac{\beta_I!}{\gamma!\,(\beta_I - \gamma)!\,\delta!} \times \right.$$
$$\times \frac{\partial^{|\alpha|+|\gamma|+|\delta|} \mathscr{H}}{\partial x_I^{\alpha_I+\delta_I}\, \partial p_{\bar I}^{\alpha_{\bar I}+\delta_{\bar I}}\, \partial p_I^\gamma} \left(-\frac{\partial S_1}{\partial \xi_I}, x_{\bar I}, \xi_I, \frac{\partial S_1}{\partial x_{\bar I}} \right) \times$$
$$\left. \times P^{k, I}_{\alpha, \beta}(S) \left(-i \frac{\partial S_2}{\partial \xi_I} \right)^{\delta_I} \left(i \frac{\partial S_2}{\partial x_{\bar I}} \right)^{\delta_{\bar I}} \frac{\partial^{|\beta|-|\gamma|}}{\partial x_{\bar I}^{\beta_{\bar I}}\, \partial \xi_I^{\beta_I - \gamma}} \right\}$$

and $R_N(S)$ *can be estimated as follows:*

$$\| R_N(S) \|_{H^{I, 1}_{l, l+N} \to H^l} \leqslant c_{n, l} h^{N - \frac{5}{2} l}.$$

Proof. We start with the note that the required expansion can be rewritten in the form

$$\mathcal{H}\left(ih\overset{2}{\frac{\partial}{\partial\xi_I}},\ \overset{2}{x_{\bar I}},\ \overset{1}{\xi_I},\ -ih\overset{1}{\frac{\partial}{\partial x_{\bar I}}}\right)\circ e^{\frac{i}{h}S(\xi_I,\,x_{\bar I})}=$$

$$=e^{\frac{i}{h}S(\xi_I,\,x_{\bar I})}\sum_{k=0}^{N-1}(-ih)^k\Phi^N_{k,\,I}(S)+\bar R_N(S),\tag{2.17}$$

where $\bar R_N(S)=F_{x_I\to\xi_I}\circ R_N(S)$. In fact,

$$\mathcal{H}\left(\overset{2}{x},\ \overset{1}{p}\right)\circ F^{-1}_{\xi_I\to x_I}=F^{-1}_{\xi_I\to x_I}\circ\mathcal{H}\left(ih\overset{2}{\frac{\partial}{\partial\xi_I}},\ \overset{2}{x_{\bar I}},\ \overset{1}{\xi_I},\ -ih\overset{1}{\frac{\partial}{\partial x_{\bar I}}}\right).$$

Since $H^{I,\,h}_{l,\,l}$ is continuously embedded into $F_{x_I\to\xi_I}H^l$, it suffices to check the estimate

$$\|\bar R_N(S)\|_{H^{I,\,1}_{l,\,l+N}\to H^{I,\,h}_{l,\,l}}\leqslant c_{l,\,N}h^{N-\frac{5}{2}l}.\tag{2.18}$$

The proof will be given in three steps.

(1) By an obvious modification of the proof of Lemma 2.1 we arrive at

$$\mathcal{H}\left(ih\overset{2}{\frac{\partial}{\partial\xi_I}},\ \overset{2}{x_{\bar I}},\ \overset{1}{\xi_I},\ -ih\overset{1}{\frac{\partial}{\partial x_{\bar I}}}\right)\circ e^{-\frac{1}{h}S_2(\xi_I,\,x_{\bar I})}=$$

$$=\sum_{|\alpha|=0}^{k-1}\frac{(-ih)^\alpha}{\alpha!}(-1)^{|\alpha_I|}\frac{\partial^\alpha e^{-\frac{1}{h}S_2(\xi_I,\,x_{\bar I})}}{\partial\xi_I^{\alpha_I}\,\partial x_{\bar I}^{\alpha_{\bar I}}}\times$$

$$\times\frac{\partial^\alpha\mathcal{H}}{\partial x_I^{\alpha_I}\,\partial p_{\bar I}^{\alpha_{\bar I}}}\left(ih\overset{2}{\frac{\partial}{\partial\xi_I}},\ \overset{2}{x_{\bar I}},\ \overset{1}{\xi_I},\ -ih\overset{1}{\frac{\partial}{\partial x_{\bar I}}}\right)+(-ih)^k r_k(S_2)\tag{2.19}$$

with the remainder

$$r_k(S_2)=\sum_{j=1}^{n}\sum_{|\alpha|=k-1}\left\{\frac{(-1)^{|\alpha_I|+\varepsilon_j}}{\alpha!}\overset{3}{[\![}\frac{\partial^k e^{-\frac{1}{h}S_2(\xi_I,\,x_{\bar I})}}{\partial y^\alpha\,\partial y^j}]\!]\times\right.$$

$$\left.\times\frac{\partial}{\partial z^\alpha}\frac{\delta}{\delta z_j}\bar{\mathcal{H}}\left(\overset{1}{\xi_I},\ \overset{5}{x_{\bar I}},\ \overset{2}{z_1},\ \dots,\ \overset{2}{z_j},\ \overset{4}{z_j},\ \dots,\ \overset{4}{z_n}\right)\right\},\tag{2.20}$$

where

$$y=(p_I,\ x_{\bar I}),\quad \hat z=\left(ih\frac{\partial}{\partial\xi_I},\ -ih\frac{\partial}{\partial x_{\bar I}}\right),$$

$$\overline{\mathscr{H}}(y,\ z)=\mathscr{H}((z_I,\ y_{\bar I}),\ (y_I,\ z_{\bar I})),$$

$$\varepsilon_j=\begin{cases}1 & \text{for } j\in I\\ 0 & \text{for } j\in \bar I.\end{cases}$$

Expression (2.20) implies the following estimate:

$$\| r_k(S_2)\|_{H^{I,h}_{l,l}\to H^{I,h}_{l,l}}\leqslant c_{k,l}h^{-\frac{k+l}{2}}. \tag{2.21}$$

(2) By the obvious modification of the proof of Lemma 2.2, we arrive at

$$\mathscr{H}\left(ih\overset{2}{\frac{\partial}{\partial\xi_I}},\ \overset{2}{x_{\bar I}},\ \overset{1}{\xi_I},\ -ih\overset{1}{\frac{\partial}{\partial x_{\bar I}}}\right)\circ e^{\frac{i}{h}S_1(\xi_I,\ x_{\bar I})}=$$

$$=e^{\frac{i}{h}S_1(\xi_I,\ x_{\bar I})}\left\{\sum_{l=0}^{k-1}(-ih)^l\,\Phi_{l,I}(S_1)+h^k\bar r_k(S_1)\right\}, \tag{2.22}$$

where

$$\bar r_k(S_1)\in \operatorname{Hom}(H^{I,h}_{l,l+k},\ H^{I,h}_{l,l})$$

for every non-negative integer l, and

$$\Phi_{l,I}(S)=\sum_{l\leqslant|\alpha|\leqslant 2l}\ \sum_{0\leqslant|\beta|\leqslant l}\left\{\frac{\partial^{|\alpha|}\mathscr{H}}{\partial x_I^{\alpha_I}\,\partial p_{\bar I}^{\alpha_{\bar I}}}\left(-\overset{3}{\frac{\partial S}{\partial \xi_I}},\ \overset{3}{x_{\bar I}},\ \overset{1}{\xi_I},\ \overset{3}{\frac{\partial S}{\partial x_{\bar I}}}\right)\times\right.$$

$$\left.\times\overset{3}{[\![P^{l,I}_{\alpha,\beta}(S)]\!]}\left(\overset{2}{\frac{\partial}{\partial\xi_I}}\right)^{\beta_I}\left(\overset{2}{\frac{\partial}{\partial x_{\bar I}}}\right)^{\beta_{\bar I}}\right\}.$$

Commuting ξ_I and $\dfrac{\partial}{\partial\xi_I}$ in the last formula, we obtain

$$\Phi_{l,I}(S)=\sum_{|\alpha|=l}^{2l}\ \sum_{|\beta|=0}^{k}\ \sum_{\gamma\leqslant\beta_I}\left\{\frac{\beta_I!}{\gamma!\,(\beta_I-\gamma)!}\times\right.$$

$$\times\frac{\partial^{|\alpha|+|\gamma|}\mathscr{H}}{\partial x_I^{\alpha_I}\,\partial p_{\bar I}^{\alpha_{\bar I}}\,\partial p_I^{\gamma}}\left(-\frac{\partial S}{\partial\xi_I},\ x_{\bar I},\ \xi_I,\ \frac{\partial S}{\partial x_{\bar I}}\right)\times$$

$$\left.\times P^{l,I}_{\alpha,\beta}(S)\frac{\partial^{|\beta|-|\gamma|}}{\partial x_{\bar I}^{\beta_{\bar I}}\,\partial\xi_I^{\beta_I-\gamma}}\right\}. \tag{2.23}$$

(3) By a slight modification of the proof of Theorem 2.1 (in particular, we must replace (2.1) and (2.5) used there by (2.19) and (2.20), respectively) we obtain (2.17) with the remainder having the following estimate:

$$\| \overline{R}_N (S) \|_{H^{I,h}_{l,l+N} \to H^{I,h}_{l,l}} \leqslant c_{n,l} h^{N-\frac{3}{2}l}.$$

It remains to be noted that the bound of the identity operator $I : H^{I,1}_{l,l+N} \to H^{I,h}_{l,l+N}$ does not exceed h^{-l}.

Corollary 2.1. *Under the conditions of Theorem 2.2 we have for any* $\varphi \in C_0^\infty$:

$$\begin{aligned} &\mathscr{H}\left(\overset{2}{x}, \overset{1}{p}\right) F^{-1}_{\xi_I \to x_I} e^{\frac{i}{h} S(\xi_I, x_{\bar I})} \varphi(\xi_I, x_{\bar I}) \approx \\ &\approx F^{-1}_{\xi_I \to x_I} e^{\frac{i}{h} S(\xi_I, x_{\bar I})} \sum_{k=0}^{\infty} \sum_{|\alpha|=k}^{2k} \sum_{|\beta|=0}^{k} \sum_{\gamma \leqslant \beta_I} \sum_{|\delta|=0}^{\infty} \Bigg\{ (-ih)^k \times \\ &\times \frac{\beta_I!}{\gamma!\,(\beta_I-\gamma)!\,\delta!} \frac{\partial^{|\alpha|+|\gamma|+|\delta|}\mathscr{H}}{\partial x_I^{\alpha_I+\delta_I}\, \partial p_{\bar I}^{\alpha_{\bar I}+\delta_{\bar I}}\, \partial p_I^{\gamma}} \left(-\frac{\partial S_1}{\partial \xi_I}, x_{\bar I}, \xi_I, \frac{\partial S_1}{\partial x_{\bar I}}\right) \times \\ &\times P^{k,I}_{\alpha,\beta}(S) \left(-i\frac{\partial S_2}{\partial \xi_I}\right)^{\delta_I} \left(i\frac{\partial S_2}{\partial x_{\bar I}}\right)^{\delta_{\bar I}} \times \\ &\times \frac{\partial^{|\beta|-|\gamma|}}{\partial x_{\bar I}^{\beta_{\bar I}}\, \partial \xi_I^{\beta_I-\gamma}} \varphi(\xi_I, x_{\bar I}) \Bigg\} \stackrel{\text{def}}{=} F^{-1}_{\xi_I \to x_I} e^{\frac{i}{h} S(\xi_I, x_{\bar I})} \hat{\mathscr{H}} \varphi(\xi_I, x_{\bar I}), \end{aligned} \tag{2.24}$$

$\hat{\mathscr{H}}$ *being an* S_2*-asymptotic differential operator.*

The same formula is valid in the case where φ is an S_2-asymptotic series.

Example 2.1. Let $\psi(\alpha) \in C_0^\infty$ be supported in a γ-domain u_γ of the zone $\Omega_{\bar I}$ of a Lagrangean manifold with a complex germ, and let X be the image of $\Gamma \cap \operatorname{supp}\psi$ under the mapping $\alpha \to q(\alpha)$. Then, for any $\chi(x) \in C_0^\infty$ supported outside of X, we have

$$\chi(x) F^{-1}_{\xi_I \to x_I} \left[e^{\frac{i}{h} \Phi^{\gamma}_{\bar I}(\alpha)} \psi(\alpha) \Big|_{\alpha=(\pi^I_\gamma)^{-1}(\xi_I, x_{\bar I})} \right] \approx 0.$$

In fact, setting $S=\Phi_{\bar{I}}^{\gamma}\circ(\pi_{\gamma}^{I})^{-1}$, $\varphi=\psi\circ(\pi_{\gamma}^{I})^{-1}$, we obtain by Corollary 2.1:

$$\chi(x)F^{-1}_{\xi_I\to x_I}e^{\frac{i}{h}S(\xi_I,\,x_{\bar{I}})}\varphi(\xi_I,\,x_{\bar{I}})\approx$$
$$\approx F^{-1}_{\xi_I\to x_I}e^{\frac{i}{h}S(\xi_I,\,x_{\bar{I}})}\sum_{k=0}^{\infty}\sum_{|\alpha|=k}^{2k}\sum_{|\beta|=0}^{k}\sum_{|\delta|=0}^{\infty}\Big\{\frac{(-ih)^k}{\delta!}\times$$
$$\times\frac{\partial^{|\alpha|+|\delta|}\chi}{\partial x^{\alpha+\delta}}\left(-\frac{\partial S_1}{\partial\xi_I},\,x_{\bar{I}}\right)P^{k,\,I}_{\alpha,\,\beta}(S)\times$$
$$\times\left(-i\frac{\partial S_2}{\partial\xi_I}\right)^{\delta}\frac{\partial^{|\beta|}}{\partial x_{\bar{I}}^{\beta_{\bar{I}}}\,\partial\xi_I^{\beta_I}}\varphi(\xi_I,\,x_{\bar{I}})\Big\}.$$

It is obvious that the support of each term of the h-asymptotic series on the right is a subset of the support of the function

$$\chi\left(-\frac{\partial S_1}{\partial\xi_I}(\xi_I,\,x_{\bar{I}})\right)\varphi(\xi_I,\,x_{\bar{I}}).$$

Set

$$\hat{\chi}(\alpha)=\chi\left(-\frac{\partial S_1}{\partial\xi_I}(p_I(\alpha)+\gamma_I(\alpha),\,q_{\bar{I}}(\alpha)+\gamma_{\bar{I}}(\alpha)),\,q_{\bar{I}}(\alpha)+\gamma_{\bar{I}}(\alpha)\right).$$

It suffices to show that $\hat{\chi}(\alpha)\psi(\alpha)$ vanishes in the vicinity of Γ. We have

$$\hat{\chi}(\alpha)=\chi(q(\alpha)+f(\alpha)),$$

where $f=O_D(h^{1/2})$ (see (3.3) of Chapter IV). Let u be a neighborhood of X such that $u\cap\operatorname{supp}\chi=\varnothing$. Then the inverse image of u under the mapping $\alpha\to q(\alpha)+f(\alpha)$ is a neighborhood of $\Gamma\cap\operatorname{supp}\psi$, on which $\hat{\chi}$ vanishes.

Problem 2.1. Given $\mathscr{H}$ and S, let $\varphi(\xi_I,\,x_{\bar{I}})$ be an S_2-asymptotic series with the principal monomial of the form $h^s\varphi_0(\xi_I,\,x_{\bar{I}})$, and let $\hat{\mathscr{H}}$ be the S_2-asymptotic operator defined by (2.24). Then the principal monomial of $\hat{\mathscr{H}}\varphi$ equals

$$h^s\mathscr{H}\left(-\frac{\partial S_1}{\partial\xi_I},\,x_{\bar{I}},\,\xi_I,\,\frac{\partial S_1}{\partial x_{\bar{I}}}\right)\varphi_0(\xi_I,\,x_{\bar{I}})$$

provided that this functions does not vanish identically.

Problem 2.2. The expansion (2.24) remains valid in the cbse $\mathscr{H}(x,\,p)\in S^{\infty}$.

Hint: represent $\mathscr{H}$ in the form

$$\mathscr{H}(x,\,p)=(x^2+1)^l(p^2+1)^l\mathscr{H}_0(x,\,p)$$

with $\mathscr{H}_0\in\mathscr{B}_{\infty}$.

Sec. 3. C-Lagrangean Manifolds and the Index of a Complex Germ

Let M be a real manifold of dimension n, and let f be a smooth mapping of M into $\mathbf{C}^{2n}$ (identified with $\mathbf{R}^{4n}$), the $2n$-dimensional complex phase space with the coordinates $P = (P_1, \ldots, P_n)$, $Q = (Q_1, \ldots, Q_n)$. If f is given by

$$P = P(\alpha), \qquad Q = Q(\alpha)$$

in local coordinates $\alpha = (\alpha_1, \ldots, \alpha_n)$ in M, then we set

$$P_\alpha = B, \qquad Q_\alpha = C,$$
$$\frac{\partial (Q_I, P_{\bar I})}{\partial \alpha} = C_I, \qquad \frac{\partial (P_I, -Q_{\bar I})}{\partial \alpha} = B_I. \tag{3.1}$$

Definition 3.1. *We say that a point $\alpha_0 \in M$ is C-Lagrangean with respect to f if the following conditions are satisfied:*

(C1)
$$\operatorname{rank}\begin{pmatrix} B(\alpha_0) \\ C(\alpha_0) \end{pmatrix} = n;$$
in other words, M is locally embedded at α_0 into $\mathbf{C}^{2n} = \mathbf{R}^{4n}$, and the tangent space to M at α_0 is real-like.

(C2) *The Lagrange brackets for the functions $P(\alpha)$, $Q(\alpha)$ vanish at α_0:*
$$\left[\left\langle \frac{\partial P}{\partial \alpha_j}, \frac{\partial Q}{\partial \alpha_k}\right\rangle - \left\langle \frac{\partial P}{\partial \alpha_k}, \frac{\partial Q}{\partial \alpha_j}\right\rangle\right]_{\alpha=\alpha_0} = 0;$$

(C3) *The imaginary part of the quadratic form*
$$F(g) = (B(\alpha_0)\, g,\ C(\alpha_0)\, g), \qquad g \in \mathbf{C}^n, \tag{3.2}$$
is non-negative.

The conditions (C1)-(C3) are, of course, independent of the choice of local coordinates.

Note that (C3) is equivalent to the following:

(C3′) For some I, the imaginary part of the quadratic form
$$F_I(g) = (B_I(\alpha_0)\, g,\ C_I(\alpha_0)\, g) \tag{3.3}$$
is non-negative.

In fact, a straightforward calculation shows that $\operatorname{Im} F_I(g)$ does not depend on I.

A subset of M consisting of C-Lagrangean points will be called C-Lagrangean.

Example 3.1. The subset Γ of a Lagrangean manifold with a complex germ is C-Lagrangean with respect to the mapping

$$Q(\alpha) = q(\alpha) + z(\alpha), \qquad P(\alpha) = p(\alpha) + w(\alpha). \tag{3.4}$$

In fact, (C1) and (C2) follow immediately from Definitions 2.1. 2.2 of Chapter IV. Let us verify (C3′) for an arbitrary point $\alpha \in \Gamma$.

Let α be in Ω_I and set $\mathscr{E}_I = B_I C_I^{-1}$, $\mathscr{E}_I = \mathscr{E}_I^{(1)} + \mathscr{E}_I^{(2)}$, where both $\mathscr{E}_I^{(1)}$ and $\mathscr{E}_I^{(2)}$ are real. We have shown before that $\mathscr{E}_I^{(1)}$ and $\mathscr{E}_I^{(2)}$ are symmetric, $\mathscr{E}_I^{(2)}$ being non-negative because of the dissipativity condition. Hence we have for any $h = h^{(1)} + ih^{(2)}$, $h^{(1)}$, $h^{(2)} \in \mathbf{R}^n$:

$$\begin{aligned}\operatorname{Im}(\mathscr{E}_I h, h) &= -(\mathscr{E}^{(1)}h^{(1)}, h^{(2)}) + (\mathscr{E}^{(1)}h^{(2)}, h^{(1)}) + \\ &+ (\mathscr{E}^{(2)}h^{(1)}, h^{(1)}) + (\mathscr{E}^{(2)}h^{(2)}, h^{(2)}) = \\ &= \langle \mathscr{E}^{(2)}h^{(1)}, h^{(1)}\rangle + \langle \mathscr{E}^{(2)}h^{(2)}, h^{(2)}\rangle \geqslant 0.\end{aligned}$$

Putting $h = C_I g$ here, we obtain the required result

$$\operatorname{Im}(B_I g, C_I g) \geqslant 0.$$

Now we shall produce some elementary consequences of (C1)-(C3).

Lemma 3.1. *If $\alpha \in M$ is C-Lagrangean, then the matrix $B_I(\alpha) + it\, C_I(\alpha)$ is non-degenerate for $t > 0$.*

Proof. Suppose that this is not the case and let g be a non-zero vector such that

$$(B_I + it\, C_I)\, g = 0$$

for some $t > 0$. Since $B_I g = -it\, C_I g$, it follows that $F(g) = -it \,\| C_I g \|^2$. Then $C_I g = 0$ by (C3), so $B_I g = 0$, contradicting (C1). The lemma is proved.

Lemma 3.2. *Let $\alpha_0 \in M$ be C-Lagrangean with respect to the mapping $P = P(\alpha)$, $Q = Q(\alpha)$, let $H(p, q)$ be a real quadratic form, and let*

$$P = P(\alpha, t), \qquad Q = Q(\alpha, t) \tag{3.5}$$

be the solution of the Hamiltonian system

$$\dot{P} = -H_q(P, Q), \qquad \dot{Q} = H_p(P, Q), \tag{3.6}$$

satisfying the initial condition $P(\alpha, 0) = P(\alpha)$, $Q(\alpha, 0) = Q(\alpha)$. Then α_0 is C-Lagrangean with respect to the mapping (3.5) *for any fixed $t \in \mathbf{R}$.*

Proof.

(1) First we show that

$$\operatorname{rank}\begin{pmatrix} P_\alpha(\alpha_0, t) \\ Q_\alpha(\alpha_0, t)\end{pmatrix} = n.$$

Let $P(\alpha, \beta)$, $Q(\alpha, \beta)$ be linear in $\beta \in \mathbf{R}^n$ functions satisfying the conditions

$$P(\alpha, 0) = P(\alpha), \quad Q(\alpha, 0) = Q(\alpha),$$
$$\det \left.\frac{\partial(P, Q)}{\partial(\alpha, \beta)}\right|_{\alpha=\alpha_0} \neq 0.$$

Further, let $P(\alpha, \beta, t)$, $Q(\alpha, \beta, t)$ be the solution of (3.6) with the initial conditions

$$P(\alpha, \beta, 0) = P(\alpha, \beta), \quad Q(\alpha, \beta, 0) = Q(\alpha, \beta),$$

and set

$$J(\alpha, \beta, t) = \det \frac{\partial (P(\alpha, \beta, t), Q(\alpha, \beta, t))}{\partial (\alpha, \beta)}.$$

In the same way as in (the proof of) Lemma 2.2. of Chapter IV, we conclude that $\frac{\partial}{\partial t} J(\alpha, \beta, t) = 0$. Thus, $J(\alpha_0, 0, t) \neq 0$ which implies the required assertion.

(2) Next we check (C2) for (3.5). We have

$$\begin{aligned} \dot{Q}_\alpha(\alpha, t) &= H_{pp} P_\alpha(\alpha, t) + H_{pq} Q_\alpha(\alpha, t), \\ \dot{P}_\alpha(\alpha, t) &= -H_{qp} P_\alpha(\alpha, t) - H_{qq} Q_\alpha(\alpha, t). \end{aligned} \tag{3.7}$$

A straightforward calculation yields

$$\frac{d}{dt} \{Q, P\}_{jk} = 0.$$

(3) Now we prove that

$$\operatorname{Im}(P_\alpha(\alpha_0, t) h, Q_\alpha(\alpha_0, t) h) \geqslant 0.$$

Equation (3.6) implies that

$$\begin{aligned} \frac{d}{dt}(P_\alpha(\alpha, t) h, Q_\alpha(\alpha, t) h) &= -(H_{pq} P_\alpha(\alpha, t) h, Q_\alpha(\alpha, t) h) - \\ &- (H_{qq} Q_\alpha(\alpha, t) h, Q_\alpha(\alpha, t) h) + \\ &+ (P_\alpha(\alpha, t) h, H_{pp} P_\alpha(\alpha, t) h) + (P_\alpha(\alpha, t) h, \\ &H_{pq} Q_\alpha(\alpha, t) h) = (P_\alpha(\alpha, t) h, H_{pp} P_\alpha(\alpha, t) h) - \\ &- (Q_\alpha(\alpha, t) h, H_{qq} Q_\alpha(\alpha, t) h). \end{aligned}$$

Thus the derivative $\frac{d}{dt}(P_\alpha(\alpha, t) h, Q_\alpha(\alpha, t) h)$ is real, as the sum of two quadratic forms corresponding to Hermitian bilinear forms. Hence $\operatorname{Im}(P_\alpha(\alpha, t) h, Q_\alpha(\alpha, t) h)$ does not depend on t, which completes the proof.

Now let us consider the Hamiltonian function $H = \frac{1}{2}(p_{\bar{I}}^2 + q_{\bar{I}}^2)$. The trajectory of H starting from $(Q(\alpha), P(\alpha))$ is given by

$$\begin{aligned} Q_{\bar{I}}(\alpha, t) &= Q_{\bar{I}}(\alpha) \cos t + P_{\bar{I}}(\alpha) \sin t, \\ Q_I(\alpha, t) &= Q_I(\alpha), \\ P_{\bar{I}}(\alpha, t) &= P_{\bar{I}}(\alpha) \cos t - Q_{\bar{I}}(\alpha) \sin t, \\ P_I(\alpha, t) &= P_I(\alpha). \end{aligned} \tag{3.8}$$

Let $\alpha_0 \in M$ be C-Lagrangean with respect to the mapping $Q = Q(\alpha)$, $P = P(\alpha)$, and suppose that the conditions

$$J(\alpha) \overset{\text{def}}{=} \frac{DQ}{D\alpha} \neq 0,$$

$$J_I(\alpha) \overset{\text{def}}{=} \frac{D(Q_I,\ P_{\bar{I}})}{D\alpha} \neq 0$$

are satisfied at α_0. Let a certain value $\operatorname{Arg} J(\alpha_0)$ of the phase argument of the Jacobian $J(\alpha_0)$ be chosen. Using the transformation (3.8) we define $\operatorname{Arg} J_I(\alpha_0)$ correlated with $\operatorname{Arg} J(\alpha_0)$ as follows. First we define the family $(Q^\tau(\alpha), P^\tau(\alpha))$, $\tau \geqslant 0$, of mappings $M \to \mathbf{C}^{2n}$ by

$$Q^\tau(\alpha) = Q(\alpha) - i\tau P(\alpha),$$
$$P^\tau(\alpha) = P(\alpha) + i\tau Q(\alpha). \tag{3.9}$$

Next we set

$$J(\alpha,\ \tau) = \frac{DQ^\tau}{D\alpha}$$

and define $\operatorname{Arg} J(\alpha_0,\ \tau)$ as a continuous function satisfying the condition

$$\operatorname{Arg} J(\alpha_0,\ 0) = \operatorname{Arg} J(\alpha_0).$$

The transformation (3.8) induces the following homotopy of the mappings

$$\alpha \to Q^\tau(\alpha)$$

and

$$\alpha \to (Q_I(\alpha) - i\tau P_I(\alpha),\quad P_{\bar{I}}(\alpha) + i\tau Q_{\bar{I}}(\alpha)):$$

$$Q^\tau(\alpha,\ t) = (Q_I^\tau(\alpha),\quad Q_{\bar{I}}^\tau(\alpha)\cos t + P_{\bar{I}}^\tau(\alpha)\sin t),$$

$$0 \leqslant t \leqslant \frac{\pi}{2}.$$

For a fixed τ, the mapping $\left[0,\ \frac{\pi}{2}\right] \to \mathbf{C}$ defined by

$$t \to \left.\frac{DQ^\tau(\alpha,\ t)}{D\alpha}\right|_{\alpha=\alpha_0} \overset{\text{def}}{=} J(\alpha_0,\ \tau,\ t),\qquad 0 \leqslant t \leqslant \frac{\pi}{2} \tag{3.10}$$

is a curve in $\mathbf{C}$ starting from $J(\alpha_0,\ \tau)$ and finishing at $J_I(\alpha_0,\ \tau) \overset{\text{def}}{=} \overset{\text{def}}{=} \frac{DQ_I^\tau}{D\alpha}(\alpha_0)$. Let us show that $J(\alpha_0,\ \tau,\ t) \neq 0$ for $\tau > 0$. In fact, it follows from Lemma 3.2 that α_0 is C-Lagrangean with respect

to the mapping

$$\alpha \to (Q(\alpha, t), P(\alpha, t)),$$

where $Q(\alpha, t)$ and $P(\alpha, t)$ are defined by (3.8). By Lemma 3.1, for $\tau > 0$ we have

$$\det\left(\frac{\partial Q(\alpha, t)}{\partial \alpha} - i\tau \frac{\partial P(\alpha, t)}{\partial \alpha}\right)_{\alpha=\alpha_0} \neq 0.$$

It remains to be noted that

$$Q^\tau(\alpha, t) = Q(\alpha, t) - i\tau P(\alpha, t).$$

Since the curve (3.10) does not pass through the origin, it defines a certain value $\operatorname{Arg} J_I(\alpha_0, \tau)$ of the argument of the Jacobian $J_I(\alpha_0, \tau)$. Finally, we define $\operatorname{Arg} J_I(\alpha_0)$ by

$$\operatorname{Arg} J_I(\alpha_0) = \lim_{\tau \to +0} \operatorname{Arg} J_I(\alpha_0, \tau). \tag{3.11}$$

The limit in (3.11) must exist as is seen from the following lemma.

Lemma 3.3. *If* $J_I(\alpha_0, 0) \neq 0$, *then* $\operatorname{Arg} J_I(\alpha_0, \tau)$ *is uniformly continuous on* $(0, \tau_0)$ *for any* $\tau_0 > 0$.

Proof. Let $0 < \tau_1 < \tau_2 < \tau_0$. Consider the following four paths in $\mathbf{C}$:

$$l_1 : \tau \to J(\alpha_0, \tau), \quad \tau_1 \leqslant \tau \leqslant \tau_2,$$

$$l_2 : t \to J(\alpha_0, \tau_1, t), \quad 0 \leqslant t \leqslant \frac{\pi}{2},$$

$$l_3 : t \to J(\alpha_0, \tau_2, t), \quad 0 \leqslant t \leqslant \frac{\pi}{2},$$

$$l_4 : \tau \to J_I(\alpha_0, \tau), \quad \tau_1 \leqslant \tau \leqslant \tau_2.$$

Let Δ_j be the increment of the phase argument corresponding to l_j, $j = 1, \ldots, 4$. Since the increment of the phase argument corresponding to a closed path is a homotopic invariant and the path $l_1 + l_3 - l_4 - l_2$ in $\mathbf{C}\,\{0\}$ is homotopic to a point, we have $\Delta_1 + \Delta_3 = \Delta_2 + \Delta_4$. It is clear that

$$\operatorname{Arg} J_I(\alpha_0, \tau_2) - \operatorname{Arg} J_I(\alpha_0, \tau_1) = \Delta_1 + \Delta_3 - \Delta_2 = \Delta_4.$$

Let $m = \min\limits_{0 \leqslant \tau \leqslant \tau_0} |J_I(\alpha_0, \tau)|$. If

$$\max_{\tau_1 \leqslant \tau \leqslant \tau_2} |J_I(\alpha_0, \tau_1) - J_I(\alpha_0, \tau)| < m,$$

then a simple calculation yields

$$|\sin \Delta_4| \leqslant m^{-1} |J_I(\alpha_0, \tau_1) - J_I(\alpha_0, \tau_2)|.$$

Thus uniform continuity of $\operatorname{Arg} J_I(\alpha_0, \tau)$ follows from that of $J_I(\alpha_0, \tau)$, and the lemma is proved.

Definition 3.2. Arg $J_I(\alpha_0)$ *defined by* (3.11) *will be considered concordant with* Arg $J(\alpha_0)$.

Let $\alpha_0 \in \Omega \cap \Omega_I \cap \Gamma$ be a point of a Lagrangean manifold with a complex germ. For this situation we have introduced in Sec. 1 another rule of correlating the argument Arg $J_I(\alpha_0)$ with Arg $J(\alpha_0)$, that rule depending on bypassing focuses operation. Let us show that these two rules are equivalent. In the case where α_0 is in the intersection of non-singular patches of the zones Ω and Ω_I, this is an immediate consequence of the following result.

Lemma 3.4. *Let $\alpha_0 \in M$ be C-Lagrangean with respect to the mapping $\alpha \to (Q(\alpha), P(\alpha))$, and let $Q^\tau(\alpha)$, $P^\tau(\alpha)$ be given by* (3.9). *Further let $\bar{I} = \{1, \ldots, k\}$, and set*

$$Q_{\bar{I}}^{\tau, \varepsilon}(\alpha, t) \overset{\text{def}}{=\!=} Q_{\bar{I}}^\tau(\alpha)\cos t + \varepsilon P_{\bar{I}}^\tau(\alpha)\sin t,$$

$$Q_I^{\tau, \varepsilon}(\alpha, t) \overset{\text{def}}{=\!=} Q_I^\tau(\alpha),$$

$$P_I^{\tau, \varepsilon}(\alpha, t) \overset{\text{def}}{=\!=} P_{\bar{I}}^\tau(\alpha)\cos t - \varepsilon Q_{\bar{I}}^\tau(\alpha)\sin t,$$

$$P^{\tau, \varepsilon}(\alpha, t) \overset{\text{def}}{=\!=} P_I^\tau(\alpha)$$

$$J_\varepsilon(\alpha, \tau, t) \overset{\text{def}}{=\!=} \frac{DQ^{\tau, \varepsilon}(\alpha, t)}{D\alpha} \tag{3.12}$$

for any symmetrical orthogonal $k \times k$ matrix ε. Define Arg $J_\varepsilon(\alpha_0, \tau_0, 0)$ *for $\tau > 0$ as a continuous function independent of ε* (note that $J_\varepsilon(\alpha_0, \tau, 0)$ does not depend on ε). *Further define* Arg $J_\varepsilon(\alpha_0, \tau, t)$ *as a continuous function of t for a fixed τ. Then*

$$\operatorname{Arg} J_\varepsilon\left(\alpha_0, \tau, \frac{\pi}{2}\right) + \pi\sigma_\varepsilon,$$

where σ_ε is the number of negative eigenvalues of ε, does not depend on ε.

Proof. An argument similar to that used in the proof of Lemma 3.3 shows that Arg $J_\varepsilon(\alpha_0, \tau, \pi/2)$ is continuous in τ for $\tau > 0$. Consider the difference

$$\Delta_{\varepsilon, \varepsilon'}(\tau) = \left[\operatorname{Arg} J_\varepsilon\left(\alpha_0, \tau, \frac{\pi}{2}\right) + \pi\sigma_\varepsilon\right] - \left[\operatorname{Arg} J_{\varepsilon'}\left(\alpha_0, \tau, \frac{\pi}{2}\right) + \pi\sigma_{\varepsilon'}\right].$$

Since

$$(-1)^{\sigma_\varepsilon} J_\varepsilon\left(\alpha_0, \tau, \frac{\pi}{2}\right) = (-1)^{\sigma_{\varepsilon'}} J_{\varepsilon'}\left(\alpha_0, \tau, \frac{\pi}{2}\right),$$

we have

$$\Delta_{\varepsilon, \varepsilon'}(\tau) \equiv 0 \pmod{2\pi},$$

which implies by continuity that $\Delta_{\varepsilon,\varepsilon'}(\tau)$ does not depend on τ. Therefore, it suffices to show that $\Delta_{\varepsilon,\varepsilon'}(1) = 0$ for any ε and ε', i.e., that $\operatorname{Arg} J_\varepsilon(\alpha_0, 1, \pi/2) + \pi\sigma_\varepsilon$ does not depend on ε. We have

$$J_\varepsilon(\alpha_0, \tau, t) = v(t) \det (C^\tau \cos t + \hat{\varepsilon} C_I^\tau \sin t),$$

where

$$v(t) = \det \begin{pmatrix} 1_k & 0 \\ 0 & (\cos t + \sin t)^{-1} 1_{n-k} \end{pmatrix} > 0,$$

$$C^\tau = \frac{DQ^\tau}{D\alpha}(\alpha_0), \quad C_I^\tau = \frac{D(P_I^\tau, Q_I^\tau)}{D\alpha}(\alpha_0),$$

$$\hat{\varepsilon} = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1_{n-k} \end{pmatrix},$$

and 1_j is the $j \times j$ identity matrix. In particular,

$$J_\varepsilon(\alpha_0, 1, t) = v(t) \det C^1 \det \left(\cos t + \hat{\varepsilon} \frac{\partial (P_I^1, Q_I^1)}{\partial \alpha} \alpha_0 \sin t \right).$$

Note that $P^1 = iQ^1$, hence

$$\frac{\partial (P_I^1, Q_I^1)}{\partial Q^1} = \begin{pmatrix} i1_k & 0 \\ 0 & 1_{n-k} \end{pmatrix},$$

so we obtain

$$J_\varepsilon(\alpha_0, 1, t) =$$

$$= v(t) \det C^1 (\cos t + \sin t)^{n-k} \prod_{s=1}^{k} (\cos t + i\lambda_s \sin t) =$$

$$= \det C^1 \prod_{s=1}^{k} (\cos t + i\lambda_s \sin t),$$

where $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of ε.

For any $z \in \mathbf{C} \setminus \{0\}$, define $\arg z$, the *reduced* phase argument of z, in such a way that $-\pi < \arg z \leqslant \pi$. Since

$$\arg (\cos t + i\lambda_s \sin t) \neq \pi$$

for $t \in \left[0, \frac{\pi}{2}\right]$, it follows that

$$\sum_{s=1}^{k} \arg (\cos t + i\lambda_s \sin t)$$

is continuous in t on $\left[0, \frac{\pi}{2}\right]$. Therefore,

$$\operatorname{Arg} J_\varepsilon(\alpha_0, 1, t) = \operatorname{Arg} J_\varepsilon(\alpha_0, 1, 0) + \sum_{s=1}^{k} \arg (\cos t + i\lambda_s \sin t).$$

Putting $t=\frac{\pi}{2}$, we get

$$\operatorname{Arg} J_\varepsilon\left(\alpha_0,\ 1,\ \frac{\pi}{2}\right)=\arg J_\varepsilon(\alpha_0,\ 1,\ 0)+\frac{\pi k}{2}-\pi\sigma_\varepsilon, \tag{3.13}$$

so $\Delta_\varepsilon(1)$ equals $\frac{\pi k}{2}$ which is independent of ε, Q.E.D.

Now we turn to the case where $\alpha_0 \in \Omega \cap \Omega_I \cap \Gamma$ is not in the intersection of non-singular patches of the zones Ω and Ω_I.

Lemma 3.5. *The limit in* (1.22) *exists,* $\operatorname{Arg} J_I(\alpha_0)$ *defined by* (1.22) *being concordant with* $\operatorname{Arg} J(\alpha_0)$.

Proof. Let α_0 satisfy the assumptions of Proposition 1.2 and let $Q^{\tau,\varepsilon}(\alpha, t, s)$, $P^{\tau,\varepsilon}(\alpha, t, s)$ be defined in the same way as $Q^{\tau,\varepsilon}(\alpha, t)$ and $P^{\tau,\varepsilon}(\alpha, t)$ in (3.12) but with $Q(\alpha)$, $P(\alpha)$ replaced by $q(\alpha, s)+z(\alpha, s)$ and $p(\alpha, s)+w(\alpha, s)$, respectively, where

$$(p(\alpha,\ s),\ q(\alpha,\ s))=g_{H_0}^s(p(\alpha),\ q(\alpha)),$$

$$(w(\alpha,\ s),\ z(\alpha,\ s))=dg_{H_0}^s(w(\alpha),\ z(\alpha)),$$

$$H_0(p,\ q)=\frac{1}{2}(p^2+q^2).$$

Set

$$J_\varepsilon(\alpha,\ \tau,\ t,\ s)=\det\frac{\partial Q^{\tau,\varepsilon}(\alpha,\ t,\ s)}{\partial\alpha}$$

and choose for any $s \in (0, s_0)$ a symmetric orthogonal matrix $\varepsilon(s)$ in such a way that

$$J_{\varepsilon(s)}(\alpha_0,\ 0,\ t,\ s)\neq 0$$

for $0 \leqslant t \leqslant \frac{\pi}{2}$.

Let a certain value of the phase argument of $J(\alpha_0)$ be chosen. We first define by continuity $\operatorname{Arg} J_{\varepsilon(s)}(\alpha_0, 0, 0, s)$, next $\operatorname{Arg} J_{\varepsilon(s)}(\alpha_0, \tau, 0, s)$, $\tau \geqslant 0$ for a fixed $s \in (0, s_0)$, and then $\operatorname{Arg} J_{\varepsilon(s)}(\alpha_0, \tau, t, s)$ for fixed s and τ. The statement of the lemma means that

$$\lim_{s\to+0}\left[\operatorname{Arg} J_{\varepsilon(s)}\left(\alpha_0,\ 0,\ \frac{\pi}{2},\ s\right)+\pi\sigma(\varepsilon)\right]=$$

$$=\lim_{\tau\to+0}\operatorname{Arg} J_1\left(\alpha_0,\ \tau,\ \frac{\pi}{2},\ 0\right).$$

To prove this we observe that by Lemma 3.4,

$$\pi\sigma_{\varepsilon(s)}+\operatorname{Arg} J_{\varepsilon(s)}\left(\alpha_0,\ \tau,\ \frac{\pi}{2},\ s\right)=\operatorname{Arg} J_1\left(\alpha_0,\ \tau,\ \frac{\pi}{2},\ s\right) \tag{3.14}$$

for $\tau>0$. Since

$$\lim_{\substack{\tau\to 0\\ s\to 0}} J\left(\alpha_0,\ \tau,\ \frac{\pi}{2},\ s\right)=J_I(\alpha_0)\neq 0,$$

we obtain by an argument similar to that of the proof of Lemma 3.3 that $\operatorname{Arg} J_1\left(\alpha_0, \tau, \frac{\pi}{2}, s\right)$ is uniformly continuous on the open square $0 < \tau < \tau_0$, $0 < s < s_0$. This, together with (3.14), imply the required relation.

Definition 3.3. *Given a real-smooth mapping $\alpha \to (P(\alpha), Q(\alpha))$ from a manifold M into $\mathbf{C}^{2n}$, the zone Ω_I is the set of all the points $\alpha \in M$ such that α is C-Lagrangean and*

$$\det C_I(\alpha) \overset{\text{def}}{=\!=} J_I(\alpha) \neq 0.$$

It follows from Lemma 2.4 of Chapter IV that the family $\{\Omega_I\}$ with I running through all subsets of $\{1, \ldots, n\}$ covers the set of all C-Lagrangean points of M.

Let $\alpha_0 \in \Omega_I \cap \Omega_K$. We shall give the rule of correlating the phase argument of $J_K(\alpha_0)$ with that of $J_I(\alpha_0)$. Let a certain value of the phase argument of $J_I(\alpha_0)$ be chosen. We set

$$J_I(\alpha, \tau) = \frac{D\left(Q_I^\tau(\alpha), P_{\bar{I}}^\tau(\alpha)\right)}{D\alpha}$$

(Q^τ, P^τ being introduced in (3.9)), and define $\operatorname{Arg} J_I(\alpha_0, \tau)$ as a continuous function satisfying the condition

$$\operatorname{Arg} J_I(\alpha_0, 0) = \operatorname{Arg} J_I(\alpha_0).$$

Consider the Hamiltonian function

$$H(p, q) = \frac{1}{2}\left(p_{I_1}^2 + q_{I_1}^2 - p_{K_1}^2 - q_{K_1}^2\right),$$

where

$$I_1 = I \setminus K, \quad I_2 = K \setminus I.$$

The trajectory of this H starting from $(Q(\alpha), P(\alpha))$ is given by

$$Q_{I_1}(\alpha, t) = Q_{I_1}(\alpha)\cos t + P_{I_1}(\alpha)\sin t,$$

$$P_{I_1}(\alpha, t) = P_{I_1}(\alpha)\cos t - Q_{I_1}(\alpha)\sin t,$$

$$Q_{K_1}(\alpha, t) = Q_{K_1}(\alpha)\cos t - P_{K_1}(\alpha)\sin t,$$

$$P_{K_1}(\alpha, t) = P_{K_1}(\alpha)\cos t + Q_{K_1}(\alpha)\sin t,$$

$$Q_{\bar{I}_1 \cap \bar{K}_1}(\alpha, t) = Q_{\bar{I}_1 \cap \bar{K}_1}(\alpha),$$

$$P_{\bar{I}_1 \cap \bar{K}_1}(\alpha, t) = P_{\bar{I}_1 \cap \bar{K}_1}(\alpha).$$

The transformation (3.15) induces the following homotopy of the curves $\tau \to J_I(\alpha_0, \tau)$ and $\tau \to J_K(\alpha_0, \tau)$:

$$J_I(\alpha, \tau, t) = \\ = \det \frac{\partial (Q^\tau_{I_1}(\alpha)\cos t + P^\tau_{I_1}(\alpha)\sin t,\ Q^\tau_{I\cap K}(\alpha),\ P^\tau_{K_1}(\alpha)\cos t + Q^\tau_{K_1}(\alpha)\sin t,\ P^\tau_{\overline{I}\cap\overline{K}}(\alpha))}{\partial\alpha},$$

$$0 \leqslant t \leqslant \frac{\pi}{2}. \tag{3.16}$$

Lemma 3.1 and Lemma 3.2 imply that $J_I(\alpha_0, \tau, t) \neq 0$ for $\tau > 0$. Therefore, for $\tau > 0$, one can define $\operatorname{Arg} J_I(\alpha_0, \tau, t)$ as a function continuous in t. Finally, let us define $\operatorname{Arg} J_K(\alpha_0)$ by

$$\operatorname{Arg} J_K(\alpha_0) = \lim_{\tau \to +0} \operatorname{Arg} J_I\left(\alpha_0, \tau, \frac{\pi}{2}\right). \tag{3.17}$$

The proof of existence of the last limit is quite similar to that of Lemma 3.3.

Definition 3.4. $\operatorname{Arg} J_K(\alpha_0)$ *defined by* (3.17) *will be said to be concordant with* $\operatorname{Arg} J_I(\alpha_0)$.

Lemma 3.6. *If* $\operatorname{Arg} J_K(\alpha_0)$ *is concordant with* $\operatorname{Arg} J_I(\alpha_0)$ *and* $\operatorname{Arg} J_I(\alpha_0)$ *is concordant with* $\operatorname{Arg} J_L(\alpha_0)$, *then* $\operatorname{Arg} J_K(\alpha_0)$ *is concordant with* $\operatorname{Arg} J_L(\alpha_0)$.

Proof. We may assume without loss of generality that $\overline{L} = \varnothing$. Set

$$J(\alpha, \tau, t) = \det \frac{\partial (Q^\tau_I(\alpha),\ Q^\tau_{\overline{I}}(\alpha)\cos t + P^\tau_{\overline{I}}(\alpha)\sin t)}{\partial\alpha},$$

$$\overline{J}(\alpha, \tau, t) = \det \frac{\partial (Q^\tau_K(\alpha),\ Q^\tau_{\overline{K}}(\alpha)\cos t + P^\tau_K(\alpha)\sin t)}{\partial\alpha}.$$

It suffices to show that

$$\Delta(\tau) \overset{\text{def}}{=} \operatorname{Arg} J\left(\alpha_0, \tau, \frac{\pi}{2}\right) - \operatorname{Arg} J(\alpha_0, \tau, 0) + \\ + \operatorname{Arg} J_I\left(\alpha_0, \tau, \frac{\pi}{2}\right) - \operatorname{Arg} J_I(\alpha_0, \tau, 0) - \\ - \operatorname{Arg} \overline{J}\left(\alpha_0, \tau, \frac{\pi}{2}\right) + \operatorname{Arg} \overline{J}(\alpha_0, \tau, 0) = 0$$

for every $\tau > 0$; here $J_I(\alpha, \tau, t)$ is defined by (3.16), and all the phase arguments of Jacobians are assumed to be continuous functions in τ and t. This, in turn, is equivalent to the equality

$$\Delta(1) = 0,$$

because $\Delta(\tau)$ is continuous for $\tau > 0$ and

$$\Delta(\tau) \equiv 0 \pmod{2\pi}.$$

Let m_1, m_2, m_3 be the numbers of elements in $I \setminus K$, $K \setminus I$, $I \setminus K$, respectively. We have already shown (see (3.13)) that

$$\operatorname{Arg} J\left(\alpha_0, 1, \frac{\pi}{2}\right) - \operatorname{Arg} J(\alpha_0, 1, 0) = \frac{\pi}{2}(n - m_1 - m_3),$$

$$\operatorname{Arg} \bar{J}\left(\alpha_0, 1, \frac{\pi}{2}\right) - \operatorname{Arg} \bar{J}(\alpha_0, 1, 0) = \frac{\pi}{2}(n - m_2 - m_3). \quad (3.18)$$

It is easy to verify that

$$J_I(\alpha, t, \tau) = (\sin t + \cos t)^{-n+m_1+m_2} \times$$

$$\times \left[\frac{\partial(Q_I^\tau(\alpha), P_{\bar{I}}^\tau(\alpha))}{\partial a}\cos t + \frac{\partial(Q_K^\tau(\alpha), P_{\bar{K}}^\tau(\alpha))}{\partial \alpha}\sin t\right].$$

Putting $\tau = 1$ here, one obtains

$$J_I(\alpha, 1, t) = \frac{D(Q(\alpha) - iP(\alpha))}{D\alpha} \cdot i^{n-m_1-m_2-m_3} \times$$

$$\times (\cos t + i\sin t)^{m_1}(i\cos t + \sin t)^{m_2},$$

hence

$$\operatorname{Arg} J_I\left(\alpha_0, 1, \frac{\pi}{2}\right) - \operatorname{Arg} J_I(\alpha_0, 1, 0) = \frac{\pi}{2}(m_1 - m_2). \quad (3.19)$$

The required result $\Delta(1) = 0$ now follows from (3.18) and (3.19), and the lemma is proved.

We can now define the index of a complex germ. To make the treatment more transparent, however, we shall first define the index for a class of objects which are closely related to Lagrangean manifolds with complex germs. For example, this class contains all (real) Lagrangean manifolds.

Definition 3.5. *A C-Lagrangean manifold is a real manifold together with a real-smooth mapping $f : M \to \mathbb{C}^{2n}$ such that each point of M is C-Lagrangean with respect to f.*

We start with the index of a closed path in a C-Lagrangean manifold. First consider a path l (not necessarily closed) lying in a coordinate neighborhood $u \in M$. Let $\Delta(l, \tau)$, $\tau > 0$, be the increment of the phase argument of $\det(C(\alpha) - i\tau B(\alpha))$ corresponding to l. It is obvious that $\Delta(l, \tau)$ is independent of the choice of local coordinates; in fact, on making a change of coordinates in u, $\det(C(\alpha) - i\tau B(\alpha))$ acquires a positive or a negative factor. For the case where l is not in any coordinate neighborhood, we define $\Delta(l, \tau)$ by additivity. It is clear that $\Delta(l, \tau)$ is independent of τ

provided that l is closed. In this case we set

$$\operatorname{Ind} l = \frac{1}{2\pi}\Delta(l, \pi), \quad \tau > 0.$$

One can rewrite this as follows:

$$\operatorname{Ind} l = -\frac{i}{2\pi}\oint_l d\ln\frac{\frac{D(Q-i\tau P)}{D\alpha}}{\left|\frac{D(Q-i\tau P)}{D\alpha}\right|}. \tag{3.20}$$

It is easily seen that if l is homotopic to a point, then $\operatorname{Ind} l = 0$; hence, some 1-dimensional (characteristic) cohomology class of M is defined.

Now, let l be a path joining two non-singular points (i.e., points where $DQ/D\alpha \neq 0$). Set

$$\Delta l = \lim_{\tau\to+0}\Delta(l, \tau); \tag{3.21}$$

Δl will be called the *rotation of Jacobian along* l. Existence of the limit in (3.21) can be proved in the same way as in (3.11).

Now suppose that $\operatorname{Ind} l = 0$ for every closed path l in M, in other words, that the cohomology class introduced above is trivial. Then the rotation of Jacobian along a path joining a non-singular point α^1 to a non-singular point α^2 depends only on α^1 and α^2 (but not on the choice of a path). Let a point $\alpha^0 \in M$, called initial, be fixed; to be definite, assume α^0 to be non-singular. Let us choose a certain value of the argument of $\det Q_\alpha(\alpha^0)$, which determines, in particular, a local orientation of M at α^0. We can uniquely define $\operatorname{Arg}\frac{DQ}{D\alpha}$ for any non-singular point α_1 which belongs to the connected component of M containing α_0 by the formula

$$\operatorname{Arg}\frac{DQ}{D\alpha}(\alpha^1) = \operatorname{Arg}\frac{DQ}{D\alpha}(\alpha^0) + \Delta(l[\alpha^0, \alpha^1]), \tag{3.22}$$

where $l[\alpha^0, \alpha^1]$ is a path joining α^0 to α^1. In particular, (3.22) determines a local orientation of M at α^1. Thus, we have defined a local orientation of M at every point of the non-singular zone of a connected component of M.

We shall show that if M is connected, then the local orientations introduced above determine a global orientation of the zone Ω. Let α^1, α^2 be two points of Ω, let $u_1 \ni \alpha^1$, $u_2 \ni \alpha^2$ be two coordinate neighborhoods in Ω, and let x, y be coordinates in u_1, u_2 compatible with local orientations at α^1 and α^2, respectively. If $\alpha \in u_1 \cap u_2$, then both x and y are compatible with the local orientation at α. To see this it suffices to consider the rotations of Jacobian along the following two paths joining α_0 to α:

(1) $\quad l[\alpha^0, \alpha^1] + l[\alpha^1, \alpha]$,

where the whole of $l\,[\alpha^2, \alpha]$ lies in u_2;

(2) $\quad l\,[\alpha^0, \alpha^2] + l\,[\alpha^2, \alpha]$,

where the whole of $l\,[\alpha^2, \alpha]$ lies in u_2.

So $\frac{Dy}{Dx} > 0$, the global orientation of Ω being actually defined.

Thus, we have shown that triviality of the characteristic class Ind implies orientability of the non-singular zone. But, in fact, triviality of this class implies orientability of the whole of M. To show this consider the mapping

$$\alpha \to (Q(\alpha) - i\tau P(\alpha), \quad P(\alpha) + i\tau Q(\alpha)) \tag{3.23}$$

for some $\tau > 0$. With respect to this mapping, M is again C-Lagrangean (this is an easy exercise for the reader), and all points of M are now non-singular as seen from Lemma 3.1

We now define uniquely $\operatorname{Arg} J_I(\alpha)$ for every point $\alpha \in \Omega_I$ provided that the initial point $\alpha^0 \in M$ is fixed and a certain value $\operatorname{Arg} \frac{DQ}{D\alpha}(\alpha^0)$ of the phase argument of $\frac{DQ}{D\alpha}(\alpha_0)$ is chosen, by the following procedure (correct if $\mathrm{Ind} = 0$):

(1) For $\tau \geqslant 0$, define $\operatorname{Arg} J(\alpha^0, \tau)$, where

$$J(\alpha^0, \tau) = \frac{D(Q - i\tau P)}{D\alpha}(\alpha^0),$$

by continuity.

(2) For $\tau > 0$, define $\operatorname{Arg} J(\alpha, \tau)$ by

$$\operatorname{Arg} J(\alpha, \tau) = \operatorname{Arg}(\alpha^0, \tau) + \Delta(l\,[\alpha^0, \alpha], \tau),$$

so $\operatorname{Arg} J(\alpha, \tau)$ is continuous in α.

(3) For $\tau > 0$, define $\operatorname{Arg} J_I(\alpha, \tau)$, where

$$J_I(\alpha, \tau) = \frac{D(Q_I - i\tau P_I,\ P_{\bar{I}} + i\tau Q_{\bar{I}})}{D\alpha},$$

to be concordant with $\operatorname{Arg} J(\alpha, \tau)$.

(4) Set

$$\operatorname{Arg} J_I(\alpha) = \lim_{\tau \to +0} \operatorname{Arg} J_I(\alpha, \tau).$$

Problem 3.1. If $\alpha \in \Omega_I \cap \Omega_{\bar{K}}$, then $\operatorname{Arg} J_I(\alpha)$ is concordant with $\operatorname{Arg} J_K(\alpha)$.

Returning to a Lagrangean manifold with a complex germ, we recall that it can be regarded as a manifold with the C-Lagrangean subset Γ (see Example 3.1). For such a manifold, one could define Ind_Γ, the index on Γ, by considering paths lying in a small neighborhood of Γ. However, it is more convenient to define Ind_Γ in terms of the Čech cohomology theory, where the index of a closed

chain of patches plays the same role as before the index of a closed path did. To clarify the connection between these two approaches, we shall describe Ind as a Čech cohomology class of a C-Lagrangean manifold M.

We start with the case where $M = \Omega$, the non-singular zone. Let $\{u_j\}$ be a covering of M by open subsets u_j such that the increment of Arg J corresponding to any path lying in u_j is less than $\pi/2$; such a covering will be said to be *admissible*. For every j, fix a point $\alpha^j \in u_j$ which will be called the central point of u_j. Define the 1-cochain Δ Arg J with real coefficients as follows: for every pair of intersecting sets (u_j, u_k), we let $(\Delta \text{ Arg } J)_{jk}$ be the increment of Arg J according to a path $l\,[\alpha^j, \alpha^k]$ which joins α^j to α^k and consists of two arcs, the first lying in u_j, the second in u_k. Obviously, $(\Delta \text{ Arg } J)_{jk}$ does not depend on the choice of paths $l\,[\alpha^j, \alpha^k]$, since $|\,\Delta \text{ Arg } J\,|_{jk} < \pi$ because of the admissibility property of the covering $\{u_j\}$. It is easy to check that Δ Arg J is a cocycle.

Definition 3.6. *We define* Ind *to be the cohomology class induced by the 1-cocycle* $\frac{1}{2\pi}\,\Delta$ Arg J.

Let us show that Ind is independent of the choice of central points. In fact, let $[\bar{\alpha}^j]$ be another family of central points, and let δ_j be the increment of Arg J corresponding to a path joining α^j to $\bar{\alpha}^j$ and lying in u_j. Then the coboundary of the 0-cochain $[\delta_j]$ is the difference of the 1-cocycles Δ Arg J corresponding to the families $\{\bar{\alpha}^j\}$ and $\{\alpha^j\}$, so these two 1-cocycles are cohomologic.

Problem 3.2. The class Ind is independent of the choice of an admissible covering $\{u_j\}$.

Lemma 3.7. *The class* Ind *of a C-Lagrangean manifold $M \subset M$ is trivial if, and only if, there exists a continuous branch of the phase argument of the Jacobian J.*

Proof. 1°. Sufficiency. Let Arg J be a continuous branch of the phase argument of J. Then $\{\text{Arg } J\,(\alpha^j)\}$ is a 0-cochain with the coboundary Δ Arg J.

2°. Necessity. Let Ind be trivial. Then for some admissible covering $\{u_j\}$ (with any fixed family of central points), there is such a family $\{a_j\}$ of real numbers that $(\Delta \text{ Arg } J)_{jk} = a_k - a_j$, where $(\Delta \text{ Arg } J)_{jk}$ is the 1-cochain corresponding to $\{u_j\}$. Without loss of generality we may assume that M is connected and that a^{j_0} is one of the possible values of Arg $J\,(\alpha^{j_0})$ for some j_0. Then a_j is obviously one of the possible values of Arg $J\,(\alpha^j)$ for any j (to see this it suffices to consider a chain of elements of the covering joining u_{j_0} to u_j). This value of Arg $J\,(\alpha^j)$ gives rise to a continuous branch of Arg J in u_j, so Arg J

is defined as a 0-cochain with coefficients in the sheaf of germs of smooth functions. To prove necessity it remains to show that this cochain is a cocycle. But this is an obvious consequence of the fact that

$$|a_j - a_k| < \pi \quad \text{if} \quad u_j \cap u_k \neq \varnothing.$$

We shall now remove the assumption made above that the whole of M is in the zone Ω. Let $\alpha \to (P(\alpha), Q(\alpha))$ be, as formerly, the mapping that determines on M the structure of a C-Lagrangean manifold. Recall that M is in the non-singular zone with respect to the mapping (3.9) with $\tau > 0$. So we obtain the family $\{\mathrm{Ind}^\tau\}$ of cohomology classes of M. But, in fact, Ind^τ is independent of τ. To see this, let τ_1 and τ_2 be positive numbers, $\tau_1 < \tau_2$, and let $\{u_j\}$ be a covering of M by such open subsets that for every j, the increment of $\mathrm{Arg}\, J(\alpha, \tau)$ corresponding to any path lying in u_j is less than $\pi/2$ if $0 < \tau_1 \leqslant \tau \leqslant \tau_2$. Further, let $\{\delta_{ik}^{\tau_1}\}$ and $\{\delta_{ik}^{\tau_2}\}$ be the cocycles $\Delta\, \mathrm{Arg}\, J$ corresponding to the mapping (3.9) with $\tau = \tau_1$ and $\tau = \tau_2$, respectively. Then we have

$$\delta_{jk}^{\tau_2} - \delta_{jk}^{\tau_1} = \Delta_{\tau_1\tau_2}^{k} - \Delta_{\tau_1\tau_2}^{j},$$

where $\Delta_{\tau_1\tau_2}^{j}$ is the increment of $\mathrm{Arg}\, J(\alpha^j, \tau)$ corresponding to the change of τ from τ_1 to τ_2. So the cocycles $\{\delta_{jk}^{\tau_2}\}$ and $\{\delta_{ik}^{\tau_1}\}$ are cohomologic.

We may now define the index for a C-Lagrangean manifold by

$$\mathrm{Ind} \overset{\text{def}}{=} \mathrm{Ind}^\tau, \quad \tau > 0.$$

Lemma 3.7 leads to the following criterion for triviality of the class Ind:

Proposition 3.1. *The class* Ind *of a C-Lagrangean manifold M is trivial if, and only if, for each zone $\Omega_I \subset M$, there exists a continuous branch* $\mathrm{Arg}\, J_I$ *of the phase argument of the Jacobian J_I, these branches being correlated in the intersections of zones.*

Now let M be a smooth real manifold with a given mapping $M \to \mathbf{C}^{2n}$, and let $\Gamma \subset M$ be C-Lagrangean with respect to this mapping.

Consider the following "almost constant" pre-sheaf Π on M:

$$\Pi(u) = \begin{cases} \mathbf{R} \text{ if } u \cap \Gamma \neq \varnothing \\ 0 \text{ if } u \cap \Gamma \neq \varnothing \end{cases}$$

with the naturally defined restriction homomorphism. We will define Ind_Γ as a Čech cohomology class with coefficients in Π.

First consider the special case where the Jacobian J does not vanish on Γ. Let $\{u_j\}$ be a covering of M by open subsets with the following *admissibility property*: if $u_j \cap \Gamma \neq \varnothing$, then J does not vanish on u_j

and, for any path lying in u_j, the increment of Arg J corresponding to this path is less than $\pi/2$. For any j, fix a central point $\alpha^j \in u_j$, and for j, k so that $u_j \cap u_k \cap \Gamma \neq \varnothing$, define $\{\Delta \operatorname{Arg} J\}_{jk}$ as in the case of a C-Lagrangean manifold. Thus a 1-cocycle with coefficients in Π is defined, which will be denoted by $\Delta_\Gamma \operatorname{Arg} J$. *We now define* $\operatorname{Ind}_\Gamma$ *to be the cohomology class induced by* $\frac{1}{2\pi} \Delta_\Gamma \operatorname{Arg} J$.

Problem 3.3. $\operatorname{Ind}_\Gamma$ is independent of the choice of a covering $\{u_j\}$ and a family of central points $\{\alpha^j\}$.

Lemma 3.8. *Let* $J(\alpha) \neq 0$ *on* Γ. *Then* $\operatorname{Ind}_\Gamma = 0$ *if, and only if, there exists a continuous on* Γ *branch of* Arg J.

Proof. 1°. Sufficiency. Let Arg J be a continuous on Γ branch of the phase argument of Arg J. Choose an admissible covering $\{u_j\}$ of M so that

$$|\operatorname{Arg} J(\bar{\alpha}) - \operatorname{Arg} J(\bar{\bar{\alpha}})| < \frac{\pi}{2}$$

whenever there is a j such that both $\bar{\alpha}$ and $\bar{\bar{\alpha}}$ are in $u_j \wedge \Gamma$. Further, choose a family of central points $\{\alpha^j\}$ so that $\alpha^j \in \Gamma \cap u_j$ if $\Gamma \cap u_j \neq \varnothing$. Thus, a 0-cochain $\{\operatorname{Arg} J(\alpha^j)\}$ is defined, whose boundary is $\Delta_\Gamma \operatorname{Arg} J$.

2°. Necessity. Let $\operatorname{Ind}_\Gamma = 0$. Then there are an admissible covering $\{u_j\}$ of M and real numbers a_j (dependent on the choice of central points $\alpha^j \in u_j$) so that

$$(\Delta \operatorname{Arg} J)_{jk} = a_k - a_j$$

whenever $u_j \cap u_k \cap \Gamma \neq \varnothing$. Moreover, for some fixed j_0 such that $u_{j_0} \cap \Gamma \neq \varnothing$, we can let a_{j_0} be one of the possible values of Arg $J(\alpha^{j_0})$. As before, we choose the family of central points $\{\alpha^j\}$ so that $\alpha^j \in \Gamma$ if $u_j \cap \Gamma \neq \varnothing$. Assume that any two points α' and α'' of Γ can be joined by a chain $(u_{j1}, \ldots, u_{jm})$ of elements of the covering $\{u_j\}$ in such a way that

$$\alpha' \in u_1, \ \alpha'' \in u_m, \ u_{js} \cap u_{js+1} \cap \Gamma \neq 0, \quad s = 1, \ldots, m-1$$

(which does not affect the generality). Putting $\alpha' = \alpha^{j_0}$, $\alpha'' = \alpha^j$, we see that a_j is one of the possible values of Arg $J(\alpha^j)$ whenever $u_j \cap \Gamma \neq 0$. Define the 0-cochain with coefficients in the sheaf of germs of smooth functions on Γ by assigning to each j with $u_j \cap \Gamma \neq 0$ the continuous on u_j branch of Arg J satisfying the condition Arg $J(\alpha^j) = a_j$. The same argument as in the case of a C-Lagrangean manifold shows that this cochain is a cocycle, and the proof is complete.

We now remove the assumption that $\Gamma \subset \Omega$. Again using the family of the mappings (3.9), we obtain the family of cohomology classes Ind_Γ^τ, $\tau > 0$. It is easy to verify that Ind_Γ^τ is independent of τ.
So we define the index on Γ by

$$\mathrm{Ind}_\Gamma = \mathrm{Ind}_\Gamma^\tau, \qquad \tau > 0.$$

In particular, Ind_Γ is now defined for any Lagrangean manifold with a complex germ.

Lemma 3.8 implies the following criterion for triviality of Ind_Γ.

Proposition 3.2. *The class* Ind_Γ *of* M *is trivial if, and only if, for each* $I \subset \{1, \ldots, n\}$, *there exists a continuous on* $\Omega_I \cap \Gamma$ *branch* $\operatorname{Arg} J_I$ *of the phase argument of* J_I, $\operatorname{Arg} J_I$ *being concordant with* $\operatorname{Arg} J_K$ *on* $\Omega_I \cap \Omega_K \cap \Gamma$.

The construction of the canonical operator in the next section will depend on the existence of correlated continuous branches of the roots of Jacobians corresponding to various zones (but not of such branches of the phase arguments of Jacobians). Since $\sqrt{J} = | J | e^{\frac{i}{2} \operatorname{Arg} J}$ is defined uniquely, even if $\operatorname{Arg} J$ is defined only modulo 4π, the condition of triviality of Ind_Γ is, in general, too restrictive. Therefore, it is natural to introduce Ind_Γ modulo 2.

Given $a \in \mathbf{R}$, set $\mathbf{R}_a = \mathbf{R}/_a$, where $(x \overset{a}{\sim} y) \Leftrightarrow (x - y \equiv 0 \pmod{a})$. Obviously, $\mathbf{R}_a$ is a module over $\mathbf{Z}$. In the definition of Ind_Γ, let us change the pre-sheaf Π by using $\mathbf{R}_2$ instead of $\mathbf{R}$. Suppose that $J \neq 0$ on Γ, which is essentially the general case as we have already seen. Let ind_{jk} be the element of $\mathbf{R}_2$ generated by the number $\{\Delta \operatorname{Arg} J\}_{jk}$. Then $\{\mathrm{ind}_{jk}\}$ is a cocycle. The cohomology class induced by this cocycle will be denoted by $\mathrm{Ind}_\Gamma \pmod 2$. The criterion for triviality of $\mathrm{Ind}_\Gamma \pmod 2$ coincides word for word with that for Ind_Γ (see Proposition 3.2) if $\operatorname{Arg} J_I$ is regarded as a function with values in $\mathbf{R}_{4\pi}$, $\operatorname{Arg} J_I(\alpha_0)$ and $\operatorname{Arg} J_K(\alpha_0)$ being considered concordant if they have concordant representatives in $\mathbf{R}$.

For some applications of the canonical operator (but not for the proof of the Main Theorem) it is not sufficient to consider the class $\mathrm{Ind}_\Gamma \pmod 2$ only, and the notion of index needs to be generalized as follows. Let $\partial/\partial l = (\partial/\partial l_1, \ldots, \partial/\partial l_n)$ be an n-tuple of commuting linearly independent at every point complex vector fields on M. Then on replacing $J(\alpha)$ by $\det \frac{\partial Q}{\partial l}$ in the above construction of the index, we define a cohomology class of M which will be denoted by $\mathrm{Ind}_{\Gamma, \partial/\partial l} \pmod 2$. It is easy to see that this new class differs from $\mathrm{Ind}_\Gamma \pmod 2$, in general. The triviality criterion for $\mathrm{Ind}_{\Gamma, \partial/\partial l} \pmod 2$ is quite similar to that for $\mathrm{Ind}_\Gamma \pmod 2$.

Sec. 4. Canonical Operator

In this section we assume all the phase arguments of Jacobians to be defined modulo 4π.

1. Definition of local canonical operators. We need some notation and terminology.

Let us denote by $\mathcal{A}(\Lambda^n, r^n)$ the set of all equivalency classes of D-asymptotic series on the Lagrangean manifold Λ^n with the complex germ r^n. $\mathcal{A}(\Lambda^n, r^n)$ is obviously a left module over the ring of D-asymptotic operators. A point $\alpha \in \Lambda^n$ will be called *essential* for $\varphi \in \mathcal{A}(\Lambda^n, r^n)$ if φ has a representative non-equivalent to zero at α_0. It is clear that all the representatives of φ are non-equivalent to zero at α_0 provided that α_0 is essential for φ. The set of all points essential for φ will be called *the support* of this element, supp φ in symbol. For any $\varphi \in \mathcal{A}(\Lambda^n, r^n)$, the support of φ is a compact subset of Γ.

Let u be a subset of Λ^n. We denote by $\mathcal{A}(u)$ the set of all elements of $\mathcal{A}(\Lambda^n, r^n)$ supported in u. We shall use the notation $\mathcal{A}(\mathbf{R}^n)$ for the set of all equivalency classes of h-asymptotic series in $\mathbf{R}^n$.

Let us say that a γ-patch of a Lagrangean manifold with a complex germ is *admissible* if a dissipativity inequality with $\varepsilon = 0$:

$$c(\alpha)\,\Phi_I^\gamma(\alpha) \geqslant D(\alpha)$$

holds in this patch. Let (u, π_γ^I) be an admissible γ-patch of (Λ^n, r^n). Fix an n-tuple $\frac{\partial}{\partial l} = \left(\frac{\partial}{\partial l_1}, \ldots, \frac{\partial}{\partial l_n}\right)$ of commuting linearly independent at every point vector fields (in general, complex) on u. Set $J_I = \det \dfrac{\partial(q_I + z_I,\ p_{\bar I} + w_{\bar I})}{\partial l}$ and fix a continuous on u branch of the phase argument of J_I. Then we define *the local canonical operator*

$$\mathcal{K}_I^\gamma:\ \mathcal{A}(u) \to \mathcal{A}(\mathbf{R}^n)$$

by

$$(\mathcal{K}_I^\gamma\varphi)(x) = F^{-1}_{\xi_{\bar I}\to x_{\bar I}}\{e^{i\left(\frac{1}{h}\Phi_h^\gamma(\alpha) - \frac{1}{2}\operatorname{Arg} J_I(\alpha)\right)} \times$$
$$\times |J_I(\alpha)|^{-1/2} L_I^\gamma\varphi(\alpha)|_{\alpha = (\pi_\gamma^I)^{-1}(x_I,\, \xi_{\bar I})}\} \tag{4.1}$$

(for definition of L_I^γ see (1.27)).

Proposition 4.1. $\mathcal{K}_I^\gamma\varphi \overset{x}{\approx} 0$ *provided that* x *does not belong to the image of* Γ *under the mapping* $\alpha \to q(\alpha)$.

This proposition is an immediate consequence of Example 2.1.

Proposition 4.2. $\mathcal{K}_I^\gamma$ *is a monomorphism.*

To prove this proposition we need the following lemma.

Lemma 4.1. *There is a function* $k(\rho)$, $\rho \geqslant \frac{1}{2}$, *such that*

$$|a_j| e^{-\frac{D}{h}} \leqslant k(\rho)\, ch^{\rho - \frac{j}{2}}, \qquad j = 0, \ldots, r$$

for all $h > 0$ *and* $i = 0, 1, \ldots, [2\rho - 1] \overset{\text{def}}{=} r$ *whenever* $D > 0$, $c \geqslant 0$, $a_i \in \mathbb{C}$ *satisfy the inequality*

$$\left| \sum_{j=0}^{r} a_j h^{j/2} \right| e^{-\frac{D}{h}} \leqslant ch^{\rho} \tag{4.2}$$

for all $h > 0$.

Proof. Putting in (4.2) $h = Dy^2$, where y is a positive variable, we get

$$\left| \sum_{j=0}^{r} a_j y^j D^{\frac{j}{2}} \right| \leqslant cy^{2\rho} e^{1/y^2} D^{\rho}.$$

It follows that for any $(r+1)$-tuple $\vec{y} = (y_0, \ldots, y_r)$ of positive numbers, and $(r+1)$-tuple $b = (b_0, \ldots, b_r)$ of real numbers, the following inequality holds:

$$\left| \sum_{j=0}^{r} \sum_{l=0}^{r} b_j a_j y_j^l D^{l/2} \right| \leqslant cf(\vec{y}, b) D^{\rho}. \tag{4.3}$$

Fix $\vec{y}$ such that $y_j \neq y_l$ for $j \neq l$ and consider the non-homogeneous linear system of equations

$$\sum_{j=0}^{r} y_j^l b_j = \delta_{ls}, \qquad l = 0, \ldots, r, \tag{4.4}$$

where s is a given integer and b is to be found. Since the determinant of this system is exactly the Vandermonde determinant

$$\Delta = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ y_0 & y_1 & \ldots & y_r \\ y_0^r & y_1^r & \ldots & y_r^r \end{vmatrix}$$

corresponding to the $(r+1)$-tuple $\vec{y}$, it is nonzero, so (4.4) has a solution $b = b(s)$. Substituting this solution into (4.3), we obtain

$$|a_s| \leqslant ck'(\rho) D^{\rho - s/2},$$

where

$$k'(\rho) = \max_{0 \leqslant s \leqslant r} f(\vec{y}, b^{(s)}).$$

To complete the proof use the result of Problem 1.1 of Chapter IV.

Proof of Proposition 4.2. It is clear that the operator

$$e^{-\frac{i}{2}\operatorname{Arg} J_I(\alpha)}\,|J_I(\alpha)|^{-1/2}L_I^{\gamma}$$

is a monomorphism of $\mathcal{A}(u)$ to itself and that $F^{-1}_{\xi_{\bar I}\to x_{\bar I}}$ is a monomorphism of $\mathcal{A}(\mathbf{R}^n)$ to itself. So it is to be shown that the multiplication operator by $e^{\frac{i}{h}\Phi_I^{\gamma}(\alpha)}$ cannot send a D-asymptotic series non-equivalent to zero and supported in u into an h-asymptotic series equivalent to zero.

Let $\psi = \sum_{j\in J}\psi_j$ be a D-asymptotic series supported in u.

1°. First we show that

$$\left(e^{\frac{i}{h}\Phi_I^{\gamma}}\psi\sim 0\right)\Rightarrow(\psi\sim 0).$$

In fact, if $e^{\frac{i}{h}\Phi_I^{\gamma}}\psi\sim 0$, then, for any N, there is a finite set $J_0\subset J$ such that

$$\left|\sum_{j\subset K} e^{\frac{i}{h}\Phi_I^{\gamma}(\alpha)}\psi_j(\alpha,\ h)\right|\leqslant\text{const}\cdot h^N$$

for any finite $K\supset J_0$. Since

$$\sum_{j\in K}\psi_j(\alpha,\ h)=h^{-s/2}\sum_{l=0}^{l_0}h^{l/2}a_l(\alpha),$$

it follows from Lemma 4.1 that

$$a_l=\hat{O}_D\left(h^{N+\frac{s-l}{2}}\right)$$

for $l<2N+s$. Thus,

$$\sum_{j\in K}\psi_j=\hat{\mathcal{O}}_D(h^N),$$

which means that $\psi\sim 0$.

2°. Now let $e^{\frac{i}{h}\Phi_I^{\gamma}}\psi\approx 0$, i.e.,

$$\left(\frac{\partial}{\partial\alpha}\right)^k e^{\frac{i}{h}\Phi_I^{\gamma}(\alpha)}\psi(\alpha)\sim 0$$

for any multi-index k. We shall prove by induction that $\left(\frac{\partial}{\partial\alpha}\right)^k\psi(\alpha)\sim 0$ for any k. For $k=0$, this follows from 1°. Suppose that $\left(\frac{\partial}{\partial\alpha}\right)^l\psi\sim 0$ for $|l|\leqslant N$ and let $|k|=N+1$. Then

$$\left(\frac{\partial}{\partial\alpha}\right)^k e^{\frac{i}{h}\Phi_I^{\gamma}}\psi=e^{\frac{i}{h}\Phi_I^{\gamma}}\left(\frac{\partial}{\partial\alpha}\right)^k\psi+e^{\frac{i}{h}\Phi_I^{\gamma}}L^{-N-1}\hat{L}\psi,$$

where $\hat{L}$ is an Nth order differential operator with coefficients smooth in α and polynomial in h. By the induction hypothesis, $\hat{L}\psi \sim 0$, hence

$$e^{\frac{i}{h}\Phi_I^\gamma} h^{-N-1}\hat{L}\psi \sim 0,$$

$$e^{\frac{i}{h}\Phi_I^\gamma}\left(\frac{\partial}{\partial\alpha}\right)^k \psi \sim 0,$$

which implies by 1° that $\left(\frac{\partial}{\partial\alpha}\right)^k \psi \sim 0$. The proposition is proved.

2. Transition operators. Consider two admissible γ-patches (u, π_γ^I) and $(u', \pi_{\gamma'}^K)$ of a Lagrangean manifold with a complex germ, together with fixed continuous branches of the phase arguments of J_I and J_K defined on u and u', respectively, these branches being concordant on $u \cap u'$. Let an n-tuple $\frac{\partial}{\partial l}$ of commuting linearly independent at every point complex vector fields on $u \cup u'$ be also fixed. Then the local canonical operators $\mathcal{K}_I^\gamma : \mathcal{A}(u) \to \mathcal{A}(\mathbf{R}^n)$ and $\mathcal{K}_K^{\gamma'} : \mathcal{A}(u') \to \mathcal{A}(\mathbf{R}^n)$ are uniquely determined. Define the local operator $V_{IK}^{\gamma\gamma'}$ acting on $\mathcal{A}(u \cap u')$ to itself (the transition operator from (u, π_γ^I) to $(u', \pi_{\gamma'}^K)$) by the formula

$$\mathcal{K}_I^\gamma \varphi = \mathcal{K}_K^{\gamma'} V_{IK}^{\gamma\gamma'} \varphi. \tag{4.5}$$

The existence of $V_{IK}^{\gamma\gamma'}$ is a consequence of the following result which generalizes Proposition 1.2:

Proposition 4.3. *Let $\alpha_0 \in \Gamma \cap u \cap u'$. Then for any $\varphi(\alpha) \in C_0^\infty$ supported near α_0, we have*

$$F_{x_{I_1} \to \xi_{I_1}} F^{-1}_{\xi_{K_1} \to x_{K_1}} \Big[e^{\frac{i}{h}\Phi_I^\gamma(\alpha)} |J_I(\alpha)|^{-\frac{1}{2}} \times$$
$$\times e^{-\frac{i}{2}\operatorname{Arg} J_I(\alpha)} L_I^\gamma \varphi(\alpha)\Big]_{\alpha = (\pi_\gamma^I)^{-1}(x_I, \xi_{\bar{I}})} \approx$$
$$\approx \Big[e^{\frac{i}{h}\Phi_K^{\gamma'}(\alpha)} |J_K(\alpha)| e^{-\frac{i}{2}\operatorname{Arg} J_K(\alpha)} \times$$
$$\times L_K^{\gamma'} V_{IK}^{\gamma\gamma'} \varphi(\alpha)\Big]_{\alpha = (\pi_{\gamma'}^K)^{-1}(x_K, \xi_{\bar{K}})},$$

where $I_1 = I \setminus K$, $K_1 = K \setminus I$, $\operatorname{Arg} J_K$ *is concordant with* $\operatorname{Arg} J_I$ *and V is a D-asymptotic differential quasi-identity operator.*

This proposition can be proved in the same way as Proposition 2.2 and Lemma 3.5, or by using the canonical transformation corresponding to the Hamiltonian function H_I (see Example 2.1 of Chapter IV).

The definition of $V_{IK}^{\gamma'\gamma}$ is correct in the sense that there is only one operator satisfying (4.5). This is due to the fact that local canonical operators are monomorphisms. Moreover, it is easily seen that $V_{IK}^{\gamma\gamma'}$ is independent of the choice of vector fields $\partial/\partial l$ and concordant branches of Arg J_I and Arg J_K.

It follows from the definition of transition operators that

$$V_{IK}^{\gamma\gamma'}V_{KJ}^{\gamma'\gamma''}\varphi = V_{IJ}^{\gamma'\gamma''}\varphi$$

for any three admissible γ-patches (u, π_γ^I), $(u', \pi_{\gamma'}^K)$, $(u'', \pi_{\gamma''}^J)$ and any $\varphi \in \mathcal{A}\,(u \cap u' \cap u'')$. In particular, $V_{KI}^{\gamma\gamma'}$ is invertible, $(V_{IK}^{\gamma\gamma'})^{-1}$ being equal to $V_{KI}^{\gamma'\gamma}$.

3. Definition of the canonical operator on (Λ^n, r^n). Consider a Lagrangean manifold Λ^n with a complex germ r^n. Let us fix the following objects:

(1) An admissible (i.e., consisting of admissible patches) γ-atlas $\{\omega_j, \pi_{\gamma(j)}^{I(j)}\}$ of a neighborhood of Γ such that the covering $\{\omega_j\}$ is locally finite. This γ-atlas will be called *weighting*.

(2) A "weighting" C^∞-smooth partition of unity $\{\rho_j\}$:

$$\sum_j \rho_j = 1, \quad \operatorname{supp}\, \rho_j \subset \omega_j.$$

(3) An n-tuple $\frac{\partial}{\partial l} = \left(\frac{\partial}{\partial l_1}, \ldots, \frac{\partial}{\partial l_n}\right)$ of commuting complex vector fields on Λ^n, linearly independent at every point. Note that such vector fields always exist: for instance, one can put

$$\frac{\partial}{\partial l} = \frac{\partial}{\partial (p+iq)}.$$

(4) A set of concordant continuous branches of the phase arguments of $J_I \mid_{\Omega_I \cap \Gamma}$ (I runs over all subsets of $\{1, \ldots, n\}$). Thus, the cohomology class $\operatorname{Ind}_{\Gamma, \partial/\partial l}$ (mod 2) is assumed to be trivial. We define the canonical operator

$$\mathcal{K} : \mathcal{A}\,(\Lambda^n, r^n) \to \mathcal{A}\,(\mathbf{R}^n)$$

by the formula

$$\mathcal{K}\varphi = \sum_j \mathcal{K}_{I(j)}^{\gamma(j)} \rho_j \varphi. \tag{4.6}$$

Note that the number of non-zero summands in the sum in the right-hand member of (4.6) is obviously finite.

Proposition 4.4. *Let* $\varphi \in \mathcal{A}\,(\Lambda^n, r^n)$, *let* $\{u_k, \pi_{\gamma(k)}^{I(k)}\}$ *be an admissible γ-atlas of a neighborhood of* supp φ, *and let* $\{e_k\}$ *be a* C^∞ *smooth par-*

tition of unity subordinate to the covering $\{u_k\}$. *Then*

$$\mathcal{K}\varphi = \sum_{j,k} \mathcal{K}_{I_k}^{\gamma_k} V_{I(j) I_k}^{\gamma(j)\gamma_k} \rho_j e_k \varphi.$$

Proof. By the definition of transition operators,

$$\mathcal{K}_{I_k}^{\gamma_k} V_{I(j) I_k}^{\gamma(j)\gamma_k} \rho_j e_k \varphi = \sum_j \mathcal{K}_{I(j)}^{\gamma(j)} \rho_j \sum_k e_k \varphi = \sum_j \mathcal{K}_{I(j)}^{j(j)} \rho_j \varphi = \mathcal{K}\varphi.$$

Note 4.1. So far we have assumed that there exists an s-action on Λ^n (or at least in the vicinity of Γ). This assumption can be expressed in terms of triviality of a certain cohomology class of Λ^n. Namely, let $\{u_j\}$ be a locally finite covering of Λ^n by open subsets satisfying the condition: $u_j \cap u_k \cap \Gamma \neq \varnothing$, then $\oint_l p\,dq$ vanishes for any cycle l lying in $u_j \cap u_k$. Choose a central point $\alpha^j \in u_j$ for any j, and set

$$\delta_{jk} = \frac{1}{\pi h} \oint_{l[\alpha^j, \alpha^k]} p\,dq \pmod 2,$$

where $l[\alpha^j, \alpha^k]$ is a path lying in $u_j \cup u_k$ and joining α^j to α^k. The family $\{\delta_{jk}\}$ determines a cocycle (depending on h) with coefficients in the same pre-sheaf as for the class $\mathrm{Ind}_\Gamma \pmod 2$. This cocycle induces a cohomology class of Λ^n, which will be de noted by $\frac{1}{\pi h}\oint_\Gamma p\,dq$. If h is a continuous parameter, then the existence of s-action is equivalent to triviality of this cohomology class for all $h > 0$.

In the case, where s-action does not exist in the vicinity of Γ, the condition

$$\frac{1}{\pi h}\oint_\Gamma p\,dq = 0$$

determines a certain set M of (permissible) values of h. If M has 0 as a limit point, then we can define the canonical operator by allowing h to take on permissible values only.

Note 4.2. The case where (Λ^n, r^n) depends on parameters can be considered in the same way as in Chapter III. If both the classes $\frac{1}{\pi h}\oint_\Gamma p\,dq$ an d $\mathrm{Ind}_{\Gamma, \partial/\partial l} \pmod 2$ are non-trivial, then, to define the canonical operator and to satisfy the quantization condition, it is required that

$$\frac{1}{\pi h}\oint_\Gamma p\,dq = \mathrm{Ind}_{\Gamma, \partial/\partial l} \pmod 2, \tag{4.8}$$

which generalizes (4.7) and determines the permissible values of h and the other parameters. For example, if (Λ^n, r^n) depends on a parameter β, then the condition (4.8) may be satisfied by a substitution of the form $\beta = \beta(\varepsilon, h)$, where ε is a new parameter. To substitute operators for the parameters ε and h, one can use the regularized canonical operator in the same way as in Chapter III.

4. Commutation of a canonical operator with a Hamiltonian.

Proposition 4.5. *Any canonical operator is a monomorphism.*

Proof. Let $\chi(x)$ be an infinitely smooth function which equals 1 for $|x| < 1$ and vanishes for $|x| > 2$. Set

$$\chi_\varepsilon(x) = \chi(\varepsilon^{-1}x),$$
$$e_\varepsilon(\alpha) = \chi_\varepsilon(q(\alpha)),$$
$$f_\varepsilon(\alpha) = \chi_\varepsilon(p(\alpha)).$$

Let φ be a non-zero element of $\mathcal{A}(\Lambda^n, r^n)$ and let $\varphi_0 \in \operatorname{supp} \varphi$; we may assume without loss of generality that $p(\alpha_0) = q(\alpha_0) = 0$. Suppose $\mathcal{K}\varphi = 0$. Then, for any $\varepsilon > 0$, $\chi_\varepsilon \mathcal{K}\varphi = 0$ and $\chi_\varepsilon \mathcal{K} \times$ $\times (1 - e_{2\varepsilon})\varphi = 0$ (see Proposition 4.1).

We have

$$\chi_\varepsilon \mathcal{K} e_{2\varepsilon}\varphi = \sum_j \chi_\varepsilon \mathcal{K}^{\gamma(j)}_{I(j)} \rho_j e_{2\varepsilon}\varphi = \sum_j \mathcal{K}^{\gamma(j)}_{I(j)} \hat{\chi}^{(j)}_\varepsilon \rho_j e_{2\varepsilon}\varphi = 0,$$

where $\hat{\chi}^{(j)}_\varepsilon$ is a D-asymptotic differential operator preserving the principal monomial at α_0 (see Problem 2.1). This equation implies that

$$\sum_j F_{x\to\xi} \mathcal{K}^{\gamma(j)}_{I(j)} \hat{\chi}^{(j)}_\varepsilon \rho_j e_{2\varepsilon}\varphi = 0.$$

Set

$$\tilde{\mathcal{K}}^\gamma_I = F_{x\to\xi}\mathcal{K}^\gamma_I.$$

Since the operator $\tilde{\mathcal{K}}^\gamma_I$ can be explicitly expressed by the formula

$$(\tilde{\mathcal{K}}^\gamma_I \psi)(\xi) = F_{x_I \to \xi_I} \Big\{ e^{\frac{i}{h}\Phi^\gamma_I(\alpha)} e^{-\frac{i}{2}\operatorname{Arg} J_I(\alpha)} \times$$
$$\times |J_I(\alpha)|^{-\frac{1}{2}} L^\gamma_I \varphi(\alpha) \Big|_{\alpha = (\pi^I_\gamma)^{-1}(x_I,\, \xi_{\bar{I}})} \Big\}.$$

Proposition 4.1 remains valid if $\mathcal{K}^\gamma_I$ is replaced by $\tilde{\mathcal{K}}^\gamma_I$ and the mapping $\alpha \to q(\alpha)$ is simultaneously replaced by the mapping $\alpha \to p(\alpha)$. Therefore,

$$\chi_\varepsilon \sum_j \tilde{\mathcal{K}}^{\gamma(j)}_{I(j)} f_{2\varepsilon} \hat{\chi}^{(j)}_\varepsilon \rho_j e_{2\varepsilon}\varphi = 0.$$

Commuting χ_ε with $\widetilde{\mathcal{K}}_{I(j)}^{\gamma(j)}$ and making the inverse Fourier transformation, we obtain

$$\sum_j \widetilde{\mathcal{K}}_{I(j)}^{\gamma(j)} \overline{\chi}^{(j)} f_{2\varepsilon} \hat{\chi}_\varepsilon^{(j)} e_{2\varepsilon} \rho_j \varphi = 0, \tag{4.9}$$

where $\overline{\chi}_\varepsilon^{(j)}$ is a D-asymptotic differential operator preserving the principal monomial at α_0.

Let ε be so small that the whole of supp $(f_{2\varepsilon} e_{2\varepsilon})$ is in an admissible γ-patch (u, π_γ^I). Then (4.9) can be rewritten in the form

$$\mathcal{K}_{I_0}^{\gamma_0} \sum_j V_{I_0 I(j)}^{\gamma_0 \gamma(j)} \overline{\chi}_\varepsilon^{(j)} f_{2\varepsilon} \hat{\chi}_\varepsilon^{(j)} e_{2\varepsilon} \rho_j \varphi = 0.$$

The element

$$\psi = \sum_j V_{I_0 I(j)}^{\gamma_0 \gamma(j)} \overline{\chi}_\varepsilon^{(j)} f_{2\varepsilon} \hat{\chi}_\varepsilon^{(j)} e_{2\varepsilon} \rho_j \varphi$$

is non-zero since the classes φ and ψ have representatives with the same principal monomial. But this contradicts the fact that $\mathcal{K}_I^\gamma$ is a monomorphism, and the proof is complete.

Let $\mathcal{H}(p, x) \in S^\infty(\mathbf{R}^{2n})$. Since the operator $\mathcal{H}\left(\overset{1}{p}, \overset{2}{x}\right)$ preserves the property of h-asymptotic series to be equivalent to zero, it induces an operator acting on $\mathcal{A}(\mathbf{R}^n)$ to itself which will be also denoted by $\mathcal{H}\left(\overset{1}{p}, \overset{2}{x}\right)$.

Theorem 4.1. *Given an $\mathcal{H} \in S^\infty(\mathbf{R}^{2n})$ and a canonical operator $\mathcal{K}$ on (Λ^n, r^n), there is such an operator $P_{\mathcal{H}}$ acting on $\mathcal{A}(\Lambda^n, r^n)$ to itself, that*

$$\mathcal{H}\left(\overset{1}{p}, \overset{2}{x}\right) \mathcal{K} = \mathcal{K} P_{\mathcal{H}}.$$

The following lemma is an essential ingredient in our proof of this theorem:

Lemma 4.2. *Every D-asymptotic quasi-identity operator is an automorphism of $\mathcal{A}(\Lambda^n, r^n)$.*

Proof. Let V be a D-asymptotic quasi-identity operator regarded as an endomorphism of $\mathcal{A}(\Lambda^n, r^n)$. It is clear that V is a monomorphism. It remains to be shown that V is any.

Let $\varphi \in \mathcal{A}(\Lambda^n, r^n)$. We have to prove that the equation $V\psi = \varphi$ (where φ is to be found) has a solution. Let $\hat{\varphi}$ be a D-asymptotic series representing φ. We shall seek a D-asymptotic series $\hat{\psi}$ representing ψ in the form $\hat{\psi} = \sum_{j=0}^{\infty} \psi_j$. Setting $\psi_0 = m[\hat{\varphi}]$, we define

ψ_j, $j = 1, 2, \ldots$ by the following recurrence formula:

$$\psi_j = m\left[\hat{\varphi} - \sum_{k=0}^{j-1} V\psi_k\right].$$

Let us check that the series $\sum_{j=0}^{\infty} \psi_j$ is D-asymptotic. To do this we observe that since

$$\psi_{j+1} = m\left[\hat{\varphi} - \sum_{k=0}^{j-1} V\psi_k - Vm\left[\hat{\varphi} - \sum_{k=0}^{j-1} V\psi_k\right]\right]$$

and V is a quasi-identity, it follows that for any j, at least one of the following three statements is valid:

(1) ord $\psi_{j+1} >$ ord ψ_j;

(2) the type of ψ_{j+1} is greater than that of ψ_j;

(3) $\psi_j = 0$ (in this case $\psi_k = 0$ for $k \geqslant j$).

Since the type of a monomial does not exceed its order it follows that for any natural N, there is j_0 such that ord $\psi_j \geqslant \frac{N}{2}$ if $j \geqslant j_0$, which means that the series under consideration is D-asymptotic.

We now show that $V\tilde{\psi} \approx \hat{\varphi}$. Let $j_0\,(N)$ be an integer-valued function of such a natural variable, that ord $\psi_j \geqslant N/2$ for $j \geqslant j_0\,(N)$. Then

$$\operatorname{ord}\left(V \sum_{j=j_0(N)}^{\infty} \psi_j\right) \geqslant \frac{N}{2}$$

and

$$\operatorname{ord}\left(V \sum_{j=0}^{j_0(N)-1} \psi_j - \hat{\varphi}\right) = \operatorname{ord}\ (-\psi_{j_0}(N)) \geqslant \frac{N}{2},$$

so ord $(V\hat{\psi} - \hat{\varphi}) \geqslant N/2$. Hence ord $(V\hat{\psi} - \hat{\varphi}) = \infty$, and the proof is complete.

Note. It is easy to see that if V is a quasi-identity then the same is true of V^{-1}.

Proof of Theorem 4.1. We begin by commuting $\mathscr{H}\left(\overset{1}{p}, \overset{2}{x}\right)$ with a local canonical operator. Let (u, π_γ^I) be an admissible γ-patch and let $\varphi \in \mathscr{A}\,(u)$. By the commutation formula for a Hamiltonian and the composition of the multiplication operator by a complex exponential with the Fourier transformation, we have

$$\mathscr{H}\left(\overset{1}{p}, \overset{2}{x}\right) \mathscr{K}_I^\gamma \varphi =$$

$$= F^{-1}_{\xi_{\bar{I}} \to x_{\bar{I}}}\left[e^{\frac{i}{h}\Phi_I^\gamma(\alpha)}\, \hat{\mathscr{H}}\, \frac{e^{-\frac{i}{2}\operatorname{Arg} J_I(\alpha)}}{\sqrt{|J_I(\alpha)|}}\, L_I^\gamma \varphi\,(\alpha)\right]_{\alpha = (\pi_\gamma^I)^{-1}(x_I,\, \xi_{\bar{I}})},$$

where $\hat{\mathscr{H}}$ is the D-asymptotic differential operator defined by (2.24) ("lifted" to Λ^n by the diffeomorphism π^I_γ). It follows that

$$\mathscr{H}\left(\overset{1}{p}, \overset{2}{x}\right) \mathscr{K}^\gamma_I = \mathscr{K}^\gamma_I P^{I,\gamma}_{\mathscr{H}}, \tag{4.10}$$

where

$$P^{I,\gamma}_{\mathscr{H}} = R^\gamma_I \sqrt{J_I}\, \hat{\mathscr{H}} \frac{1}{\sqrt{J_I}}, \quad \sqrt{J_I} \overset{\text{def}}{=\!=} |J_I|^{\frac{1}{2}} e^{\frac{i}{2} \operatorname{Arg} J_I},$$

and R^γ_I is the inverse of L^γ_I in the group of automorphisms of $\mathscr{A}(u)$.

Equation (4.10) implies that for any $\varphi \in \mathscr{A}(\Lambda^n, r^n)$, there is such a family $\{\psi_j\}$ of elements of $\mathscr{A}(\Lambda^n, r^n)$ that

$$\mathscr{H}\left(\overset{1}{p}, \overset{2}{x}\right) \mathscr{K}\varphi = \sum_j \mathscr{K}^{\gamma(j)}_{I(j)} \psi_j, \quad \psi_j \in \mathscr{A}(\omega_j).$$

Let us show that there exists an element $\psi \in \mathscr{A}(\Lambda^n, r^n)$ satisfying the condition

$$\sum_j \mathscr{K}^{\gamma(j)}_{I(j)} \psi_j = \mathscr{K}\psi. \tag{4.11}$$

To do this, it suffices to find, for any j, such an element $\overline{\psi}_j \in \mathscr{A}(\omega_j)$ that

$$\mathscr{K}^{\gamma(j)}_{I(j)} \psi_j = \mathscr{K}\overline{\psi}_j. \tag{4.12}$$

Rewrite (4.12) in the form

$$\mathscr{K}^{\gamma(j)}_{I(j)} \psi_j = \sum_i \mathscr{K}^{\gamma(j)}_{I(j)} V^{\gamma(i)\gamma(j)}_{I(i)I(j)} \rho_j \overline{\psi}_j. \tag{4.13}$$

The last equation is equivalent to the following one:

$$\mathscr{A}_j \overline{\psi}_j = \psi_j \tag{4.14}$$

with

$$\mathscr{A}_j = \sum_i V^{\gamma(i)\gamma(j)}_{I(i)I(j)} \rho_i.$$

But note that $\mathscr{A}_j$ is a quasi-identity operator, so the equation (4.14) has a solution by Lemma 4.2.

Thus we have shown that for any $\varphi \in \mathscr{A}(\Lambda^n, r^n)$, there is a $\psi \in \mathscr{A}(\Lambda^n, r^n)$ satisfying the equation

$$\mathscr{H}\left(\overset{1}{p}, \overset{2}{x}\right) \mathscr{K}\varphi = \mathscr{K}\psi,$$

such a ψ being unique by virtue of the fact that $\mathscr{K}$ is a monomorphism. It is obvious that the operator $P_{\mathscr{H}}$ taking φ to ψ is linear.

5. Canonical operator on the family $\{\Lambda^n_t, r^n_t\}$. Let $\{\Lambda^n_t, r^n_t\}$, $0 \leqslant t \leqslant T$, be the family of Lagrangean manifolds with complex

germs obtained from (Λ^n, r^n) by the family of complex canonical transformations associated with a Hamilton function $\mathcal{H}^\circ$, and let M^{n+1} be the $(n+1)$-dimensional manifold (with the complex germ $w(\alpha, t)$, $z(\alpha, t)$, the dissipation $D(\alpha, t)$ and the potential $E(\alpha, t)$) associated with $\{\Lambda_t^n, r_t^n\}$. The canonical operator $\mathcal{K}_{M^{n+1}}$ on M^{n+1} is defined almost word by word as that on Λ^n. Here we shall mention only a few additions and modifications required.

(1) One must replace the space $\mathcal{A}(\Lambda^n, r^n)$ and $\mathcal{A}(\mathbf{R}^n)$ by $\mathcal{A}(M^{n+1})$ and $\mathcal{A}(\mathbf{R}^n \times [0, T])$, respectively; the definition of these new spaces is obvious. Note that there is a natural embedding of $\mathcal{A}(\Lambda^n, r^n)$ into $\mathcal{A}(M^{n+1})$.

(2) The complex vector fields $\partial/\partial l$ must be chosen so that they would be expressed locally via α only.

(3) One must replace the diffeomorphisms π_γ^I in the definition of local canonical operators by Π_γ^I (see Sec. 4 of Chapter IV).

It is not difficult to verify that for the canonical operators on M^{n+1}, the results similar to that of Propositions 4.1, 4.2, 4.4, 4.5 and Theorem 4.1 hold.

It is of special interest to consider the commutation of the canonical operator $\mathcal{K}_{M^{n+1}}$ with a pseudodifferential operator of the form

$$\hat{H} = -ih\frac{\partial}{\partial t} + \mathcal{H}\left(\overset{1}{p}, \overset{2}{x}, t, h\right), \quad \mathcal{H} \in \mathcal{S}^\infty \tag{4.15}$$

in the case with

$$\mathcal{H}(p, q, t, 0) = \mathcal{H}^\circ(p, q, t). \tag{4.16}$$

We shall agree to say that M^{n+1} is subordinated to the operator $\hat{H}$ defined by (4.15) if the condition (4.16) is satisfied.

Theorem 4.2. *Let M^{n+1} be subordinated to the operator $\hat{H}$ defined by* (4.15). *Then*

$$\hat{H}\mathcal{K}_{M^{n+1}} = -ih\mathcal{K}_{M^{n+1}}\left(\frac{d}{dt} + G(p, q, t)\right)\varkappa,$$

where $\varkappa$ is a quasi-identity operator and

$$G(p, q, t) = -\frac{1}{2}\operatorname{tr}\mathcal{H}^0_{pq}(p, q, t) + i\mathcal{H}_h(p, q, t, 0).$$

Proof. We have

$$\hat{H}\mathcal{K}_{M^{n+1}} = \hat{H}\sum_j \mathcal{K}_{I(j)}^{\gamma(j)}\rho_j = \sum_j \mathcal{K}_{I(j)}^{\gamma(j)}\overline{H}_j\rho_j,$$

where $\overline{H}_j$ is the operator that arises when commuting $\hat{H}$ with the local canonical operator $\mathcal{K}_{I(j)}^{\gamma(j)}$. By virtue of (2.24) and the results

of Sec. 6 of Chapter IV, $\bar{H}_j$ has the form

$$\bar{H}_j = -ih\left(\frac{d}{dt} + G + A_j\right),$$

where A_j is a D-asymptotic differential operator belonging to the class $\mathcal{F}$. Let P be the operator taking any $\varphi \in \mathcal{A}(M^{n+1})$ to the solution ψ of the equation

$$\hat{H}\mathcal{K}_{M^{n+1}}\varphi = \mathcal{K}_{M^{n+1}}\psi \tag{4.19}$$

(recall that this equation has a unique solution). Set

$$\psi = -ih\frac{d\varphi}{dt} - ih\varepsilon\varphi - ihG\varphi, \tag{4.20}$$

where ε is an operator to be found. Substituting $\psi = P\varphi$ into (4.19) and calculating the left-hand member of (4.19) by (4.17) and (4.18) we obtain the following equation to be satisfied by ε:

$$\mathcal{K}_{M^{n+1}}\varepsilon\varphi = \sum_j \mathcal{K}^{\gamma(j)}_{I(j)}[\dot{\rho}_j\varphi + A_j\rho_j\varphi]. \tag{4.21}$$

Let $\{V_j\}$ be such a family of quasi-identity operators that

$$\mathcal{K}^{\gamma(j)}_{I(j)}(\dot{\rho}_j\varphi + A_j\rho_j\varphi) = \mathcal{K}_{\Lambda^{n+1}}V_j(\dot{\rho}_j\varphi + A_j\rho_j\varphi).$$

Then (4.21) can be rewritten in the form

$$\varepsilon\varphi = \sum_j V_j(\dot{\rho}_j\varphi + A_j\rho_j\varphi).$$

Since $\sum_j \dot{\rho}_j = 0$, one of the following two statements concerning the element $\chi = \sum_j V_j\rho_j\varphi$ is true: (1) the order of χ is greater than that of φ; (2) the order of χ is equal to that of φ but typ $\chi >$ typ φ. The same is true if χ is replaced by $I = \sum_j A_j\rho_j\varphi$, where I is the operator on $\mathcal{A}(M^{n+1})$ into itself induced by the operator of integration with respect to t:

$$f(\alpha, t) \to \int_0^t f(\alpha, \tau)\,d\tau.$$

It follows that

$$\varkappa = 1 + e^{-\int_0^t G\,dt} I e^{\int_0^t G\,dt}\varepsilon$$

is a quasi-identity. But this $\varkappa$ satisfies the condition

$$\frac{d}{dt} + G + \varepsilon = \left(\frac{d}{dt} + G\right)\varkappa,$$

so the theorem is proved.

The equation $\left(\frac{d}{dt}+G\right)\varkappa\varphi=0$, where $\varkappa$ and G are the same as in Theorem 4.2, will be called a *transfer equation*.

For any $\varphi\in\mathcal{A}(M^{n+1})$, we shall write $\varphi\,|_{t=0}$ to denote the element of $\mathcal{A}(\Lambda^n, r^n)$ induced by $\sum\limits_{j\in J}\varphi_j\,|_{t=0}$, where $\sum\limits_{j\in J}\varphi_j$ is a D-asymptotic series representing φ. The definition of $\varphi\,|_{t=0}$ is obviously correct, .e., it does not depend on the choice of a representative of φ. Simiiarly we define $\psi\,|_{t=0}\in\mathcal{A}(\mathbf{R}^n)$ for any $\psi\in\mathcal{A}(\mathbf{R}^n\times[0, T])$. lurther, we set

$$\mathcal{K}_{\Lambda^n}(\varphi\,|_{t=0})\overset{\text{def}}{=\!=}(\mathcal{K}_{M^{n+1}}\varphi)\,|_{t=0}.$$

It is readily seen that $\mathcal{K}_{\Lambda^n}$ is a canonical operator on (Λ^n, r^n). We shall say that $\mathcal{K}_{M^{n+1}}$ is *correlated with* $\mathcal{K}_{\Lambda^n}$. Given a canonical operator $\mathcal{K}$ on (Λ^n, r^n), there is always a canonical operator $\mathcal{K}_{M^{n+1}}$ correlated with $\mathcal{K}$ (to construct $\mathcal{K}_{M^{n+1}}$, it suffices to co-ordinate, in the natural sense, the branches of the phase arguments of Jacobians, the weighting atlases, the weighting partitions of unity and the vector fields $\partial/\partial l$).

Consider the Cauchy problem for the transfer equation:

$$\left(\frac{d}{dt}+G\right)\varkappa\varphi=0,\quad \varphi\,|_{t=0}=\varphi_0. \tag{4.23}$$

The (unique) solution of this problem is

$$\varphi=\varkappa^{-1}V^{-1}\varphi_0,$$

where V is the quasi-identity operator in $\mathcal{A}(\Lambda^n, r^n)$ defined by

$$V\psi=(\varkappa^{-1}\psi)\,|_{t=0}.$$

As a direct consequence of Theorem 4.2, we have the following result:

Theorem 4.3. *The Cauchy problem*

$$-ih\frac{\partial\psi}{\partial t}+\mathcal{H}\left(\overset{1}{p},\overset{2}{x}, t, h\right)\psi=0,\quad \psi\,|_{t=0}=\mathcal{K}_{\Lambda^n}\varphi_0$$

has the asymptotic solution

$$\psi=\mathcal{K}_{M^{n+1}}\varphi,$$

where φ *is the solution of* (4.23).

In conclusion, we note that Theorem 4.3 can be generalized in an obvious way to the case where the Hamiltonian function $\mathcal{H}$ may depend on an m-dimensional parameter ω.

Sec. 5. Proof of the Main Theorem

Notation. If

$$A = (A_1, \ldots, A_n), \quad B = (B_1, \ldots, B_m),$$

the components of B commuting, then we write

$$f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right) = f\left(\overset{1}{A}, \overset{2}{B}\right).$$

If $a = (a_1, \ldots, a_n)$, $\rho = (\rho_1, \ldots, \rho_n)$, then we write

$$\left(a_1 \lambda^{\rho_1}, \ldots, a_n\lambda^{\rho_n}\right) = a\lambda^{\rho};$$

here λ is "one-dimensional".

Here we shall prove the Main Theorem (see Introduction) using the lemmas of Appendix to this section. Let us first deal with the most elementary version of the Main Theorem which is stated on page 132.

Let $f(x, \alpha)$ be the symbol of the operator to be quasi-inverted and let $L_j(\eta, x, \beta)$ be the symbol of the differential operator $\hat{L}_j = L_j\left(\overset{2}{i\frac{\partial}{\partial x}}, \overset{1}{x}, \overset{1}{-i\frac{\partial}{\partial \alpha}}\right)$ representing A_j. According to the conditions of the theorem, the composite function $f(L(\eta, x, \beta), \alpha)$ is asymptotically ρ-quasi-homogeneous in x and β, where $\rho = (\rho_1, \ldots, \rho_{n+m})$. One can check that $L_j(0, x, 0) = x_j$, so that $f(x, \alpha)$ is asymptotically ρ'-quasi-homogeneous in x, where $\rho' = (\rho_1, \ldots, \rho_n)$. Lemma 3 of Appendix implies that the problem of quasi-invertion reduces to the following: one has to find a function $g_N(x, \alpha)$ satisfying (9.6) of Introduction with 1 replaced by a ρ'-quasi-homogeneous in x function $F(x, \alpha)$, supported in the vicinity of the zero-set of f_0. Let S_ρ^{n-1} be the unit "quasi-sphere" in $\mathbf{R}^n$:

$$S_\rho^{n-1} = \{x \in \mathbf{R}^n \mid \Lambda_0(x) = 1\},$$

where $\Lambda_0(x) = \left(\sum\limits_j x^{2/\rho_j}\right)^{1/2}$. Using a partition of unity we can additionally suppose the support of $F(x, \alpha)$ restricted to $S_\rho^{n-1} \times M^m$ to be so small, that there exists such a $T > 0$ that $0 < T < \tau'(q^0, \omega, 0, 0)$ and

$$\tilde{\tilde{H}}(p(q^0, \omega, p^0, \eta, T), \; q(q^0, \omega, p^0, \eta, T), \omega, \eta) < 0$$

for $(q^0, \omega) \in \operatorname{supp} F|_{S_\rho^{n-1} \times M^m}$ and $(q^0, \omega, p^0, \eta) \in \Omega_\varepsilon$ (see the absorption condition). By the rule of reduction (see p. 108), it suffices to find, for any natural N, a function $\psi_N(\alpha, \eta, x, t)$ vanishing if x is in the neighbourhood of 0 (say, for $|x| < c$), belonging to $C^\infty_{\mathscr{L}}$

for any fixed t and satisfying the following conditions:

$$\psi_N(\alpha, \eta, x, T) = O_{\mathscr{L}}(|x|^{-N}), \tag{5.1}$$

$$\begin{cases} -i\Lambda_0^{r-1}\left(\overset{1}{x}\right)\dfrac{\partial\psi_N}{\partial t}\left(\overset{2}{\alpha}, i\overset{2}{\dfrac{\partial}{\partial x}}, \overset{1}{x}, t\right) + \\ +[\![f\left(\overset{1}{L_1}, \ldots, \overset{n}{L_n}, \overset{n+1}{\alpha}\right)]\!]\psi_N\left(\overset{2}{\alpha}, i\overset{2}{\dfrac{\partial}{\partial x}}, \overset{1}{x}, t\right) = \\ = \mathscr{B}_N\left(\overset{2}{\alpha}, i\overset{2}{\dfrac{\partial}{\partial x}}, \overset{1}{x}, t\right), \\ \mathscr{B}_N(\alpha, \eta, x, t) = O_{\mathscr{L}}(|x|^{-N}) \text{ uniformly in } t, \end{cases} \tag{5.2}$$

$$\psi_N(\alpha, \eta, x, 0) = F(x, \alpha)\rho(\eta) \text{ for } |x| > c, \tag{5.3}$$

where $\rho(\eta) = 0$ for $|\eta| > \varepsilon$ and $\rho(\eta) = 1$ in a neighbourhood of the origin. Note that $F\left(\overset{1}{x}, \overset{2}{\alpha}\right)\rho\left(i\overset{2}{\dfrac{\partial}{\partial x}}\right)1 = F(x, \alpha) + O_{\mathscr{L}}(|x|^{-\infty})$ which can be shown by partial integration with respect to y in the formula

$$F\left(\overset{1}{x}, \overset{2}{\alpha}\right)\rho\left(i\overset{2}{\frac{\partial}{\partial x}}\right)1 = F(x, \alpha) + \\ + (2\pi)^{-n}\int_{\mathbf{R}^{2n}} e^{i\langle\eta, y-x\rangle}[-1+\rho(\eta)]F(y, \alpha)\,d\eta\,dy.$$

Further, it is convenient to replace $\Lambda_0(x)$ by such a C^∞ function $\Lambda(x)$ that $\Lambda > 0$ and $\Lambda(x) = \Lambda_0(x)$ for large x.

Now let us transform the Hamiltonian $f\left(\overset{1}{L}, \overset{2}{d}\right)$. First of all, note that the condition for asymptotic quasi-homogeneity of a function $\varphi(\xi)$, $\xi = (\xi_1, \ldots, \xi_n)$, may be written as follows: for all $\lambda > 0$,

$$\varphi(\lambda^{\rho_1}\xi_1, \ldots, \lambda^{\rho_n}\xi_n) = \sum_{j=0}^{k}\lambda^{r_j}\varphi_j(\xi) + \lambda^{r_0-1}\varphi_{k+1}(\xi, \lambda),$$

where $r_0 > r_1 > \ldots > r_k$, φ_j is quasi-homogeneous of degree r_j for $j = 0, \ldots, k$, $\varphi_{k+1} \in \mathscr{S}^\infty(\mathbf{R}^n \times (0, \infty))$ being bounded as $\lambda \to \infty$ as well as all its derivatives with respect to ξ. Thus, any asymptotically quasi-homogeneous function $\varphi(\xi)$ can be expressed in the form

$$\varphi(\xi) = \lambda^{r_0}\left\{\sum_{j=0}^{k}\lambda^{r_j-r_0}\varphi_j(\lambda^{-\rho}\xi) + \lambda^{-1}\varphi_{k+1}(\lambda^{-\rho}\xi, \lambda)\right\}. \tag{5.4}$$

Set $\varepsilon_j = r_0 - r_j$, $j = 1, 2, \ldots, k$, so we correlate a k-tuple $(\varepsilon_1, \ldots, \varepsilon_k)$ of positive numbers with any asymptotically quasi-homogeneous function.

Using the representation (5.3), the composite function $f(L(\eta, x, \beta), \alpha)$ can be expressed in the form

$$\begin{aligned} f(L(\eta, x, \beta), \alpha) &= \\ &= \lambda^r [\mathscr{H}_1(\alpha, \eta, \beta, \omega, \varkappa) + \lambda^{-1}\mathscr{H}_2(\alpha, \eta, \beta, \omega, \lambda)], \end{aligned} \tag{5.5}$$

where $\omega = \lambda^{-\rho}x$, $\varkappa = (\lambda^{-\varepsilon_1}, \ldots, \lambda^{-\varepsilon_k})$, $\mathscr{H}_1$ and $\mathscr{H}_2$ belong to $\mathscr{S}^\infty$, $\operatorname{Im} \mathscr{H}_1 \leqslant 0$, and $\mathscr{H}_2$ is bounded as $\lambda \to \infty$, together with all its derivatives in α, η, β, ω.

Taking into account (5.5) and using the note following Lemma 2 of Appendix, we obtain the following expression of the Hamiltonian

$$\begin{aligned} f\left(\overset{1}{L}, \overset{2}{\alpha}\right) &= \\ &= \overset{1}{\Lambda}{}^{\overset{2}{r}} [\![\mathscr{H}_1 \left(\overset{2}{\alpha}, i\frac{\overset{2}{\partial}}{\partial x}, -i\overset{1'}{\Lambda}{}^{-1}\frac{\overset{1}{\partial}}{\partial \alpha}, \overset{1}{x}\overset{1'}{\Lambda}{}^{-\rho}, \overset{1'}{\Lambda}{}^{-\varepsilon}\right) + \\ &+ \left(-i\overset{1'}{\Lambda}{}^{-1}\right) \mathscr{H}_3 \left(\overset{2}{\alpha}, i\frac{\overset{2}{\partial}}{\partial x}, -i\overset{1'}{\Lambda}{}^{-1}\frac{\overset{1}{\partial}}{\partial \alpha}, \overset{1}{x}\overset{1'}{\Lambda}{}^{-1}, \overset{1'}{\Lambda}\right)]\!], \end{aligned} \tag{5.6}$$

where $\varepsilon > 0$, $\mathscr{H}_3(\alpha, \eta, \beta, x, \lambda) \in \mathscr{S}^\infty$ is bounded as $\lambda \to \infty$, together with all its derivatives with respect to α, η, β, x, and $\Lambda^{-\varepsilon} = (\Lambda^{-\varepsilon_1}, \ldots, \Lambda^{-\varepsilon_k})$.

To solve the problem (5.2) with the Hamiltonian function $f\left(\overset{1}{L}, \overset{2}{\alpha}\right)$ of the form (5.6) we use the theory of canonical operator on a Lagrangean manifold with a complex germ developed in this chapter.

Set

$$\begin{aligned} \mathscr{H}_0(X_0, X; P_0, P; \omega, \varkappa_1, \varkappa_2) &\overset{\text{def}}{=\!=} \\ &= \mathscr{H}_1(X_0, X, P_0, \omega + \Lambda^{1-\rho}P, \Lambda^{-\varepsilon}), \end{aligned}$$

where

$$\begin{aligned} &X_0 \in M^m, \ X = (X_1, \ldots, X_n) \in \mathbf{R}^n, \ P_0 \in \mathbf{R}^m, \\ &P = (P_1, \ldots, P_n) \in \mathbf{R}^n, \ \omega = (\omega_1, \ldots, \omega_n) \in \mathbf{R}^n, \\ &\varkappa_1 \overset{\text{def}}{=\!=} (\Lambda^{-\varepsilon_1}, \ldots, \Lambda^{-\varepsilon_k}), \\ &\varkappa_2 \overset{\text{def}}{=\!=} (\Lambda^{1-\rho_1}, \ldots, \Lambda^{1-\rho_n}). \end{aligned}$$

The function $\mathscr{H}_0$ depends smoothly on the parameters ω, $\varkappa_1$ and $\varkappa_2$ for $\omega \in \mathbf{R}^n$ and $\varkappa_1 \to 0$, $\varkappa_2 \to 0$. Moreover, $\mathscr{H}_0$ tends to the Hamiltonian function $\mathscr{H}$ corresponding to the operator $f\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right)$ as $\varkappa_1 \to 0$, $\varkappa_2 \to 0$ (for the definition of the Hamiltonian function, see Sec. 9 of Introduction).

We will regard $\mathscr{H}_0$ as a function on the phase space

$$\{X_0, X; P_0, P\} = (M^m \times \mathbf{R}^n) \times \mathbf{R}^{m+n}.$$

Denote by M^{m+n} the space $M^m \times \mathbf{R}^n$ and by T^*M^{m+n} the phase space. It is clear that M^{m+n} is a Lagrangean manifold. The condition of the asymptotic ρ-quasi-homogeneity of the Hamiltonian implies that $\operatorname{Im} \mathscr{H}_1 \leqslant 0$, hence

$$\operatorname{Im} \mathscr{H}_0 \leqslant 0.$$

Set $H_0 = \operatorname{Re} \mathscr{H}_0$ and consider the system of bicharacteristics

$$\begin{aligned}
&\frac{\partial X_0}{\partial t} = \frac{\partial H_0}{\partial P_0}(X_0, X, P_0, \omega + \varkappa_2 P, \varkappa_1),\\
&X_0\,|_{t=0} = \alpha,\\
&\frac{\partial X_k}{\partial t} = \frac{\partial H_0}{\partial P_k}(X_0, X, P_0, \omega + \varkappa_2 P, \varkappa_1),\\
&X_k\,|_{t=0} = \eta_k,\\
&\frac{\partial P_0}{\partial t} = -\frac{\partial H_0}{\partial X_0}(X_0, X, P_0, \omega + \varkappa_2 P, \varkappa_1),\\
&P_0\,|_{t=0} = 0,\\
&\frac{\partial P_k}{\partial t} = -\frac{\partial H_0}{\partial X_k}(X_0, X, P_0, \omega + \varkappa_2 P, \varkappa_1),\\
&P_k\,|_{t=0} = 0,\\
&k = 1, \ldots, n.
\end{aligned} \tag{5.7}$$

Since the limiting function $\mathscr{H} = \lim\limits_{\Lambda\to\infty} \mathscr{H}_0$ satisfies the absorption conditions whenever the parameters of the system (5.7) lie in Ω_ε, this system has a solution for $\varkappa_1 = \varkappa_2 = 0$. Therefore, it is easy to construct by the perturbation theory an asymptotic solution of the system (5.7), i.e., functions X_0, X, P_0, $P \in C^\infty$ which satisfy (5.7) modulo Λ^{-N} with N being as large as desired.

Denote by $g^t_{H_0+i\widetilde{H}_0}(\omega, \varkappa_1, \varkappa_2)$ the shift corresponding to this asymptotic solution.

It is easy to verify that all the properties of the complex germ are preserved if one replaces the solution of the Hamilton system by the asymptotic solution of the system (5.7).

It follows that the family

$$(M_t^{m+n}, r_t^{m+n}) = g^t_{H_0+i\widetilde{H}_0}\{M^{m+n}, 0\},\ 0 \leqslant t \leqslant T,$$

obtained from M^{m+n} by the family of transformations $g^t_{H_0+i\widetilde{H}_0}(\omega, \varkappa_1, \varkappa_2)$ (here T is the same as above) is a family of Lagrangean manifolds with complex germs smoothly dependent on the parameters $\omega = (\omega_1, \ldots, \omega_n)$, $t \in (0, T)$ and $\varkappa_1$, $\varkappa_2$. As formerly

we shall identify the family $\{M_t^{m+n}\}$ with the $(m+n+1)$-dimensional manifold M^{m+n+1} which obviously smoothly depends on ω for sufficiently small $\varkappa_1$ and $\varkappa_2$.

Let $\mathscr{K}^{\Lambda}$ be the canonical operator on M^{m+n+1} corresponding to the germ r_t^{m+n}, with h replaced by Λ^{-1}*.

As is seen from (5.2), (5.3) and (5.6) we have to consider the following Cauchy problem:

$$-i\overset{1}{\Lambda^{-1}}\frac{\partial\psi}{\partial t}\left(\overset{2}{\alpha}, i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}, t\right)+$$

$$+\overset{3}{[\![}\mathscr{H}_1-i\overset{1'}{\Lambda}\mathscr{H}_3]\!]\,\psi\left(\overset{2}{\alpha}, i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}, t\right)=0, \tag{5.8}$$

$$\psi\left(\overset{2}{\alpha}, i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}, 0\right)=\rho\left(i\overset{2}{\frac{\partial}{\partial x}}\right)P\left(\overset{1}{x}, \overset{2}{\alpha}\right),$$

where

$$\mathscr{H}_1=\mathscr{H}_1\left(\overset{2}{\alpha}, i\overset{2}{\frac{\partial}{\partial x}}, -i\overset{1'}{\Lambda^{-1}}\overset{1}{\frac{\partial}{\partial \alpha}}, \overset{1}{x}\overset{1'}{\Lambda^{-\rho}}, \overset{1'}{\Lambda^{-\varepsilon}}\right),$$

$$\mathscr{H}_3=\mathscr{H}_3\left(\overset{2}{\alpha}, i\overset{2}{\frac{\partial}{\partial x}}, -i\overset{1'}{\Lambda^{-1}}\overset{1}{\frac{\partial}{\partial \alpha}}, \overset{1}{x}\overset{1'}{\Lambda^{-1}}, \overset{1'}{\Lambda}\right).$$

We will show that the problem (5.8) has an approximate solution of the form

$$\psi_N\left(\overset{2}{\alpha}, i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}, t\right)=\left(\overset{1}{\mathscr{K}^{\Lambda}}\varphi_N\right)\left(\overset{2}{\alpha}, i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{\omega}, \overset{1}{\Lambda}^{1-\rho}, \overset{1}{\Lambda}^{-\varepsilon}, t\right), \tag{5.9}$$

where φ_N is a function belonging to $\mathscr{S}^{\infty}(M^{m+n+1})$, and $\overset{1}{\omega}=[\![\overset{1}{x}\Lambda^{-\rho}]\!]$.

First note that by virtue of the choice of the initial manifold M^{m+n} we have

$$(\mathscr{K}^{\Lambda}\varphi_N)(\alpha, \eta, \omega, \Lambda^{1-\rho}, \Lambda^{-\varepsilon}, t)\,|_{t=0}=\varphi_N(\alpha, \eta, \omega, t)\,|_{t=0}.$$

Consequently, for the operator (5.9) to satisfy the initial condition of the Cauchy problem (5.8), it is sufficient to require that

$$\varphi_N(\alpha, \eta, \omega, 0)=\rho(\eta)P(\omega, \alpha). \tag{5.10}$$

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* More exactly, we interpret $\mathscr{K}^{\Lambda}$ as follows: given a C^{∞} function φ on M^{m+n+1} (identified with an element of $\mathscr{A}(M^{m+n+1})$), let $\sum_{j=0}^{\infty}\psi_j(\alpha, x, t, h)$ be an h-asymptotic series representing $\mathscr{K}\varphi$ (we omit the parameters ω, $\varkappa_1$ and $\varkappa_2$); then $(\mathscr{K}^{\Lambda}\varphi)(\alpha, x, t)$ means $\sum_{j=0}^{j_0}\psi_j(\alpha, x, t, \Lambda^{-1})$, where j_0 is large enough.

Making the Fourier transformation $F^{-1}_{x\to y}$ in (5.8), we obtain

$$-i\overset{1'}{h}\frac{\partial\psi}{\partial t}\left(\overset{2}{\alpha},\ \overset{2}{y},\ -i\overset{1}{\frac{\partial}{\partial y}},\ t\right)+$$
$$+\overset{3}{[\![}\mathscr{H}_1\left(\overset{2}{\alpha},\ \overset{2}{y},\ -i\overset{1'}{h}\overset{1}{\frac{\partial}{\partial\alpha}},\ -i\overset{1'}{h^{\rho}}\overset{1}{\frac{\partial}{\partial y}},\ \overset{1'}{h^{\varepsilon}}\right)-$$
$$-i\overset{1'}{h}\mathscr{H}_3\left(\overset{2}{\alpha},\ \overset{2}{y},\ -i\overset{1'}{h}\overset{1}{\frac{\partial}{\partial\alpha}},\ -i\overset{1'}{h}\overset{1}{\frac{\partial}{\partial y}},\ \overset{1'}{h^{-1}}\right)]\!]\times$$
$$\times\psi\left(\overset{2}{\alpha},\ \overset{2}{y},\ -i\overset{1}{\frac{\partial}{\partial y}},\ t\right)=0,\qquad(5.11)$$

where h stands for the operator $\Lambda^{-1}\left(-i\frac{\partial}{\partial x}\right)$.

Now consider $\mathscr{K}^{1/h}$, i. e., $\mathscr{K}^{\Lambda}$ with Λ replaced by h^{-1}. By Theorem 4.2,

$$\left[-ih\frac{\partial}{\partial t}+\mathscr{H}_1\left(\overset{2}{\alpha},\ \overset{2}{y},\ -ih\overset{1}{\frac{\partial}{\partial\alpha}},\ -ih^{\rho}\overset{1}{\frac{\partial}{\partial y}}+h^{\rho-1}\omega,\ h^{\varepsilon}\right)-\right.$$
$$\left.-ih\mathscr{H}_3\left(\overset{2}{\alpha},\ \overset{2}{y},\ -ih\overset{1}{\frac{\partial}{\partial\alpha}},\ -ih\overset{1}{\frac{\partial}{\partial y}}+\omega,\ h^{-1}\right)\right]\left(\mathscr{K}^{\overset{1}{\bar{h}}}\varphi_N\right)=$$
$$=\left(-ih\mathscr{K}^{\overset{1}{\bar{h}}}\left(\frac{d}{dt}+q\right)\varkappa\varphi_N\right)(\alpha,\ y,\ \omega,\ h^{\rho-1},\ h^{\varepsilon},\ t),\qquad(5.12)$$

where q is a smoth function on M^{m+n+1} and $\varkappa$ is a quasi-identity operator.

Now we substitute the operator

$$\psi_N\left(\overset{2}{\alpha},\ \overset{2}{y},\ -ih\overset{1}{\frac{\partial}{\partial y}},\ t\right)=$$
$$=\left(\mathscr{K}^{\overset{1}{\bar{h}}}\varphi_N\right)\left(\overset{2}{\alpha},\ \overset{2}{y},\ -ih^{\rho}\overset{1}{\frac{\partial}{\partial y}},\ h^{\rho-1},\ h^{\varepsilon},\ t\right)$$

into (5.11). Noting Lemma 5 of Appendix and (5.12), we obtain

$$-ih\frac{\partial\left(\mathscr{K}^{\overset{1}{\bar{h}}}\varphi_N\right)}{\partial t}\left(\overset{2}{\alpha},\ \overset{2}{y},\ -ih^{\rho}\overset{1}{\frac{\partial}{\partial y}},\ h^{\rho-1},\ h^{\varepsilon},\ t\right)+$$
$$+\overset{3}{[\![}\mathscr{H}_1\left(\overset{2}{\alpha},\ \overset{2}{y},\ -ih\overset{1}{\frac{\partial}{\partial\alpha}},\ -ih^{\rho}\overset{1}{\frac{\partial}{\partial y}},\ h^{\varepsilon}\right)-$$
$$-ih\mathscr{H}_3\left(\overset{2}{\alpha},\ \overset{2}{y},\ -ih\overset{1}{\frac{\partial}{\partial\alpha}},\ -ih\overset{1}{\frac{\partial}{\partial y}},\ h^{-1}\right)]\!]\times$$
$$\times\left(\mathscr{K}^{\overset{1}{\bar{h}}}\varphi_N\right)\left(\overset{2}{\alpha},\ \overset{2}{y},\ -ih^{\rho}\overset{1}{\frac{\partial}{\partial y}},\ h^{\rho-1},\ h^{\varepsilon},\ t\right)=$$
$$=-ih\left[\mathscr{K}^{\overset{1}{\bar{h}}}\left(\frac{d}{dt}+q\right)\varkappa\varphi\right]\left(\overset{2}{\alpha},\ \overset{2}{y},\ -ih^{\rho}\overset{1}{\frac{\partial}{\partial y}},\ h^{\rho-1},\ h^{\varepsilon}\right).\qquad(5.13)$$

Substituting $h = [\![\overset{1'}{\Lambda^{-1}} \left(-i \frac{\partial}{\partial y} \right)]\!]$ here and making the Fourier transform $F_{y \to x}$, we get

$$-i\overset{1}{\Lambda^{-1}} \frac{\partial \left(\mathscr{K}^{\overset{1}{\Lambda}} \varphi_N \right)}{\partial t} \left(\overset{2}{\alpha}, i \frac{\overset{1}{\partial}}{\partial x}, \overset{1}{\omega}, \overset{1}{\Lambda}^{1-\rho}, \overset{1}{\Lambda}^{-\varepsilon}, t \right) + \overset{3}{[\![} \mathscr{H}_1 - i\mathscr{H}_3]\!] \times$$

$$\times \left[\mathscr{K}^{\overset{1}{\Lambda}} \left(\frac{d}{dt} + q \right) \varkappa \varphi_N \right] \left(\overset{2}{\alpha}, i \frac{\overset{2}{\partial}}{\partial x}, \overset{1}{\omega}, \overset{1}{\Lambda}^{1-\rho}, \overset{1}{\Lambda}^{-\varepsilon}, t \right), \quad (5.14)$$

where $\omega_k = \Lambda^{-\rho_k} x_k$, $k = 1, \ldots, n$, and we have dropped the arguments

$$\left(\overset{2}{\alpha}, i \frac{\overset{2}{\partial}}{\partial x}, -i\overset{1'}{\Lambda^{-1}} \frac{\overset{1}{\partial}}{\partial \alpha}, \overset{1}{x}\overset{1'}{\Lambda}^{-\rho}, \overset{1'}{\Lambda}^{-\varepsilon} \right),$$

and

$$\left(\overset{2}{\alpha}, i \frac{\overset{2}{\partial}}{\partial x}, -i\overset{1'}{\Lambda^{-1}} \frac{\overset{1}{\partial}}{\partial \alpha}, \overset{1}{x}\overset{1'}{\Lambda}^{-1}, \overset{1'}{\Lambda} \right)$$

of the operators $\mathscr{H}_1$ and $\mathscr{H}_3$, respectively.

By virtue of (5.14) and (5.6), to solve approximately the Cauchy problem (5.2), (5.3) it suffices to find a function φ_N satisfying the transfer equation

$$\left(\frac{d}{dt} + q \right) \varkappa \varphi_N = 0 \ (\mathrm{mod}\, \mathcal{O}_D (\Lambda^{-N})),$$

which is possible by Theorem 4.3.

Thus, we have constructed the required approximate solution of the Cauchy problem (5.2), (5.3) for the operator $f\left(\overset{1}{L}, \overset{2}{\alpha}\right)$ (take into account the estimates given in Lemma 4 of Appendix).

In fact, it remains only to verify that for $t = T$, the function

$$\psi_N (\alpha, \eta, x, t) = (\mathscr{K}^{\Lambda} \varphi_N) (\alpha, \eta, \omega, \Lambda^{1-\rho}, \Lambda^{-\varepsilon}, t) \quad (5.15)$$

decreases as $x \to \infty$ faster than any power of $\frac{1}{|x|}$. Note that the bicharacteristics (X, P) corresponding to H_0 (see (5.7)), which depend on the parameters $\varkappa_1$ and $\varkappa_2$ coincide in the limiting case $\varkappa_1 = \varkappa_2 = 0$ with the bicharacteristics corresponding to $H \overset{\mathrm{def}}{=} \mathrm{Re}\, \mathscr{H}$, where $\mathscr{H}$ is the Hamiltonian function of the operator $f(A, B)$. According to the condition of the theorem the latter satisfy the absorption condition. It follows that the absorption condition is satisfied as well for the bicharacteristics corresponding to H_0, provided that $\min_{x \in \mathbf{R}^n} \Lambda(x)$ is large enough. Namely, the following inequality holds for $t = T$:

$$\mathrm{Im}\, \mathscr{H}_0 (X_0, X; P_0, P; \omega, \varkappa_1, \varkappa_2) \leqslant -\delta < 0.$$

It follows from this inequality and the construction of the canonical operator, that the function $\mathscr{K}^{\Lambda}\varphi_N \,|_{t=T}$ decreases faster than any negative power of Λ as $x \to \infty$. One can check directly that (9.8) of Introduction with ψ defined by (5.15) gives a solution of the problem

$$\begin{cases} [\![f\left(\overset{1}{A}, \overset{2}{B}\right)]\!] \, [\![g_N\left(\overset{1}{A}, \overset{2}{B}\right)]\!] = F\left(\overset{1}{A}, \overset{2}{B}\right) + R_N\left(\overset{1}{A}, \overset{2}{B}\right), \\ \qquad R_N(x, \alpha) = O_{\mathscr{L}}(|x|^{-N}). \end{cases}$$

The Main Theorem in the general form stated in item 4 of Sec. 9 of Introduction can be proved in the same way. Here we consider in detail only the reduction of the generalized absorption condition (see item 4 of Sec. 9 of Introduction) to the ordinary one (see item 3 of Sec. 9 of Introduction) and sketch a modification of the above proof for the case of an operator $f\left(\overset{1}{A}, \overset{2}{A}, \xi\right)$ dependent on parameter (but independent of the standard r).

The generalized absorption condition differs essentially from ordinary one only in the case where the set Ω_0 is non-connected. In such a case Ω_ε is non-connected as well, the function

$$\tilde{\tilde{H}}\,(p\,(q^0, \omega, p^0, \eta, \nu, \tau'),\ q\,(q^0, \omega, p^0, \eta, \nu, \tau'),\ \omega, \nu, \eta)$$

being strictly positive or strictly negative by the absorption condition on every connectivity component $\Omega_\varepsilon^{(j)}$ of Ω_ε (which corresponds to a connectivity component of Ω_0). Therefore, using the same argument as that at the beginning of this section, we can choose a covering of M_1 and a partition of unity corresponding to this covering, so that the quasi-inversion problem reduces to the equations

$$fg_N^{(i)} = F^{(i)} + R_N^{(i)}, \qquad i = 1, \ldots, s,$$

where s is the number of connectivity components of Ω_0 and $F^{(i)}$ is such that the imaginary part of the Hamiltonian function

$$\tilde{\tilde{H}}\,(p\,|_{\tau=\tau'},\ q\,|_{\tau=\tau'},\ \omega, \nu, \eta)$$

is strictly negative on supp $F^{(i)}$ (if it is positive we multiply the equation by -1). Thus the general case is reduced to the case of an ordinary absorption condition.

Now consider a Hamiltonian

$$f\left(\overset{1}{[\![}L\left(\overset{1}{x}, i\frac{\overset{2}{\partial}}{\partial x}, -i\frac{\overset{1}{\partial}}{\partial \alpha}\right)]\!];\ \overset{2}{\alpha}, \xi\right)$$

which does not depend on the standard and is $(\rho_0, \ldots, \rho_n, \rho_{n+1})$-quasi-homogeneous of degree r with respect to the arguments $(-i\frac{\partial}{\partial\alpha} = x_0, x_1, \ldots, x_n; \xi)$, the numbers ρ_0 and ρ_{n+1} being equal

to 1. Using the same argument as above (cf. (5.6)), we can express $f\left(\overset{1}{L}, \overset{2}{\alpha}, \xi\right)$ as the sum

$$f\left(\overset{1}{L}, \overset{2}{\alpha}, \xi\right) = \overset{1}{\Lambda}{}^{\overset{2}{r}} [\![\mathscr{H}_1 \left(\overset{2}{\alpha}, i\frac{\overset{2}{\partial}}{\partial x}, -i\overset{1'}{\Lambda}{}^{-1}\frac{\overset{1}{\partial}}{\partial \alpha}, \overset{1'}{\Lambda}{}^{-\rho}\overset{1}{x}, \overset{1'}{\nu}, \overset{1'}{\Lambda}{}^{-\varepsilon}\right) +$$

$$+\left(-i\overset{1'}{\Lambda}{}^{-1}\right)\mathscr{H}_3\left(\overset{2}{\alpha}, i\frac{\overset{2}{\partial}}{\partial x}, -i\overset{1'}{\Lambda}{}^{-1}\frac{\overset{1}{\partial}}{\partial \alpha}, \overset{1'}{\Lambda}{}^{-1}\overset{1}{x}, \overset{1'}{\nu}, \overset{1'}{\Lambda}\right)]\!], \quad (5.16)$$

where $\Lambda = \Lambda(x)$ is the same as above and

$$\Lambda^{-\varepsilon} = (\Lambda^{-\varepsilon_1}, \ldots, \Lambda^{-\varepsilon_k}), \ \varepsilon_j > 0, \ j = 1, \ldots, n.$$

The function $\mathscr{H}_3(\alpha, \eta, p_0, \omega, \nu, \Lambda)$ is bounded as $\Lambda \to \infty$ uniformly in all the parameters, provided that the functional space where the Hamiltonian acts is such that the operator $\nu \overset{\text{def}}{=\!=} \xi^{-1}\Lambda$ is bounded (see below).

According to the reduction rule, the quasi-inversion problem for the operator $f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}, \xi\right)$ reduces to the Cauchy problem of the form (5.2) with the Hamiltonian $f\left(\overset{1}{L}, \overset{2}{\alpha}, \xi\right)$. Now we express the Hamiltonian in the form (5.16) and the further course of solving the Cauchy problem does not differ from the one above. In particular, one constructs the Hamiltonian function

$$\mathscr{H}_0(X_0, X; P_0, P; \omega, \nu, \Lambda^{1-\rho}, \Lambda^{-\varepsilon}) \overset{\text{def}}{=\!=}$$
$$= \mathscr{H}_1(X_0, X, P_0, \omega + \Lambda^{1-\rho}P, \nu, \Lambda^{-\varepsilon}),$$

where $\omega = (\omega_1, \ldots, \omega_n)$ are the parameters corresponding to the operators

$$x\Lambda^{-\rho} = (x_1\Lambda^{-\rho_1}, \ldots, x_n\Lambda^{-\rho_n}).$$

The function $\mathscr{H}_0$ tends as $\Lambda \to \infty$ to the Hamiltonian function $\mathscr{H}$ corresponding to the operator $f\left(\overset{1}{L}, \overset{2}{\alpha}, \xi\right)$ (see Sec. 9 of Introduction).

Further, one constructs, as above, the asymptotic solution of the Cauchy problem (5.2) of the form

$$\psi_N\left(\overset{2}{\alpha}, i\frac{\overset{2}{\partial}}{\partial x}, \overset{1}{x}, t, \xi\right) = \left(\mathscr{K}^{\overset{1}{\Lambda}}\varphi_N\right)\left(\overset{2}{\alpha}, i\frac{\overset{2}{\partial}}{\partial x}, \overset{1}{\omega}, \overset{1}{\nu}, \overset{1}{\Lambda}{}^{-\varepsilon}, \overset{1}{\Lambda}{}^{1-\rho}, t\right), \quad (5.17)$$

where $\mathscr{K}^{\Lambda}$ corresponds to $\mathscr{H}_0$ and $\omega_k = x_k\Lambda^{-\rho_k}$, $\nu = \xi^{-1}\Lambda$.

The element g_N of the quasi-inverse sequence has the symbol

$$g_N(x, \alpha, \xi) = i \int_0^T \psi_N \left(\overset{2}{\alpha}, i \frac{\overset{2}{\partial}}{\partial x}, \overset{1}{x}, t, \xi \right) \overset{1}{\Lambda^{1-r}\varphi}\, dt, \tag{5.18}$$

where

$$\varphi = \varphi(x; \alpha; \xi) = P_0(x_1\xi^{-\rho_1}, \ldots, x_n\xi^{-\rho_n}; \alpha)$$

with $P_0(x, \alpha)$ being as above.

The support of $\varphi(x; \alpha; \xi)$ is obviously contained in the domain $\Lambda(x) \leqslant c\xi$. It follows that $|\nu| \leqslant c$ on supp φ.

Thus, we have constructed a right-inverse sequence. A left-inverse one can be constructed quite similarly with the aid of the same Hamiltonian function. In fact, it suffices to note that the operator

$$[\![g_N \left(\overset{n+1}{A_1}, \ldots, \overset{2}{A_n}, \overset{1}{B} \right)]\!] \, [\![f \left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B} \right)]\!]$$

is equal to

$$r_N \left(\overset{n+1}{A_1}, \ldots, \overset{2}{A_n}, \overset{1}{B} \right)$$

with

$$r_N(x, \alpha) = f \left(\overset{n+1}{L_1^*}, \ldots, \overset{2}{L_n^*}, \overset{1}{\alpha} \right) g_N,$$

where

$$L_j^*(\eta, x, p_0) \stackrel{\text{def}}{=} L_j(-\eta, x, -p_0).$$

In conclusion, we consider the following generalization of the quasi-inversion problem: it is necessary to find an approximate solution $\hat{v} \in \mathscr{L}$, $\hat{w} \in \mathscr{L}$ of equations

$$\hat{f}\hat{v} = \hat{F}, \quad \hat{w}\hat{f} = \hat{F},$$

where

$$\hat{f} = f \left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B} \right) \in \mathscr{A}, \quad \hat{v} \in \mathscr{L}$$

and

$$\hat{F} = F \left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B} \right) \in \mathscr{A}$$

is such that supp F is contained in a fixed domain $\Delta \subset S_\rho^{n-1} \times M^m$.

More precisely, one has to find two sequences of elements of $\mathscr{L}$,

$$\hat{v}_N = v_N \left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B} \right) \text{ and } \hat{w}_N = w_N \left(\overset{n+1}{A_1}, \ldots, \overset{2}{A_n}, \overset{1}{B} \right)$$

such that

$$\hat{f}\hat{v}_N = \hat{F} + \hat{R}_N, \quad \hat{w}_N\hat{f} = \hat{F} + \hat{R}'_N,$$

where both $\hat{R}_N$ and $\hat{R}'_N$ are in $\mathcal{L}$ and have the symbols decreasing faster than $|y|^{-N}$. The elements $\hat{v}_N$ and $\hat{w}_N$ will be called respectively right and left Δ-quasi-inverse, and the corresponding sequences $\{v_n\}$, $\{w_n\}$ will be called a right- and a left-quasi-inverse sequence, respectively.

For the Δ-quasi-inversion problem the result similar to the Main Theorem holds. It is only required to modify the absorption condition slightly, namely, initial points α_0, q_0 must now belong to the intersection of Δ with a small neighbourhood of the zero-set of f_0.

Note also that the formula for a Δ-quasi-inverse element, which was given for a particular case in Introduction, leads to the following statement concerning the supports of Δ-quasi-inverse elements (both left and right):

$$\operatorname{supp} v_N \subset \pi_{S_\rho^{n-1}\times M^m} \bigcup_{0\leqslant t\leqslant T} g_H^t(\Delta\cap\Omega_\varepsilon),$$

where $\pi_{S_\rho^{n-1}\times M^m}$ is the projection onto $S_\rho^{n-1}\times M^m$, and $g_H^t(\Delta\cap\Omega_\varepsilon)$ denotes the image of $\Delta\cap\Omega_\varepsilon$ under the canonical transformation associated with the Hamiltonian function H.

Problem. Under the assumptions stated on page 138 and assuming in addition that the operators $A_1, \ldots, A_n$ commute, deduce (9.75) of Introduction from the formulas (5.17) and (5.18) for the right quasi-inverse sequence.

Note 1. In the particular case, when the operator $\hat{f}$ is not dependent on B, the relation (9.1) of Introduction will be equivalent to the following:

$$f\left(\overset{1}{\tilde{L}}_1, \ldots, \overset{n}{\tilde{L}}_n\right)\tilde{g}_N = (2\pi)^{n/2}\,\delta + \tilde{R}_N,$$

where δ is the Dirac δ-function concentrated at unity of the group G induced by the generators $iA_1, \ldots, iA_n$, $\tilde{R}_N = FR_N$ is a sufficiently smooth function on G, $\tilde{g}_N = Fg_N$, $\tilde{L}_j = FL_jF^{-1}$, and F is a Fourier transform.

Note 2. If $A_1, \ldots, A_n, B$ are generators with the defining pair of spaces (B_1, B_1), then by virtue of Note on p. 101 $R_k\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right) g \in D\left(\sum\limits_{i=1}^{n} A_i^k\right)$, $\forall g\in B_1$. In other words, the operator R_k is bounded as an operator on space B_1^γ into space $B^{\gamma+k}$, where B_1^γ is the completion D (see p. 211) in the norm

$$\|u\|B_1^\gamma = \left\|\left(1+\overset{1}{A_1^2}+\ldots+\overset{n}{A_n^2}\right)^{\gamma/2} u\right\|_B,\quad \gamma\in R.$$

Problem. Let $f(x_1, \ldots, x_n, \alpha)$ be asymptotically homogeneous function in x of degree k, where the leading term of $f(x, \alpha)\neq 0$

for $|x|=1$. Prove that if A_j, $j=1, \ldots, n$, B are generators of zero degree, then the quasi-inverse operator $g_N\left(\overset{1}{A_1}, \ldots, \overset{n}{A}, \overset{n+1}{B}\right)$ for the operator $f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)$ for any $\gamma \in R$, $\varepsilon > 0$, satisfies the estimate $\left\| g_N\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}\right)\right\|_{B_1^\gamma \to B_1^{\gamma-m-\varepsilon}} < \text{const}$.

Hint. By virtue of Theorem 3.1 of Chapter II, to prove the above inequality it will be sufficient to prove that the symbol $g_N(x, \alpha)$ satisfies the condition $(1+|x|^2)^{k/2-\varepsilon} g_N(x, \alpha) \in \mathscr{B}_0(R^{n+1})$. This fact follows from the estimate $\|(1+|x|^2)^{k/2-\varepsilon} g_N\|_{\mathscr{B}_0(R^{n+1})} = \left\| g_N\left(-i\frac{\partial}{\partial y}\right)(1-\Delta)^{k/2-\varepsilon}\right\|_{C(R^{n+1}) \to C(R^{n+1})}$. By virtue of Lemma 3 of the appendix to this section, the symbol $g_N(x, \alpha)$ is an asymptotically homogeneous function in x of degree $-k$. As is well known, the operator $g_N\left(-i\frac{\partial}{\partial y}\right)(1-\Delta)^{k/2-\varepsilon}$: $C(R^{n+1}) \to C(R^{n+1})$ is bounded in this case.

Appendix to Sec. 5

Definition 1. *Let $\Phi(y, \alpha; h)$, $y \in \mathbf{R}^n$, $\alpha \in M^m$, be a family of functions belonging to $\mathscr{S}^\infty(\mathbf{R}^n \times M^m)$. We shall say that this family is bounded in $\mathscr{S}^\infty(\mathbf{R}^n \times M^m)$ if there exists such a k that*

$$\left|\left(\frac{\partial}{\partial y}\right)^p\left(\frac{\partial}{\partial \alpha}\right)^q \Phi(y, \alpha; h)\right| \leqslant C_{p,q}(1+|y|)^k$$

for any multi-indices p and q.

Lemma 1. *Let $\Phi(y, \alpha; h)$ be a bounded family in $\mathscr{S}^\infty(\mathbf{R}^n \times M^m)$ and let*

$$L'_j(\eta, x, \beta, h) = \sum_{k=1}^n x_k P_{k,j}(\eta, \beta, h) + P_j(\eta, \beta, h),$$

where $P_{k,j}$, P_j are polynomials in η and β with bounded coefficients dependent on h.

Set

$$\overset{k}{L'_j} = [\![\overset{k}{L'_j}\left(\overset{2}{\eta}, \overset{1}{x - ih\frac{\partial}{\partial \eta}}, \overset{1}{\beta}, h\right)]\!].$$

Then we have

$$\Phi\left(\overset{1}{L'}, \overset{2}{\alpha}; h\right) = \Phi\left(L'\left(\overset{2}{\eta}, \overset{1}{x - ih\frac{\partial}{\partial \eta}}, \overset{1}{\beta}, h\right), \overset{2}{\alpha}; h\right) + hR_h, \qquad (1)$$

where $\{R_h\}$ is a bounded in $\mathscr{S}^\infty$ family of functions of the ordered operators η, $x - ih\frac{\partial}{\partial \eta}$, β, α and $[\![L'\left(\overset{2}{\eta}, \overset{1}{x - ih\frac{\partial}{\partial \eta}}, \overset{1}{\beta}, h\right)]\!]$.

Proof. By the K-formula, we get

$$\Phi\left(\overset{1}{L'},\ \overset{2}{\alpha};\ h\right)=\Phi\left(L_1'\left(\overset{2}{\eta},\ x-\overset{1}{ih\frac{\partial}{\partial\eta}},\ \overset{1}{\beta},\ h\right),\ \overset{3}{L_2'},\ \ldots,\ \overset{n+1}{L_n'},\ \overset{n+2}{\alpha};\ h\right)-$$

$$-ih\sum_{k=1}^{n}\left[\frac{\partial L_1'}{\partial\eta_k}\left(\overset{3}{\eta},\ x-\overset{1}{ih\frac{\partial}{\partial\eta}},\ \overset{1}{\beta},\ h\right)P_{k,1}\left(\overset{3}{\eta},\ \overset{1}{\beta},\ h\right)\times\right.$$

$$\times\frac{\delta^2\Phi}{\delta y_1^2}\left(L_1'\left(\overset{6}{\eta},\ x-\overset{1}{ih\frac{\partial}{\partial\eta}},\ \overset{1}{\beta},\ h\right),\ \overset{2}{L_1'},\overset{5}{L_1'},\overset{7}{L_2'},\ \ldots$$

$$\left.\ldots,\ \overset{n+5}{L_n'},\ \overset{n+6}{\alpha};\ h\right)\Big].$$

In the first term in the right-hand member let us change the order of operating of η and L_2'. Using the commutation formula, we obtain:

$$\Phi\left(L_1'\left(\overset{2}{\eta},\ x-\overset{1}{ih\frac{\partial}{\partial\eta}},\ \overset{1}{\beta},\ h\right),\ \overset{3}{L_2'},\ \ldots,\ \overset{n+1}{L_n'},\ \overset{n+2}{\alpha};\ h\right)=$$

$$=\Phi\left(L_1'\left(\overset{3}{\eta},\ x-\overset{1}{ih\frac{\partial}{\partial\eta}},\ \overset{1}{\beta},\ h\right),\ \overset{2}{L_2'},\ \overset{4}{L_3'},\ \ldots,\ \overset{n+1}{L_n'},\ \overset{n+2}{\alpha};\ h\right)-$$

$$-ih\sum_{k=1}^{n}\left[P_{k,2}\left(\overset{4}{\eta},\ \overset{1}{\beta},\ h\right)\times\right.$$

$$\times\frac{\partial L_1'}{\partial\eta_k}\left(\overset{2}{\eta_1},\ \ldots,\ \overset{2}{\eta_{k-1}},\ \overset{4}{\eta_k},\ \overset{6}{\eta_{k+1}},\ \ldots,\ \overset{6}{\eta_n},\ x-\overset{1}{ih\frac{\partial}{\partial\eta}},\ \overset{1}{\beta},\ h\right)\times$$

$$\left.\times\frac{\delta^2\Phi}{\delta y_1\delta y_2}\left(\hat{L}_1,\ \hat{L}_1,\ \overset{3}{L_2'},\ \overset{5}{L_2'},\ \overset{7}{L_3'},\ \ldots,\ \overset{n+4}{L_n'},\ \overset{n+5}{\alpha};\ h\right)\right],\tag{2}$$

where $\hat{L}_1$ stands for the operator expression

$$L_1'\left(\overset{2}{\eta_1},\ \ldots,\ \overset{2}{\eta_{k-1}},\ \overset{4}{\eta_k},\ \overset{6}{\eta_{k+1}},\ \ldots,\ \overset{6}{\eta_n},\ x-\overset{1}{ih\frac{\partial}{\partial\eta}},\ \overset{1}{\beta},\ h\right).$$

We now transform the expression of the first term in the right-hand member of (2), first applying the K-formula to L_2' and then changing the order of operating η and L_3' in the first summand of the sum obtained.

Repeating this argument n times gives (1) with

$$R=-i\left(\sum_{j=1}^{n}r_j'+\sum_{j=1}^{n-1}r_j''\right),$$

where

$$r_j' = \sum_{k=1}^{n} \left[\frac{\partial L_j'}{\partial \eta_k} \left(\overset{3}{\eta}, x - \overset{1}{i h \frac{\partial}{\partial \eta}}, \overset{1}{\beta}, h \right) P_{k,j} \left(\overset{3}{\eta}, \overset{1}{\beta}, h \right) \times \right.$$

$$\times \frac{\delta^2 \Phi}{\delta y_j^2} \left(L_1' \left(\overset{6}{\eta}, x - \overset{1}{i h \frac{\partial}{\partial \eta}}, \overset{1}{\beta}, h \right), \ldots \right.$$

$$\left. \left. \ldots, L_j' \left(\overset{6}{\eta}, x - \overset{1}{i h \frac{\partial}{\partial \eta}}, \overset{1}{\beta}, h \right), \overset{2}{L_j'}, \overset{5}{L_j'}, \overset{7}{L_{j+1}'}, \ldots, \overset{n+6-j}{L_n'}, \overset{n+7-j}{\alpha}; h \right) \right],$$

$$r_j'' = \sum_{l=1}^{j} \sum_{k=1}^{n} \left[P_{k,j+1} \left(\overset{4}{\eta}, \overset{1}{\beta}, h \right) \times \right.$$

$$\times \frac{\partial L_j'}{\partial \eta_k} \left(\overset{2}{\eta_1}, \ldots, \overset{2}{\eta_{k-1}}, \overset{4}{\eta_k}, \overset{6}{\eta_{k+1}}, \ldots, \overset{6}{\eta_n}, x - \overset{1}{i h \frac{\partial}{\partial \eta}}, \overset{1}{\beta}, h \right) \times$$

$$\times \frac{\delta}{\delta y_{j+1}} \frac{\partial}{\partial y_l} \Phi \left(\hat{L}_1^{(k)}, \ldots, \hat{L}_j^{(k)}, \overset{3}{L_{j+1}'}, \overset{5}{L_{j+1}'}, \overset{7}{L_{j+2}'}, \ldots \right.$$

$$\left. \left. \ldots, \overset{n+5-j}{L_n'}, \overset{n+6-j}{\alpha}; h \right) \right],$$

with $\hat{L}^{(k)}$ standing for the operator expression

$$L' \left(\overset{2}{\eta_1}, \ldots, \overset{2}{\eta_{k-1}}, \overset{4}{\eta_k}, \overset{6}{\eta_{k+1}}, \ldots, \overset{6}{\eta_n}, x - \overset{1}{i h \frac{\partial}{\partial \eta}}, \overset{1}{\beta}, h \right).$$

The assertion of the lemma now follows immediately.

Making use of estimates given in Sec. 11 of Chapter II, we obtain the following corollary to Lemma 1:

Corollary. *For any real r there is such an s that for all k, R_h is bounded uniformly in h as an operator acting from C_r^k into C_s^k.*

Lemma 2. *Let $f(y, \alpha)$ be the same as in the Main Theorem. Then*

$$f \left(\overset{1}{L}, \overset{2}{\alpha} \right) = f \left(L \left(\overset{2}{i \frac{\partial}{\partial x}}, \overset{1}{x}, - \overset{1}{i \frac{\partial}{\partial \alpha}} \right), \overset{2}{\alpha} \right) +$$

$$+ h^{1-r} f_h \left(\overset{2}{i \frac{\partial}{\partial x}}, \overset{1}{hx}, - \overset{1}{ih \frac{\partial}{\partial \alpha}}, \overset{2}{\alpha} \right),$$

where $\{f_h\}$ is a family bounded in $\mathcal{S}^\infty$.

Proof. Define $\Phi(y, \alpha; h)$ and $L'(\eta, x, \beta, h)$ by

$$h^r f(y, \alpha) = \Phi(h^\rho y, \alpha; h),$$

$$h^{\rho_j} L_j(\eta, x, \beta) = L_j'(\eta, h^\varepsilon x, h^\varepsilon \beta, h), \tag{3}$$

then $L_j'(\eta, x, \beta, h)$ satisfies the condition of Lemma 1.

We have

$$f\left(\overset{1}{L}, \overset{2}{\alpha}\right) = h^{-r}\Phi\left(h^{\rho}\overset{1}{L}, \overset{2}{\alpha}; h\right) =$$

$$= h^{-r}\Phi\left(\overset{1}{[\![}L'\left(-i\overset{2}{\frac{\partial}{\partial x}}, h^{\varepsilon}\overset{1}{x}, -ih\overset{1}{\frac{\partial}{\partial \alpha}}, h\right)]\!], \overset{2}{\alpha}; h\right). \tag{4}$$

We have shown in Sec. 11 of Chapter II that the operator

$$\Phi\left(\overset{1}{[\![}L'\left(i\overset{2}{\frac{\partial}{\partial x}}, h^{\varepsilon}\overset{1}{x}, -ih^{\varepsilon}\overset{1}{\frac{\partial}{\partial \alpha}}, h\right)]\!], \overset{2}{\alpha}; h\right)$$

can be expressed as the pseudodifferential operator

$$\psi\left(\overset{2}{\alpha}, i\overset{2}{\frac{\partial}{\partial x}}, h^{\varepsilon}\overset{1}{x}, -ih^{\varepsilon}\overset{1}{\frac{\partial}{\partial \alpha}}, h\right) \text{ with}$$

$$\psi(\alpha, \eta, x, \beta, h) = \Phi\left(\overset{1}{[\![}L'\left(\overset{2}{\eta}, x - \overset{1}{ih^{\varepsilon}}\frac{\partial}{\partial \eta}, \overset{1}{\beta}, h\right)]\!], \overset{2}{\alpha}; h\right) 1. \tag{5}$$

It follows from Lemma 1 that

$$\varphi(\alpha, \eta, x, \beta, h) =$$

$$= \Phi\left(L'\left(\overset{2}{\eta}, x - \overset{1}{ih^{\varepsilon}}\frac{\partial}{\partial \eta}, \overset{1}{\beta}, h\right), \overset{2}{\alpha}; h\right) 1 + h^{\varepsilon} f_h(\alpha, \eta, x, \beta), \tag{6}$$

where the family $\{f_h\} = \{R_{h^\varepsilon}1\}$ is bounded in $\mathcal{S}^\infty$ by the corollary to Lemma 1. It is obvious that

$$\Phi\left(L'\left(\overset{2}{\eta}, x - \overset{1}{ih^{\varepsilon}}\frac{\partial}{\partial \eta}, \overset{1}{\beta}, h\right), \overset{2}{\alpha}; h\right) 1 = \Phi(L'(\eta, x, \beta, h), \alpha; h), \tag{7}$$

so

$$f\left(\overset{1}{L}, \overset{2}{\alpha}\right) = h^{-r}\Phi\left(L'\left(i\overset{2}{\frac{\partial}{\partial x}}, h^{\varepsilon}\overset{1}{x}, -ih^{\varepsilon}\overset{1}{\frac{\partial}{\partial \alpha}}, h\right), \overset{2}{\alpha}; h\right) +$$

$$+ h^{\varepsilon - r} f_h\left(\overset{2}{\alpha}, i\overset{2}{\frac{\partial}{\partial x}}, h^{\varepsilon}\overset{1}{x}, -ih^{\varepsilon}\overset{1}{\frac{\partial}{\partial \alpha}}\right) =$$

$$= f\left(L\left(i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}, -i\overset{1}{\frac{\partial}{\partial \alpha}}\right), \overset{2}{\alpha}\right) +$$

$$+ h^{\varepsilon - r} f_h\left(\overset{2}{\alpha}, i\overset{2}{\frac{\partial}{\partial x}}, h^{\varepsilon}\overset{1}{x}, -ih^{\varepsilon}\overset{1}{\frac{\partial}{\partial \alpha}}\right),$$

and the lemma is proved.

Lemma 3. *Let $f(x, \alpha) \in \mathcal{S}^\infty(\mathbf{R}^n \times M^m)$ be the same as in the Main Theorem, and let $f_0(x, \alpha)$ be the principal part of $f(x, \alpha)$.*

Further, let $\mathscr{F}(x, \alpha)$ *be a* ρ'*-quasi-homogeneous in* x *function of degree* 0, *whose restriction to* $S_\rho^{n-1} \times M^m$ *vanishes in a neighbourhood of the zero-set of* $f_0(x, \alpha)$ *restricted to* $S_\rho^{n-1} \times M^m$. *Then, for any* N, *there exists such a* $g_N \in \mathscr{S}^\infty(\mathbf{R}^n \times M^m)$ *that*

$$f\left(\overset{1}{L}, \overset{2}{\alpha}\right) g_N = \mathscr{F} + R_N, \tag{8}$$

where $R_N \in \mathscr{S}^\infty(\mathbf{R}^n \times M^m)$ *and* $R_N = O_{\mathscr{L}}(|x|^{-N})$.

Proof. Define $F(\eta, x, \beta, \alpha; h)$ by

$$h^r f(L(\eta, x, \beta), \alpha) = F(\eta, h^{\rho'} x, h\beta, \alpha; h).$$

By Lemma 2,

$$h^r f\left(\overset{1}{L}, \overset{2}{\alpha}\right) = F\left(i\overset{2}{\frac{\partial}{\partial x}}, h^{\rho'}\overset{1}{x}, -ih\overset{1}{\frac{\partial}{\partial \alpha}}, \overset{2}{\alpha}; h\right) + \\ + h^\varepsilon f_h\left(\overset{2}{\alpha}, i\overset{2}{\frac{\partial}{\partial x}}, h^\varepsilon \overset{1}{x}, -ih^\varepsilon \overset{1}{\frac{\partial}{\partial \alpha}}\right). \tag{9}$$

Furthermore, by virtue of ρ-quasi-homogeneity of $f(L(\eta, x, \beta), \alpha)$ in x and β,

$$F\left(i\overset{2}{\frac{\partial}{\partial x}}, h^{\rho'}\overset{1}{x}, -ih\overset{1}{\frac{\partial}{\partial \alpha}}, \overset{2}{\alpha}; h\right) = \\ = F_0\left(i\overset{2}{\frac{\partial}{\partial x}}, h^{\rho'}\overset{1}{x}, -ih\overset{1}{\frac{\partial}{\partial \alpha}}, \overset{2}{\alpha}\right) + \\ + h^\delta F_h\left(i\overset{2}{\frac{\partial}{\partial x}}, h^{\rho'}\overset{1}{x}, -ih\overset{1}{\frac{\partial}{\partial x}}, \overset{2}{\alpha}\right), \tag{10}$$

where $F_0(\eta, x, \beta, \alpha)$ is the principal part of $f(L(\eta, x, \beta), \alpha)$, $0 < \delta \leqslant \varepsilon$ and $\{F_n\}$ is a family bounded in $\mathscr{S}^\infty$, $F_0(0, x, 0, \alpha)$ being obviously equal to $f_0(x, \alpha)$.

By Taylor's expansion in $ih^{-\varepsilon}\frac{\partial}{\partial x}$ and $-i\frac{\partial}{\partial \alpha}$, for any positive ν we get:

$$h^r f\left(\overset{1}{L}, \overset{2}{\alpha}\right) = f_0(h^{\rho'} x, \alpha) + \\ + h^\delta \mathscr{P}_\nu\left(\frac{i}{h^\varepsilon}\overset{2}{\frac{\partial}{\partial x}}, -i\overset{1}{\frac{\partial}{\partial \alpha}}, h^\varepsilon \overset{1}{x}, \overset{2}{\alpha}, h\right) + \\ + h^\nu Q_\nu\left(\frac{i}{h}\overset{2}{\frac{\partial}{\partial x}}, -i\overset{1}{\frac{\partial}{\partial \alpha}}, i\overset{2}{\frac{\partial}{\partial x}}, h^\varepsilon \overset{1}{x}, \overset{2}{\alpha}, h\right), \tag{11}$$

where $\mathscr{P}_\nu(u, v, x, \alpha, h)$ is a polynomial in u, v with coefficients bounded in $\mathscr{S}^\infty$ (if regarded as families of functions of x, α), and

$Q_\nu(u, v, \eta, x, \beta, \alpha)$ is a polynomial in u with coefficients bounded in $\mathscr{S}^\infty$ as $h \to +0$.

Let Λ be a C^∞ function, $\Lambda\colon \mathbf{R}^n \to \mathbf{R}^1$, such that $\Lambda > 0$ and $\Lambda(x) = |x|$ for large x. Set $\omega = \Lambda^{-\rho'} x$, $\sigma = \Lambda^{-\varepsilon} x$, $p = i\Lambda^{\varepsilon}\frac{\partial}{\partial x}$. Replace in (11) h by $\Lambda^{-1}(x)$ acting first:

$$\overset{1}{\Lambda}{}^{-r}\overset{2}{[\![} f\left(\overset{1}{L}, \overset{2}{\alpha}\right)]\!] = f_0(\omega, \alpha) + \overset{1}{\Lambda}{}^{-\delta}\mathscr{P}_\nu\left(\overset{2}{p}, -i\overset{1}{\frac{\partial}{\partial\alpha}}, \overset{1}{\sigma}, \overset{2}{\alpha}, \overset{1}{\Lambda^{-1}}\right) + \overset{1}{\Lambda}{}^{-\nu}Q_\nu\left(\overset{2}{p}, -i\overset{1}{\frac{\partial}{\partial\alpha}}, i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{\sigma}, \overset{1}{\Lambda^{-1}}\right).$$

Thus (8) becomes

$$\left[f_0(\omega, \alpha) + \overset{1}{\Lambda}{}^{-\delta}\mathscr{P}_\nu\left(\overset{2}{p}, -i\overset{1}{\frac{\partial}{\partial\alpha}}, \overset{1}{\sigma}, \overset{2}{\alpha}, \overset{1}{\Lambda^{-1}}\right) + \overset{1}{\Lambda}{}^{-\nu}Q_\nu\left(\overset{2}{p}, -i\overset{1}{\frac{\partial}{\partial\alpha}}, i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{\sigma}, \overset{2}{\alpha}, \overset{1}{\Lambda^{-1}}\right)\right] g_N = \Lambda^{-r}\mathscr{F}(\omega, \alpha) + \Lambda^{-r}\mathbf{R}_N.$$

We will find such a g_N that

$$\left[f_0(\omega, \alpha) + \overset{1}{\Lambda}{}^{-\delta}\mathscr{P}_\nu\left(\overset{2}{p}, -i\overset{1}{\frac{\partial}{\partial\alpha}}, \overset{1}{\sigma}, \overset{2}{\alpha}, \overset{1}{\Lambda^{-1}}\right)\right] g_N = \Lambda^{-r}\mathscr{F}(\omega, \alpha) + O_{\mathscr{L}}(|x|^{-N-r/\varepsilon}) \tag{12}$$

and

$$\overset{1}{\Lambda}{}^{r-\nu}Q_\nu\left(\overset{2}{p}, -i\overset{1}{\frac{\partial}{\partial\alpha}}, i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{\sigma}, \overset{2}{\alpha}, \overset{1}{\Lambda^{-1}}\right) g_N = O_{\mathscr{L}}(|x|^{-N}). \tag{13}$$

We first solve (12). By the perturbation theory, we write

$$g_N = \frac{1}{f_0(\omega, \alpha)} \times \sum_{j=0}^{N_1} (-1)^j [\![\frac{1}{f_0\left(\overset{1}{\omega}, \overset{1}{\alpha}\right)}\overset{1}{\Lambda}{}^{-\delta}\mathscr{P}_\nu\left(\overset{2}{p}, -i\overset{2}{\frac{\partial}{\partial\alpha}}, \overset{1}{\sigma}, \overset{3}{\alpha}, \overset{1}{\Lambda^{-1}}\right)]\!]^j \times \Lambda^{-r}\mathscr{F}(\omega, \alpha). \tag{14}$$

Note that the right-hand members of (14) makes sense (and belongs to $\mathscr{S}^\infty$) because $\mathscr{F}(x, \alpha)$ vanishes in the vicinity of the zero-set of $f_0(\omega, \alpha)$, and $\mathscr{P}_\nu\left(\overset{2}{p}, -i\overset{1}{\frac{\partial}{\partial\alpha}}, \overset{1}{\sigma}, \overset{3}{\alpha}, \overset{1}{\Lambda^{-1}}\right)$ is a differential (hence local) operator.

Substituting (14) into (12) gives the condition

$$[\![\frac{1}{f_0\begin{pmatrix}1 & 2\\ \omega, & \alpha\end{pmatrix}} \overset{1}{\Lambda}{}^{-\delta}\mathscr{P}_\nu\left(\overset{2}{p}, -i\overset{2}{\frac{\partial}{\partial\alpha}}, \overset{1}{\sigma}, \overset{3}{\alpha}, \overset{2}{\Lambda}{}^{-1}\right)]\!]^{N_1+1}\times$$

$$\times\Lambda^{-r}\mathscr{F}(\omega, \alpha)=O_{\mathscr{L}}(|x|^{-N-r/\varepsilon}). \qquad (15)$$

To estimate the left-hand members of (15) note that for large x,

$$p\Lambda^s\varphi(\sigma)=\Lambda^s[\mathscr{D}_s, \varphi](\sigma), \qquad (16)$$

where $\mathscr{D}_s$ is a first-order differential operator.

Using this fact, one sees by elementary calculation that

$$\left\| [\![\frac{1}{f_0\begin{pmatrix}1 & 2\\ \omega, & \alpha\end{pmatrix}} \overset{1}{\Lambda}{}^{-\delta}\mathscr{P}_\nu\left(\overset{2}{p}, -i\overset{2}{\frac{\partial}{\partial\alpha}}, \overset{1}{\sigma}, \overset{3}{\alpha}, \overset{1}{\Lambda}{}^{-1}\right)]\!]^{N_1+1}\times\right.$$

$$\left.\times\Lambda^{-r}\mathscr{F}(\omega, \alpha)\right\|_{C^k_{\frac{r+(N_1+1)\delta}{\varepsilon}}}<\infty \qquad (17)$$

for any k. Pick such N_1 that $(N_1+1)\,\delta\geqslant\varepsilon N$, then (15) is satisfied. Similarly, (13) is satisfied provided that $\nu>N$, and the lemma is proved.

Definition 2. *We say that an infinitely differentiable function $f(x)$ belongs to the class C_l^∞ if*

$$\|f\|_{C_l^s}\overset{\text{def}}{=\!=}\sup_{x\in\mathbf{R}^n,\ |\alpha|\leqslant s}\left|(1+x^2)^{\frac{l}{2}}\frac{\partial^{|\alpha|}}{\partial x^\alpha}f(x)\right|<\infty$$

for any natural s.

As before, let $\rho_j\geqslant 1$, $j=1, \ldots, n$, and $\Lambda(x)=\left(\sum_{j=1}^n x_j^{2/\rho_j}\right)^{1/2}$.

Given a C^∞-smooth function $\mathscr{S}(x, \xi)$ defined on $\mathbf{R}_x^n\times\mathbf{R}_\xi^n$, set

$$\Phi(\xi, x)=\Lambda(x)\,\mathscr{S}(\xi, [\Lambda(x)]^{-\rho}x),$$
$$\Phi_1=\operatorname{Re}\Phi,\quad \Phi_2=\operatorname{Im}\Phi.$$

Lemma 4. *Let $g(x, \xi)$ be a function belonging (in x and ξ jointly) to the class $\mathbf{C}_m^\infty$ and vanishing for $|\xi|>\xi_0$ and for $|x|<\delta$, and suppose that Φ satisfies the conditions*

(1) *for $|\xi'|\leqslant\xi_0$, $|\xi''|\leqslant\xi_0$, the mapping $x\to x+\frac{\delta\Phi_1}{\delta\xi}(\xi', \xi'', x)$ is a diffeomorphism of the outside of the ball $|x|<\delta$ into $\mathbf{R}^n$;*

(2) Φ_2 *is non-negative.*
Then the operator

$$T \overset{\text{def}}{=} e^{i\Phi\left(\overset{2}{p},\, \overset{1}{x}\right)} g\left(\overset{2}{p},\, \overset{1}{x}\right),$$

where $p = -i\dfrac{\partial}{\partial x}$, *sends every function* $f \in H^{-s}$, $s \geqslant 0$, *into a function belonging to the class* $C^{\infty}_{-2s-2m-\frac{1}{2}n-\varepsilon}$, $\forall \varepsilon > 0$.

Proof.
(1) Set

$$T_0 = e^{i\Phi_1\left(\overset{2}{p},\, \overset{1}{x}\right)} g\left(\overset{2}{p},\, \overset{1}{x}\right),$$

$$T_0^* = e^{-i\Phi_1\left(\overset{1}{p},\, \overset{2}{x}\right)} g\left(\overset{1}{p},\, \overset{2}{x}\right),$$

and consider the operator

$$A_{l,r} \overset{\text{def}}{=} T_0 (1 + x^2)^l \varphi_r (x) T_0^*,$$

where $\varphi_r \in C_0^{\infty}$, $0 \leqslant \varphi_r(x) \leqslant 1$, $\varphi_r(x) = 1$ for $|x| < r$.
We claim that

$$A_{l,r} = Q_{l,r}\left(\overset{2}{x},\, \overset{1}{p},\, \overset{3}{p}\right), \tag{18}$$

where

$$\begin{aligned} Q_{l,r}(x, \xi', \xi'') = {} & g(\xi'', \tilde{x}(x, \xi', \xi'')) \times \\ & \times \bar{g}(\xi', \tilde{x}(x, \xi', \xi''))\,(1 + \tilde{x}^2(x, \xi', \xi''))^l \times \\ & \times \varphi_r(\tilde{x}(x, \xi', \xi''))\,|J^{-1}(\xi', \xi'', \tilde{x}(x, \xi', \xi''))| \end{aligned}$$

and $\tilde{x}(x, \xi', \xi'')$ is the solution of the equation

$$\tilde{x} + \frac{\delta\Phi_1}{\delta\xi}(\xi', \xi'', \tilde{x}) = x.$$

In fact, for any $h \in C_0^{\infty}$, we have

$$\begin{aligned} A_{l,r} h(x) = {} & \\ = {} & \frac{1}{(2\pi)^{2n}} \int\limits_{\mathbf{R}^{4n}} \exp\left[i\left(-x''\xi'' + x'\xi'' - x'\xi' + x\xi' + \Phi_1(\xi', x') - \right.\right. \\ & \left.\left. - \Phi_1(\xi'', x')\right)\right] g(\xi', x')\, g(\xi'', x')\,(1 + x'^2)^l\, \varphi_r(x')\, h(x'') \times \\ & \times dx''\, d\xi''\, dx'\, d\xi'. \end{aligned}$$

On making the same change of variables in the last integral as in the proof of Theorem 1.1 of Chapter III, we obtain (18).
(2) The symbol $Q_{l,r}$ can be estimated as follows:

$$\| Q_{l,r} \|_{\mathscr{B}_{0,k,k}(\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n)} \leqslant C_{l,k},$$

where $k > l + m + n$ and $C_{l,k}$ is independent of r. Therefore, we obtain, for any $h \in C_0^\infty$:

$$\int_{\mathbf{R}^n} |T_0^* h(x)|^2 (1+x^2)^l \varphi_r(x)\,dx = |(h, A_{l,r}h)_{L_2}| \leqslant$$
$$\leqslant \|h\|_{H_k}^2 \|A_{l,r}\|_{H_k \to H_{-k}} \leqslant c_{l,k} \|h\|_{H_k}^2$$

(note that $(T^*u, v)_{L_2} = (u, Tv)_{L_2}$). Passing to the limit $r \to \infty$ in the last inequality gives

$$\|T_0^* h\|_{H_l}^2 \leqslant c_{l,k} \|h\|_{H_k}^2,$$

so

$$T_0^* \in \mathrm{Hom}\,(H_k, H_l)$$

for $k > l + m + n$. Let $u, v \in C_0^\infty$, then

$$|(T_0 u, v)_{L_2}| = |(u, T_0^* v)_{L_2}| \leqslant \|u\|_{H_{-l}} \|T_0^*\|_{H_k \to H_l} \|v\|_{H_k},$$

so

$$\|T_0 u\|_{H_{-k}} \leqslant \|T_0^*\|_{H_k \to H_l} \|u\|_{H_{-l}},$$

hence

$$T_0 \in \mathrm{Hom}\,(H_{-l}, H_{-k})$$

for $k > l + m + n$.

(3) We have

$$T = [\![T_0 \left(1 + \overset{2}{x}{}^2\right)^r]\!]\, e^{-\Phi_2\left(\overset{3}{p}, \overset{1}{x}\right)} e\left(\overset{3}{p}, \overset{1}{x}\right) \left(1 + \overset{1}{x}{}^2\right)^{-r}, \tag{19}$$

where $e(\xi, x)$ is such an infinitely differentiable function that $e(\xi, x) = 1$ for $|\xi| \leqslant \xi_0$ and $e(\xi, x) = 0$ for $|\xi| \geqslant 2\xi_0$. The operator $T_0(1 + x^2)^r$ belongs to $\mathrm{Hom}\,(H_{-l+2r}, H_{-k})$ for $k > l + m + n$. Let us estimate the symbol

$$e^{-\Phi_2(\xi, x)} e(\xi, x)(1 + x^2)^{-r} \overset{\mathrm{def}}{=\!=} h(\xi, x).$$

Let μ and ν be multi-indices of length n. Then

$$\left| \left(\frac{\partial}{\partial \xi}\right)^\mu \left(\frac{\partial}{\partial x}\right)^\nu e^{-\Phi_2(\xi, x)} e(\xi, x) \right| \leqslant c_{\mu\nu} (1 + x^2)^{|\mu|/4}$$

which implies that $h \in \mathscr{B}_{k,0}$ for $r > \frac{k}{4} + \frac{3}{8} n$. Therefore, it follows from (19) that

$$T \in \mathrm{Hom}\left(H_{-s}, H_{-2s-2m-\frac{7}{2}n-\varepsilon}\right)$$

for every $s \geqslant 0$ and $\varepsilon > 0$.

(4) Let $\mathcal{P}(\xi)$ be a polynomial, then

$$\mathcal{P}(p)\,Tf(x) = \mathcal{P}\left(\overset{2}{p}\right) e^{i\Phi\left(\overset{2}{p},\ \overset{1}{x}\right)} g\left(\overset{2}{p},\ \overset{1}{x}\right) f(x) =$$
$$= e^{i\Phi\left(\overset{2}{p},\ \overset{1}{x}\right)} g_1\left(\overset{2}{p},\ \overset{1}{x}\right) f(x),$$

where $g_1(\xi, x) = \mathcal{P}(\xi)\,g(\xi, x)$ satisfies the conditions of the lemma as well as $g(\xi, x)$. It follows that if $f \in H_{-s}$, then

$$\mathcal{P}(p)\,Tf \in H_{-2s-2m-\frac{7}{2}n-\varepsilon},$$

hence $\mathcal{P}(p)\,Tf$ belongs to $H^l_{2s-2m-\frac{7}{2}n-\varepsilon}$ for any l (see Sec. 11 of Chapter II). To complete the proof, we must note the embedding

$$H^l_k \subset C_k^{l-\left[\frac{n}{2}\right]-1}, \quad \forall k \in \mathbf{R}.$$

The lemma is proved.

Let $\mathcal{K}$ be a canonical operator on an n-dimensional Lagrangean manifold with a complex germ dependent on a parameter $\omega \in \mathbf{R}^n$ in such a way that $\mathcal{K}\varphi$ induces class $(\mathcal{K}\varphi)(y, \omega)$ of equivalent modulo h^∞ (in the natural sense) series composed of a function belonging to $C^\infty_{\mathcal{L}}(\mathbf{R}^{2n})$. Then, for any $H \in \mathcal{S}^\infty(\mathbf{R}^{2n})$,

$$H\left(\overset{2}{y},\ -ih\overset{1}{\frac{\partial}{\partial y}} + \omega\right)[(\mathcal{K}\varphi)(y, \omega)] = (\mathcal{K}R_H\varphi)(y, \omega),$$

where R_H is the transfer operator (see Theorem 4.1).

Lemma 5.

$$[\![H\left(\overset{2}{y},\ -ih\overset{1}{\frac{\partial}{\partial y}}\right)]\!]\,[\![(\mathcal{K}\varphi)\left(\overset{2}{y},\ -ih\overset{1}{\frac{\partial}{\partial y}}\right)]\!] = (\mathcal{K}R_H\varphi)\left(\overset{2}{y},\ -ih\overset{1}{\frac{\partial}{\partial y}}\right).$$

Proof. The pair (A_1, A_2), where $A_1 = -ih\frac{\partial}{\partial y}$, $A_2 = y$, is a system of generators for a nilpotent Lie algebra. Its ordered representation (see Sec. 9 of Introduction) is $\left(\overset{1}{L_1}, \overset{1}{L_2}\right)$, where $L_1 = -ih\frac{\partial}{\partial x_2} + x_1$, $L_2 = x_2$. Then one has for any $f \in \mathcal{S}^\infty(\mathbf{R}^{2n})$, $g \in C^\infty_{\mathcal{L}}(\mathbf{R}^{2n})$ (see Sec. 11 of Chapter II):

$$[\![f\left(\overset{1}{A_1}, \overset{2}{A_2}\right)]\!]\,[\![g\left(\overset{1}{A_1}, \overset{2}{A_2}\right)]\!] = \chi\left(\overset{1}{A_1}, \overset{2}{A_2}\right),$$

where $\chi(x_1, x_2) = f(L_1, L_2)\,g(x_1, x_2)$.

Using this formula and taking into account the condition of the lemma, we get the identity

$$[\![H\left(\overset{2}{y},\ -ih\overset{1}{\frac{\partial}{\partial y}}\right)]\!]\,[\![(\mathscr{K}\varphi)\left(\overset{2}{y},\ -ih\overset{1}{\frac{\partial}{\partial y}}\right)]\!]=$$

$$=\left[H\left(\overset{2}{y},\ -ih\overset{1}{\frac{\partial}{\partial y}}+\omega\right)((\mathscr{K}\varphi)(y,\ \omega))\right]_{\omega=-ih\overset{1}{\frac{\partial}{\partial y}},\ y=\overset{2}{y}}=$$

$$=[(\mathscr{K}R_H\varphi)(y,\ \omega)]_{\omega=-ih\overset{1}{\frac{\partial}{\partial y}},\ y=\overset{2}{y}}=(\mathscr{K}R_H\varphi)\left(\overset{2}{y},\ -ih\overset{1}{\frac{\partial}{\partial y}}\right),$$

and the lemma is proved.

Sec. 6. Cauchy Problem for Systems with Complex Characteristics

In this section the commutation formula of the canonical operator on a Lagrangean manifold with a complex germ and the pseudodifferential operator with a matrix valued symbol are introduced. We also consider a method for the construction of the formal asymptotic solution of the Cauchy problem for systems of the form

$$-ih\frac{\partial\Psi}{\partial t}+\mathscr{B}\left(\overset{2}{x},\ \overset{1}{\hat{p}},\ h\right)\Psi=0$$

$$\Psi|_{t=0}=\Psi_0(x,\ h) \tag{6.1}$$

where $\hat{p}=-ih\frac{\partial}{\partial x}$, $\mathscr{B}(x,p,h)$ is an $(m\times m)$-matrix with elements being smooth functions of $x\in\mathbf{R}^n$, $p\in\mathbf{R}^n$, $h\in[0,1]$ and growing (together with their derivatives) no faster than a power of $(|x|^2+|p|^2)$.

Denote

$$\left(i\frac{\partial}{\partial h}\right)^k\mathscr{B}(x,\ p,\ h)\Big|_{h=0}=\mathscr{B}^{(k)}(x,\ p).$$

We have

$$\mathscr{B}\left(\overset{2}{x},\ \overset{1}{\hat{p}},\ h\right)=\sum_{k=0}^{N}\frac{(-ih)^k}{k!}\mathscr{B}^{(k)}\left(\overset{2}{x},\ \overset{1}{\hat{p}}\right)+h^{N+1}Q'_N\left(\overset{2}{x},\ \overset{1}{\hat{p}},\ h\right), \tag{6.2}$$

where the symbol $Q'_N(x,p,h)$ of the operator $Q'_N\left(\overset{2}{x},\ \overset{1}{\hat{p}},\ h\right)$ is an $(m\times m)$-matrix with smooth elements uniformly bounded with respect to h.

Suppose the matrix $\mathscr{H}^{(0)}(x, p)$, $x \in \mathbf{R}^n$, $p \in \mathbf{R}^n$ is diagonalized by a smooth non-degenerate transformation. Denote by $\lambda_1(x, p), \ldots, \lambda_l(x, p)$, $l \leqslant m$, the eigenvalues of the matrix $\mathscr{H}^{(0)}(x, p)$. Suppose the multiplicity τ_j of the eigenvalue λ_j does not depend on (x, p). Introduce the following functions

$$\widetilde{H}^{(j)}(x, p) = \operatorname{Im} \lambda_j(x, p), \quad H^{(j)}(x, p) = \operatorname{Re} \lambda_j(x, p).$$

The inequalities

$$\widetilde{H}^{(j)}(x, p) \leqslant 0, \quad j = 1, \ldots, l$$

are assumed to be valid everywhere in the text below.

Let $E_j(x, p)$ be a projection operator on the subspace corresponding to the eigenvalue $\lambda_j(x, p)$ of the matrix $\mathscr{H}^0(x, p)$. We see that the matrix elements of $E_j(x, p)$, $j = 1, \ldots, l$ are smooth functions of the variables $x \in \mathbf{R}^n$, $p \in \mathbf{R}^n$ and the following equations are valid:

$$E_j(x, p) E_k(x, p) = \delta_{jk} E_j(x, p), \quad \sum_{j=1}^{l} \lambda_j E_j = \mathscr{H}^{(0)}. \tag{6.3}$$

Let $\Lambda^n = \{p = p(\alpha), q = q(\alpha)\}$ be a Lagrangean manifold with a complex germ r^n and $\mathscr{H}_j(q, p) = \lambda_j(q, p)$ be a Hamiltonian function. Consider the family $\{\Lambda^n_{j,t}, r^n_{j,t}\}$, $0 \leqslant t \leqslant T$ of Lagrangean manifolds with a complex germ generated by the complex canonical transformation with the Hamiltonian function $\mathscr{H}_j(q, p)$ from the Lagrangean manifold $\Lambda^n = \{p = p(\alpha), q = q(\alpha)\}$. Denote by M_j^{n+1} an $(n+1)$-dimensional manifold

$$M_j^{n+1} = \{p = p^{(j)}(\alpha, t),\ q = q^{(j)}(\alpha, t);$$
$$(p^{(j)}(\alpha, t),\ q^{(j)}(\alpha, t)) = g^t_{H^{(j)}}(p(\alpha), q(\alpha))\}$$

(with a complex germ $w^{(j)}(\alpha, t)$, $z^{(j)}(\alpha, t)$, a dissipation function $D^{(j)}$ and a potential $E^{(j)}$ corresponding to the family $\{\Lambda^n_{j,t}, r^n_{j,t}\}$).

Assume the following inequality being valid:

$$\min_{j \neq i} \inf_{(p^{(j)}, q^{(j)}) \in M_j^{n+1}} |\lambda_i(p^{(j)}, q^{(j)}) - \lambda_j(p^{(j)}, q^{(j)})| \geqslant \delta,$$

where $\delta > 0$ is a positive number.

We shall define asymptotic series in smooth m-dimensional vector-functions similarly to Sec. 1 of Ch. IV.

Let $\mathscr{A}^m(M_j^{n+1})$ be the set of all equivalence classes $D^{(j)}$-asymptotic series on M_j^{n+1}.

Let $\mathscr{A}^m(\mathbf{R}^m \times [0, T])$ be the set of all equivalence classes h-asymptotic series on $\mathbf{R}^n \times [0, T]$.

Define the canonical operator on M_j^{n+1} by the mapping

$$\mathscr{K}_{M_j^{n+1}}: \mathscr{A}^m(M_j^{n+1}) \to \mathscr{A}^m(\mathbf{R}^n \times [0, T])$$

similarly to Sec. 4, Ch. IV.

Now introduce the classes of operators defined in $\mathscr{A}^m(M_j^{n+1})$.

Definition 6.1. Operator $\mathscr{A}$ defined in $\mathscr{A}^m(M_j^{n+1})$ and depending on $h \to +0$ is of the $\mathscr{P}_j$ class if, for any admissible γ-patch $U_I^\gamma \in M_j^{n+1}$ and the corresponding functional space $\mathscr{A}^m(U_I^\gamma)$, the operator $\mathscr{A}$ is equivalent to the $D^{(j)}$-asymptotic differential operator

$$\sum_{|l|\geqslant 0} a_l(\alpha, t, h)\left(\frac{\partial}{\partial\alpha}\right)^l,$$

the coefficients a_l (α, t, h) being such $(m \times m)$-matrices that

$$(a_l)_{i,k} = \mathcal{O}_{D^{(j)}}(h^{(|l|+1)/2}) + \langle \nabla_{p,q}\widetilde{H}^j, \mathcal{O}_{D^{(j)}}(h^{|l|/2})\rangle +$$
$$+ h^{1/2} O_{D^{(j)}}(h^{(|l|-1)/2}), \ |l| > 0,$$
$$(a_0)_{i,k} = \mathcal{O}_{D^{(j)}}(h^{1/2}) + \langle \nabla_{p,q}\widetilde{H}^{(j)}, \mathcal{O}_{D^{(j)}}(h^{(0)})\rangle, \quad i, k = 1, \ldots, m.$$

Lemma 6.1. *Let $\mathscr{A} \in \mathscr{P}_j$, then the operator $1 + I\mathscr{A}$ is a quasi-identity.*

Definition 6.2. *An operator B defined in $\mathscr{A}^m(M_j^{n+1})$ and depending on the parameter $h \to +0$ is of the class $\mathscr{L}_j$ if for any admissible γ-patch $U_I^\gamma \in M_j^{n+1}$ and the corresponding functional space $\mathscr{A}^m(M_j^{n+1})$, the operator B is equivalent to the $D^{(j)}$-asymptotic differential operator*

$$\sum_{|l|\geqslant 0} b_l(\alpha, t, h)\left(\frac{\partial}{\partial\alpha}\right)^l,$$

the coefficients $b_l(\alpha, t, h)$ being such $(m \times m)$-matrices that

$$(b_l)_{i,k} = \mathcal{O}_{D^{(j)}}(h^{(|l|+1)/2}) + h^{1/2} O_{D^{(j)}}(h^{(|l|-1)/2}), \ |l| > 0,$$
$$(b_0)_{i,k} = \mathcal{O}_{D^{(j)}}(h^{1/2}), \ i, k = 1, \ldots, m.$$

The following lemma is obvious.

Lemma 6.2.

(1) *For any B from $\mathscr{L}_j$ the operator $1 + B$ is a quasi-identical $D^{(j)}$-asymptotic operator.*

(2) *$B_1, B_2 \in \mathscr{L}_j \Rightarrow B_1B_2 \in \mathscr{L}_j$.*

(3) *Let $\mathscr{A} \in \mathscr{P}_j$, then $I\mathscr{A} \in \mathscr{L}_j$.*

(4) *Let $B \in \mathscr{L}_j$, then $B \in \mathscr{P}_j$.*

Definition 6.3. *Operator C defined in $\mathscr{A}^m(M_j^{n+1})$ and depending on the parameter $h \to 0$ is of the class $\mathscr{N}_j$ if, for any admissible*

γ-patch $U_I^\gamma \subset M_j^{n+1}$ and the corresponding functional space, the operator C is equivalent to the $D^{(j)}$-asymptotic differential operator

$$C=\sum_{|l|\geqslant 0} c_l(\alpha,\ t,\ h)\left(\frac{\partial}{\partial\alpha}\right)^l,$$

the coefficients $c_l(\alpha, t, h)$ being such $(m \times m)$-matrices that

$$(c_l)_{i,k}=\mathcal{O}_{D^{(j)}}(h^{|l|/2}),\qquad |l|\geqslant 0,\quad i,\ k=1,\ \ldots,\ m.$$

Lemma 6.3. *Let $C \in \mathcal{N}_j$, then*
(1) *if $\mathcal{A} \in \mathcal{P}_j$, then $\mathcal{A}C \in \mathcal{P}_j$, $C\mathcal{A} \in \mathcal{P}_j$;*
(2) *if $B \in \mathcal{L}_j$, then $CB \in \mathcal{L}_j$, $BC \in \mathcal{L}_j$;*
(3) *if C_1, $C_2 \in \mathcal{N}_j$, then $[C_1, C_2] \in \mathcal{L}_j$.*]
The proof is immediate.

Note that we write $\mathcal{A} \in \mathcal{P}_j(u_I^\gamma)$ (corresp. $B \in \mathcal{L}_j(u_I^\gamma)$, $C \in \mathcal{N}_j(u_I^\gamma)$) if $\mathcal{A}$ (corresp. B, C) is a $D^{(j)}$-asymptotic differential operator with coefficients satisfying the conditions of Definition 6.1 (or 2, 3) in an admissible patch u_I^γ.

Let $\mathcal{K}_{\Lambda^n}$ be a canonical operator on (Λ^n, r^n). Suppose that the canonical operators $\mathcal{K}_{M_j^{n+1}}$, $j = 1, \ldots, l$ (cf. Sec. 4) correspond to the operator $\mathcal{K}_{\Lambda^n}$.

Similarly to Sec. 4, suppose that the initial value $\Psi_0(x, h)$ of problem (6.1) has the form

$$\Psi_0=\mathcal{K}_{\Lambda^n}\varphi_0, \tag{6.4}$$

where $\varphi_0 \in \mathcal{A}^m(\Lambda^n, r^n)$.

Let $F(x, p)$ be an $(m\times m)$-matrix with elements from $\mathcal{S}^\infty(\mathbf{R}^n\times\mathbf{R}^n)$. We have the following lemma.

Lemma 6.4. *For any $\varphi \in \mathcal{A}^m(M_j^{n+1})$ there is an equation*

$$[\![F\left(\overset{2}{x},\ \overset{1}{\hat{p}}\right)]\!]\,\mathcal{K}_{M_j^{n+1}}\varphi=\mathcal{K}_{M_j^{n+1}}R_j\varphi,$$

where R_j is an operator defined in $\mathcal{A}^m(M_j^{n+1})$:

$$R_j=\sum_{k\geqslant 0}(-ih)^k R_j^{(k)}$$

$$R_j^{(0)}=F(q^{(j)}(\alpha,\ t),\ p^{(j)}(\alpha,\ t))+\mathcal{E}_j$$

$$\mathcal{E}_j\in\mathcal{L}_j$$

$$\sum_{k\geqslant 1}(-ih)^k R_j^{(k)}\in\mathcal{L}_j.$$

The operator $R_j^{(k)}$ restricted on any admissible mapping of M_j^{n+1} is a differential operator with the coefficients being smooth functions of h.

Proof. Consider the local canonical operator $\mathcal{K}_I^\gamma$. For the sake of simplicity we restrict ourselves to the case of a non-singular patch u of a non-singular zone, the general case being treated along the same lines.

Let $\varphi \in \mathcal{A}(U)$ where U is a non-singular patch. We have (cf. Sec. 4)

$$\mathcal{K}^{(0)}\varphi = e^{i\left(\frac{1}{h}\Phi^{(j)}(\alpha,\ t)-\frac{1}{2}\operatorname{Arg} J(\alpha,\ t)\right)}|J(\alpha,\ t)|^{-1/2}\times \\ \times L_0\varphi(\alpha,\ t)_{\alpha=\Pi^{-1}(x)}, \tag{6.5}$$

where $\Phi^{(j)}(\alpha, t)$ is a phase in the non-singular patch U. On applying the commutation formula of Hamiltonian and complex exponential we have

$$F\left(\overset{1}{\hat{p}},\ \overset{2}{x}\right)[\mathcal{K}^{(0)}\varphi] = e^{\frac{i}{h}S^{(j)}(x,\ t)}\sum_{k\geqslant 0}(-ih)^k G_k(S^{(j)})(J)^{-\frac{1}{2}}L_0\varphi,$$

where

$$S^{(j)}(x,\ t) = \Phi^{(j)}(\alpha,\ t)\cdot\Pi^{-1}.$$

The operators G_k have the form*

$$G_k(S^{(j)}) = \sum_{|\gamma|\geqslant 0}\left\{\sum_{|\alpha|=k}^{2k}\sum_{|\beta|=0}^{k}\frac{1}{\gamma!}\left(\frac{\partial\operatorname{Im}S^{(j)}}{\partial x}\right)^\gamma\times\right. \\ \left.\times\frac{\partial^{\alpha+\gamma}F}{\partial p^{\alpha+\gamma}}\left(x,\ \frac{\partial\operatorname{Re}S^{(j)}}{\partial x}\right)\Phi_{\alpha\beta}^{(k)}\left(S^{(j)}\right)\frac{\partial^\beta}{\partial x^\beta}\right\}, \tag{6.6}$$

where $\Phi_{\alpha\beta}^{(k)}$ are polynomials with respect to the derivatives

$$\frac{\partial^{|\gamma|}S^{(j)}}{\partial x^\gamma},\qquad |\gamma| = 2,\ \ldots,\ 2k$$

having matrix coefficients. We have $\Phi_{\alpha\beta}^{(0)} = 1$. The estimates

$$\frac{\partial\operatorname{Im}S^{(j)}}{\partial x} = O_{D^{(j)}}(h^{1/2}), \tag{6.7}$$

$$\frac{\partial\operatorname{Re}S^{(j)}}{\partial x} - p^{(j)}(\alpha(x,\ t),\ t) = O_{D^{(j)}}(h^{1/2})$$

proceeding from the theorems in Ch. IV. Here the functions $\alpha(x, t)$ are solutions of the equation $q(\alpha, t) = x$.

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* From this point on, two D-asymptotic operators are defined to be equal in the sense of the equivalence of the asymptotic series (cf. page 480).

On applying (6.6) we obtain

$$G_0(S^{(j)}) = F\left(x, \frac{\partial \operatorname{Re} S^{(j)}}{\partial x}\right) + \\ + \sum_{|\gamma|\geqslant 1} \frac{1}{\gamma!}\left(i\frac{\partial \operatorname{Im} S^{(j)}(x,t)}{\partial x}\right)^{\gamma} \frac{\partial^{\gamma}}{\partial p^{\gamma}} F\left(x, \frac{\partial \operatorname{Re} S^{(j)}}{\partial x}\right). \tag{6.8}$$

On applying (6.7) and the Gårding inequality, we can put $G_0(S^{(j)})$ in the form

$$G_0 = F(q^{(j)}(\alpha(x, t), t), p^{(j)}(\alpha(x, t), t)) + G_0', \tag{6.9}$$
$$G_0' \in \mathcal{L}_j(U).$$

Note that according to Sec. 6 of Ch. IV, there exists such an operator $R \in \mathcal{N}^{*}_j(U)$ that

$$L_0 R = 1 \tag{6.10}$$

in the ring of homomorphisms over $\mathcal{A}^m(U)$.

Put (6.5) into the form

$$F\left(\overset{2}{x}, \overset{1}{\hat{p}}\right)[\mathcal{K}^{(0)}\varphi] = \mathcal{K}^{(0)}P\varphi,$$

where

$$P = R\left\{\sum_{k\geqslant 0}(-ih)^k (J)^{1/2} G_k(S^{(j)})(J)^{-1/2}\right\} L_0 \overset{\text{def}}{=} \sum_{k\geqslant 0}(-ih)^k P^{(k)}, \tag{6.11}$$

where

$$P^{(k)} = R(J)^{1/2} G_k(S^{(j)})(J)^{-1/2} L_0.$$

On applying Eq. (6.9) and the definition of the operators L and R (cf. Sec. 6, Ch. IV) we put the operator $P^{(0)}$ in the form

$$P^{(0)} = F(q^{(j)}(\alpha, t), p^{(j)}(\alpha, t)) + P', \tag{6.12}$$
$$P' \in \mathcal{L}_j(U).$$

In the same way for any patch U_I^{γ} we obtain

$$F\left(\overset{2}{x}, \overset{1}{\hat{p}}\right)\mathcal{K}_I^{\gamma}\varphi = \mathcal{K}_I^{\gamma} P_I^{\gamma}\varphi,$$

where the operator P_I^{γ} is defined in the patch U_I^{γ} and satisfies the conditions of the lemma.

Let $\{U_I^{\gamma}, \Pi_{\gamma}^{I}\}$ be an atlas of M_j^{n+1} which may be different from the weighting one, and let $\varphi \in \mathcal{A}(U_I^{\gamma})$. Then (cf. Sec. 4)

$$\mathcal{K}_{M_j^{n+1}}\varphi = \mathcal{K}_I^{\gamma}\sum_i V_{I(i)I}^{\gamma(i)\gamma}\rho_i\varphi.$$

Hence

$$[\![F\left(\overset{2}{x}, \overset{1}{\hat{p}}\right)]\!](\mathscr{K}_{M_j^{n+1}}\varphi)=\mathscr{K}_I^\gamma P_I^\gamma \sum_i V_{I(i)I}^{\gamma(i)\gamma}\rho_i\varphi.$$

Let $U_I^\gamma \cap U_k^{\gamma'} \neq \varnothing$ and let $\varphi \in \mathscr{A}^m(U_I^\gamma \cap U_K^{\gamma'})$. Now prove the equation

$$P_K^{\gamma'}V_{KI}^{\gamma'\gamma}\varphi=V_{IK}^{\gamma\gamma'}P_I^\gamma\varphi. \tag{6.13}$$

Indeed,

$$F\left(\overset{2}{x}, \overset{1}{\hat{p}}\right)[\mathscr{K}_I^\gamma\varphi]=\mathscr{K}_I^\gamma P_I^\gamma\varphi=\mathscr{K}_K^{\gamma'}V_{IK}^{\gamma\gamma'}P_I^\gamma\varphi. \tag{6.14}$$

On the other hand,

$$F\left(\overset{2}{x}, \overset{1}{\hat{p}}\right)[\mathscr{K}_I^\gamma\varphi]=F\left(\overset{2}{x}, \overset{1}{\hat{p}}\right)[\mathscr{K}_K^{\gamma'}V_{KI}^{\gamma\gamma'}]=\mathscr{K}_K^{\gamma'}P_K^{\gamma'}V_{KI}^{\gamma\gamma'}\varphi.$$

The last equation and Eq. (6.14) involve Eq. (6.13) by the monomorphism property of the canonical operator.

Now define the operator R_K^γ in the patch U_K^γ of the manifold M_j^{n+1} by the formula

$$R_K^\gamma\big|_{U_K^\gamma}=\left(\sum_i V_{KI(i)}^{\gamma'\gamma'(i)}\rho_i\right)^{-1}P_K^{\gamma'}\left(\sum_i V_{KI(i)}^{\gamma\gamma'(i)}\rho_i\right), \tag{6.15}$$

where the sum has taken over all indices i such that $\operatorname{supp}\rho_i \cap \cap \bar{U}_K^\gamma \neq \varnothing$ (cf. Sec. 4, Ch. V). Since the operator $\sum_i V_{I(i)K}^{\gamma'(i)}\rho_i$ is quasi-identical, the operator $R_K^\gamma\big|_{U_K^\gamma}$ can be put into the form

$$R_K^\gamma\big|_{U_K^\gamma}=F(q^{(j)}, p^{(j)})\big|_{U_K^\gamma}+R_K^{'\gamma}, \tag{6.16}$$

where

$$R_K^{'\gamma}\in\mathscr{L}_j\big|_{U_K^\gamma}.$$

Let the patches $U_K^{\gamma'}$ and U_I^γ intersect. Prove that the operators $R_I^\gamma\big|_{U_I^\gamma}$ and $R_K^{\gamma'}\big|_{U_K^{\gamma'}}$ coincide on the space $\mathscr{A}^m(U_K^{\gamma'}\cap U_K^\gamma)$.

On applying (6.13) we have, for any $\mathscr{V}\in\mathscr{A}^m(U_K^{\gamma'}\cap U_I^\gamma)$

$$\begin{aligned}(R_K^{\gamma'}\big|_{U_K^{\gamma'}})\mathscr{V}&=\left[\left(\sum_i V_{KI(i)}^{\gamma'\gamma'(i)}\rho_i\right)^{-1}P_K^{\gamma'}\left(\sum_i V_{KI(i)}^{\gamma'\gamma'(i)}\rho_i\right)\right]\mathscr{V}=\\&=\left[\left(\sum_i V_{KI(i)}^{\gamma'\gamma'(i)}\rho_i\right)^{-1}P_K^{\gamma'}V_{KI}^{\gamma'\gamma}\left(\sum_i V_{II(i)}^{\gamma\gamma'}\rho_i\right)\right]\mathscr{V}=\\&=\left[\left(V_{IK}^{\gamma\gamma'}\sum_i V_{KI(i)}^{\gamma'\gamma'(i)}\rho_i\right)^{-1}P_I^\gamma\left(\sum_i V_{II(i)}^{\gamma\gamma(i)}\rho_i\right)\right]\mathscr{V}=(R_I^\gamma\big|_{U_I^\gamma})\mathscr{V}.\end{aligned}$$

Thus we obtain the operators R_K^γ considering the operator R_j defined on M_j^{n+1} on the patch U_K^γ. The following equation is valid:

$$F\left(\overset{2}{x},\ \overset{1}{\hat{p}}\right)[\mathscr{K}_{M_j^{n+1}}\varphi]=\mathscr{K}_{M_j^{n+1}}R_j\varphi.$$

Now the lemma follows from (6.16).

Let B_k, $k = 1, \ldots, l$ be the operators defined by the following formula

$$E_k\left(\overset{2}{x},\ \overset{1}{\hat{p}}\right)[\mathscr{K}_{M_j^{n+1}}]=\mathscr{K}_{M_j^{n+1}}B_k.$$

On applying Lemma 6.4

$$B_k=\sum_{l\geqslant 0}(-ih)^l\,B_k^{(l)},$$

where

$$B_k^{(0)}=E_k(q^{(j)},\ p^{(j)})+\mathscr{C}_j', \tag{6.17}$$

$$\mathscr{C}_j'\in\mathscr{L}_j.$$

Lemma 6.5. *The following equations are valid*:

(i) $\sum_{j=1}^{l} B_j^{(0)}=1$,

(ii) $B_r^{(0)}B_k^{(0)}=\delta_{rk}B_k^{(0)}$.

Proof. Without any loss of generality consider a non-singular patch of a non-singular zone. On applying Eq. (6.11) and the definition of the operators $B_j^{(0)}$

$$B_r^{(0)}B_k^{(0)}=\left(\sum_i V_{0I(i)}^{0\gamma(i)}\rho_i\right)^{-1}R\,(J)^{1/2}\,g_{0,\,r}g_{0,\,k}\,(J)^{-1/2}\,L_0\left(\sum_i V_{0I(i)}^{0\gamma(i)}\rho_i\right),$$

where

$$g_{0,\,r}=\sum_{|\gamma|\geqslant 0}\frac{1}{\gamma!}\left(\frac{\partial^\gamma}{\partial p^\gamma}E_r\right)\left(i\,\frac{\partial\,\mathrm{Im}\,S^{(j)}}{\partial x}\right)^\gamma.$$

Hence $\quad g_{0,\,r}g_{0,\,k}=\delta_{rk}g_{0,\,k}.$

Thus (ii) is proved. In the same way we can prove (i).

Theorem 6.1. (1) *The following equation is true for any* $j=1,\ \ldots,\ l$:

$$\left[-ih\frac{\partial}{\partial t}+\mathscr{H}\left(\overset{2}{x},\ \overset{1}{\hat{p}},\ h\right)\right][\mathscr{K}_{M_i^{n+1}}\varphi]=\mathscr{K}_{M_j^{n+1}}R_j\varphi,$$

where $R_j=\sum_{k\geqslant 0}(-ih)^k R_j^{(k)}$ *and the restriction of the operator* R_j *on an admissible patch of* M_j^{n+1} *is* $D^{(j)}$*-asymptotic differential operator. Operators* $R_j^{(k)}$ *are differential operators on* M_j^{n+1} *having*

smooth coefficients with respect to h, such that

$$R_j^{(0)}=\sum_{k\neq j}(\lambda_k(q^{(j)},p^{(j)})-\lambda_j(q^{(j)},p^{(j)}))E_k(q^{(j)},p^{(j)})+\mathscr{E}_j^{(1)},\ \mathscr{E}_j^{(1)}\in\mathscr{L}_j.$$

$$(2)\ B_j^{(0)}R_j^{(1)}B_j^{(0)}=B_j^{(0)}\left(\frac{d}{dt_j}-\frac{1}{2}\operatorname{tr}\frac{\partial^2\lambda_j}{\partial q\,\partial p}+\right.$$
$$\left.+\frac{\partial E_j}{\partial p}\frac{\partial(\mathscr{H}^{(0)}-\lambda_j)}{\partial q}+\mathscr{H}^{(1)}+\mathscr{E}_j^{(2)}\right)B_j^{(0)},\ \mathscr{E}_j^{(2)}\in\mathscr{P}_j,$$

where $\frac{d}{dt_j}$ *is a vector field of the form*

$$\frac{d}{dt_j}=\left\langle H_p^{(j)}(q^{(j)},p^{(j)}),\frac{\partial}{\partial q}\right\rangle-\left\langle H_q^{(j)}(q^{(j)},p^{(j)}),\frac{\partial}{\partial p}\right\rangle+\frac{\partial}{\partial t}.$$

Proof. Consider the operator $\hat{H}=-ih\frac{\partial}{\partial t}+\mathscr{H}\left(\overset{2}{x},\overset{1}{\hat{p}},h\right)$. On applying Eq. (6.2) and Lemma 6.4 we obtain

$$\hat{H}[\mathscr{K}_{M_j^{n+1}}\varphi]=\mathscr{K}_{M_j^{n+1}}R_j\varphi,$$

where

$$R_j=\mathscr{H}_j^{(0)}(q^{(j)},p^{(j)})-\lambda_j(q^{(j)},p^{(j)})+\mathscr{E}_j^{(3)},$$
$$\mathscr{E}_j^{(3)}\in\mathscr{L}_j\cap\mathscr{P}_j.$$

We have taken into account that in the region Ω_K the function

$$S_K^{\gamma}(x_K,\xi_{\bar{K}},t)=\bar{\Phi}^{\gamma}(\alpha,t)(\Pi_K^{\gamma})^{-1}$$

satisfies the dissipative Hamilton-Jacobi equation with the Hamiltonian function $\lambda_j(q,p)$ (cf. Sec. 1, Ch. IV). On applying (6.17), (6.3) and Lemma 6.4 we obtain the first statement of the theorem. To prove the second, consider the product of the operators

$$[\![E_j\left(\overset{2}{x},\overset{1}{\hat{p}}\right)]\!]\hat{H}[\mathscr{K}_{M_j^{n+1}}\varphi].$$

On applying the commutation formula, we obtain

$$[\![E_j\left(\overset{2}{x},\overset{1}{\hat{p}}\right)]\!]\hat{H}=E_j\left(\overset{2}{x},\overset{1}{\hat{p}}\right)\left(-ih\frac{\partial}{\partial t}\right)+[\![E_j\mathscr{H}^{(0)}\left(\overset{2}{x},\overset{1}{\hat{p}}\right)]\!]+$$
$$+\sum_{k\geqslant 1}\frac{(-ih)^k}{k!}[\![\frac{\partial^k}{\partial p^k}E_j\frac{\partial^k\mathscr{H}^{(0)}}{\partial x^k}\left(\overset{2}{x},\overset{1}{\hat{p}}\right)]\!]+\sum_{l\geqslant 1}\sum_{k\geqslant 0}(-ih)^{k+l}\times$$
$$\frac{\partial^k E_j}{\partial p^k}\frac{\partial^k\mathscr{H}^{(l)}}{\partial x^k}\left(\overset{2}{x},\overset{1}{\hat{p}}\right)=[\![E_j\left(\overset{2}{x},\overset{1}{\hat{p}}\right)]\!][\![-ih\frac{\partial}{\partial t}+\lambda_j\left(\overset{2}{x},\overset{1}{\hat{p}}\right)]\!]+$$
$$+\sum_{k\geqslant 1}\frac{(-ih)^k}{k!}[\![\frac{\partial^k E_j}{\partial p^k}\frac{\partial^k(\mathscr{H}^{(0)}-\lambda_j)}{\partial x^k}\left(\overset{2}{x},\overset{1}{\hat{p}}\right)]\!]+$$
$$+\sum_{l\geqslant 1}\sum_{k\geqslant 0}(-ih)^{k+l}[\![\frac{\partial^k E_j}{\partial p^k}\frac{\partial^k\mathscr{H}^{(l)}}{\partial x^k}\left(\overset{2}{x},\overset{1}{\hat{p}}\right)]\!].\qquad(6.18)$$

The following equation is a consequence of Sec. 4, Ch. V:

$$\left[-ih\frac{\partial}{\partial t}+\lambda_j\left(\overset{2}{x},\ \overset{1}{\hat{p}}\right)\right]\mathscr{K}_{M_j^{n+1}}\varphi=(-ih)\,\mathscr{K}_{M_j^{n+1}}r_j\varphi, \tag{6.19}$$

where φ is an element of $\mathscr{A}^m(M_j^{n+1})$, r_j is an operator on M_j^{n+1}

$$r_j=\frac{d}{dt_j}-\frac{1}{2}\operatorname{tr}\frac{\partial^2\lambda_j}{\partial q\,\partial p}+\mathscr{E}_j^{(4)},\quad \mathscr{E}_j^{(4)}\in\mathscr{F}_j. \tag{6.20}$$

On applying the operator $E_j\left(\overset{2}{x},\ \overset{1}{\hat{p}}\right)$ to both sides of (6.19) and taking into account (6.18) we obtain

$$(-ih)\,[\![E_j\left(\overset{2}{x},\ \overset{1}{\hat{p}}\right)]\!]\,(\mathscr{K}_{M_j^{n+1}}r_j\varphi)=[\![E_j\left(\overset{2}{x},\ \overset{1}{\hat{p}}\right)]\!]\,(\mathscr{K}_{M_j^{n+1}}R_j\varphi)-$$
$$-M_j\,[\mathscr{K}_{M_j^{n+1}}\varphi], \tag{6.21}$$

where

$$M_j=\sum_{k\geqslant 1}\frac{(-ih)^k}{k!}[\![\frac{\partial^k E_j}{\partial p^k}\frac{\partial^k(\mathscr{H}^{(0)}-\lambda_j)}{\partial x^k}\left(\overset{2}{x},\ \overset{1}{\hat{p}}\right)]\!]+$$
$$+\sum_{l\geqslant 1}\sum_{k\geqslant 0}\frac{(-ih)^{k+l}}{k!}[\![\frac{\partial^k E_j}{\partial p^k}\frac{\partial^k\mathscr{H}^{(l)}}{\partial x^k}\left(\overset{2}{x},\ \overset{1}{\hat{p}}\right)]\!].$$

On applying Lemma 6.4, we obtain

$$[\![E_j\left(\overset{2}{x},\ \overset{1}{\hat{p}}\right)]\!]\,[\mathscr{K}_{M_j^{n+1}}R_j\varphi]=\mathscr{K}_{M_j^{n+1}}(B_jR_j)\,\varphi. \tag{6.22}$$

Now calculate the product B_jR_j.

On applying (6.15) in a patch U_K^γ of the zone Ω_K, we have

$$B_jR_j\,|_{U_K^\gamma}=(B_j\,|_{U_K^\gamma})\,(R_j\,|_{U_K^\gamma})=$$
$$=\left(\sum_i V_{KI(i)}^{\gamma\gamma(i)}\rho_i\right)^{-1}P_K'^{\gamma}P_K^\gamma\left(\sum_i V_{KI(i)}^{\gamma\gamma(i)}\rho_i\right). \tag{6.23}$$

Without loss of generality consider the product $P_K'^{\gamma}P_K^\gamma$ corresponding to the non-singular patch of a non-singular zone. The sign $'$ denotes the quantities of the operator B_j.

For the sake of simplicity denote $P_K'^{\gamma}$ and P_K^γ by P' and P. By (6.11)

$$P'P=R\,(J)^{1/2}\Big(\sum_{k\geqslant 0}(-ih)^k\,G_k'(S^{(j)})\sum_{l\geqslant 0}(-ih)^l\,G_l(S^{(j)})\Big)(J)^{-1/2}L_0=$$
$$=R\,\{(J)^{1/2}\,(G_0'(S^{(j)})\,G_0(S^{(j)})+(-ih)\,\times$$
$$\times[G_1'(S^{(j)})\,G_0(S^{(j)})+G_0'(S^{(j)})\,G_1(S^{(j)})+g])\,(J)^{-1/2}\}\,L_0,$$
$$g\in\mathscr{F}_j(u). \tag{6.24}$$

On applying Lemma 6.4 (cf. (6.8)), we obtain

$$G_0 = \sum_{\gamma=0}^{2} \frac{1}{\gamma!} \frac{\partial^\gamma}{\partial p^\gamma} (\mathscr{H}^{(0)} - \lambda_j)\left(i \frac{\partial \operatorname{Im} S^{(j)}}{\partial x}\right)^\gamma + $$
$$+ O_{D^{(j)}}(h^{3/2})(1+g_0) + \langle \widetilde{H}_p^{(j)},\ O_{D^{(j)}}(h)\rangle,$$
$$G_0' = \sum_{|\gamma|=0}^{2} \frac{1}{\gamma!}\left(i \frac{\partial \operatorname{Im} S^{(j)}}{\partial x}\right)^\gamma \frac{\partial^\gamma}{\partial p^\gamma} E_j\left(x, \frac{\partial \operatorname{Re} S^{(j)}}{\partial x}\right) +$$
$$+ O_{D^{(j)}}(h^{3/2})(1+g_0'),\quad g_0,\ g_0' \in \mathscr{L}_j(u).$$

From the equalities

$$(\mathscr{H}^{(0)} - \lambda_j) E_j = E_j (\mathscr{H}^{(0)} - \lambda_j) = 0$$

we obtain

$$G_0 G_0' = G_0' G_0 = \langle O_{D^{(j)}}(h),\ \widetilde{H}_p^{(j)}\rangle + O_{D^{(j)}}(h^{3/2})(1+g''),\ g'' \in \mathscr{L}_j(u). \tag{6.25}$$

Since the operators R and L_0 belong to the class $\mathscr{L}_j$, we obtain by Lemma 6.3 and Eq. (6.25)

$$P'P = (-ih) R\{(J)^{1/2} G_1' G_0 (J)^{-1/2} +$$
$$+ (J)^{1/2} G_0' G_1 (J)^{-1/2}\} L_0 + g',\quad g' \in \mathscr{P}_j(u).$$

Similarly, on applying Lemma 6.3 and (6.25), we obtain

$$P'PP' = (-ih) R (J)^{1/2} [G_0' G_1 G_0' + \widetilde{g}] (J)^{-1/2} L_0,\quad \widetilde{g} \in \mathscr{P}_j(u).$$

Hence, again applying Lemma 6.3 and (6.11), we obtain

$$B_j R_j B_j = (-ih)\ [B_j^{(0)} R_j^{(1)} B_j^{(0)} + g_j],\quad g_j \in \mathscr{P}_j. \tag{6.26}$$

Now, taking into account the monomorphism property of the canonical operator and applying (6.21), (6.22), (6.26) and Lemma 6.3, we obtain

$$B_j^{(0)} R_j^{(1)} B_j^{(0)} = \Big[B_j^{(0)} \left(\frac{d}{dt_j} - \frac{1}{2} \operatorname{tr} \frac{\partial^2 \lambda_j}{\partial q\, \partial p}\right) +$$
$$+ \frac{\partial E_j}{\partial p} \frac{\partial (\mathscr{H}^{(0)} - \lambda_j)}{\partial q} + B_j^{(0)} \mathscr{H}^{(1)} + \mathscr{E}_j^{(2)}\Big] B_j^{(0)},\quad \mathscr{E}_j^{(2)} \in \mathscr{P}_j,$$

with $\quad \varphi = B_j^{(0)} \Psi,\qquad \Psi \in \mathscr{A}^m(M_j^{n+1}).$

The following equation

$$E_j \frac{\partial E_j}{\partial p} \frac{\partial (\mathscr{H}^{(0)} - \lambda_j)}{\partial q} = \frac{\partial E_j}{\partial p} \frac{\partial (\mathscr{H}^{(0)} - \lambda_j)}{\partial q}$$

is an easy corollary of Eq. (6.3) and the equation

$$E_j \frac{\partial E_j}{\partial p} + \frac{\partial E_j}{\partial p} E_j = \frac{\partial E_j}{\partial p}.$$

Finally, on applying Lemma 6.5 and (6.27), we obtain

$$B_j^{(0)}R_j^{(1)}B_j^{(0)}=B_j^{(0)}\left[\frac{d}{dt_j}-\frac{1}{2}\operatorname{tr}\frac{\partial^2\lambda}{\partial q\,\partial p}+\right.$$
$$\left.+\frac{\partial E_j}{\partial p}\frac{\partial(\mathscr{H}^{(0)}-\lambda_j)}{\partial q}+\mathscr{H}^{(1)}+\mathscr{E}_j^{(3)}\right]B_j^{(0)},\quad \mathscr{E}_j^{(3)}\in\mathscr{P}_j,$$

Q.E.D.

By Theorem 6.1 a formal asymptotic solution of Problem 6.1 can be constructed with the elements

$$\varphi_j(\alpha,\ t,\ h)\in\mathscr{A}^m(M_j^{n+1}),\quad j=1,\ \ldots,\ l,$$

satisfying the equations

$$R_j\varphi_j=0,\quad j=1,\ \ldots,\ l \tag{6.28}$$

$$\sum_{j=1}^{l}\varphi_j(\alpha,\ 0,\ h)=\varphi_0(\alpha,\ h), \tag{6.29}$$

where the element $\varphi_0(\alpha,\ h)\in\mathscr{A}^m(\Lambda^n,\ r^n)$ verifies (6.4). Indeed, let the elements be found. Set

$$\Psi(x,\ t,\ h)=\sum_{j=1}^{l}\mathscr{K}_{M_j^{n+1}}\varphi_j.$$

Then, by Theorem 6.1, $\hat{H}\Psi=0$, and

$$\Psi(x,\ 0,\ h)=\sum_{j=0}^{l}(\mathscr{K}_{M_j^{n+1}}\varphi_j)\,|_{t=0}=\mathscr{K}_{\Lambda^n}\left(\sum_{j=1}^{l}\varphi_j(\alpha,\ 0,\ h)\right)=$$
$$=\mathscr{K}_{\Lambda^n}\varphi_0=\Psi_0(x,\ h).$$

Theorem 6.2. *There exist elements of the set* $\mathscr{A}^m(M_j^{n+1})$, $j=1,\ \ldots,\ l$, *satisfying* (6.28) *and* (6.29).

Proof. On applying Lemma 6.4 we obtain

$$R_j=B_j^{(0)}R_jB_j^{(0)}+B_j^{(0)}R_j(1-B_j^{(0)})+(1-B_j^{(0)})R_jB_j^{(0)}+$$
$$+(1-B_j^{(0)})R_j(1-B_j^{(0)}).$$

By Lemmas 6.2-6.4

$$B_j^{(0)}R_jB_j^{(0)}=(-ih)\,B_j^{(0)}(R_j^{(1)}+g_j^{(1)})\,B_j^{(0)},\quad g_j^{(1)}\in\mathscr{P}_j \tag{6.30}$$

$$B_j^{(0)}R_j(1-B_j^{(0)})=(-ih)\,[B_j^{(0)}R_j^{(1)}(1-B_j^{(0)})+$$
$$+B_j^{(0)}g_j^{(2)}(1-B_j^{(0)})],\quad g_j^{(2)}\in\mathscr{P}_j. \tag{6.31}$$

To find the vectors $\varphi_j\in\mathscr{A}^m(M_j^{n+1})$ satisfying (6.28) and (6.29), it is sufficient to calculate the "projections" $B_k^{(0)}\varphi_j$, $k=1,\ \ldots,\ l$. Consider the following equations:

$$B_j^{(0)}[R_j^{(1)}+g_j^{(1)}](B_j^{(0)}\varphi_j)=-B_j^{(0)}[R_j^{(1)}+g_j^{(2)}](1-B_j^{(0)})\,\varphi_j, \tag{6.32}$$

$$(1-B_j^{(0)})R_j(B_j^{(0)}\varphi_j)+(1-B_j^{(0)})R_j(1-B_j^{(0)})\,\varphi_j=0. \tag{6.33}$$

To solve Eqs. (6.32) and (6.33) we find the expression of $B_j^{(0)}\varphi_j$ by $(1-B_j^{(0)})\varphi_j=\sum\limits_{k\neq j}(B_k^{(0)}\varphi_j)$ with the help of (6.32) and substitute this expression in Eq. (6.33). Then we shall be able to find the values

$$B_k^{(0)}\varphi_j,\quad k=1,\ \ldots,\ l.$$

Now turn to Eq. (6.32). Consider the function

$$F_j=-B_j^{(0)}[R_j^{(1)}+g_j^{(2)}](1-B_j^{(0)})\varphi_j. \tag{6.34}$$

On applying Theorem 6.1 and Lemma 6.4 we obtain

$$B_j^{(0)}(R_j^{(1)}+g_j^{(1)})B_j^{(0)}=B_j^{(0)}\Big(\frac{d}{dt_j}-\frac{1}{2}\operatorname{tr}\frac{\partial^2\lambda_j}{\partial q\,\partial p}+\\ +\frac{\partial E_j}{\partial p}\frac{\partial(\mathscr{H}^{(0)}-\lambda_j)}{\partial q}+\mathscr{H}^{(1)}+g_j^{(3)}\Big)B_j^{(0)}, \tag{6.35}$$

$$g_j^{(3)}=g_j^{(1)}+\mathscr{C}_j^{(2)},\quad g_j^{(3)}\in\mathscr{P}_j.$$

Consider the system of equations

$$\Big(\frac{d}{dt_j}-\frac{1}{2}\operatorname{tr}\frac{\partial^2\lambda_j}{\partial q\,\partial p}B_j^{(0)}+B_j^{(0)}\frac{dB_j^{(0)}}{dt}+\\ +B_j^{(0)}\frac{\partial E_j}{\partial p}\frac{\partial(\mathscr{H}^{(0)}-\lambda_j)}{\partial q}B_j^{(0)}+B_j^{(0)}\mathscr{H}^{(1)}B_j^{(0)}+\\ +B_j^{(0)}g_j^{(3)}B_j^{(0)}\Big)\theta_j=F_j,\theta_j|_{t=0}=\theta_{0j}, \tag{6.36}$$

where F_j is defined by (6.34) and the element $\theta_{0j}\in\mathscr{A}^m(\Lambda^n,\ r^n)$ will be chosen later. We shall prove that if θ_j is a solution of Eq. (6.36), then $B_j^{(0)}\theta_j$ is a solution of the system

$$B_j^{(0)}(R_j^{(1)}+g_j^{(1)})B_j^{(0)}\theta_j=F_j.$$

For this purpose it is sufficient to prove the following equality

$$B_j^{(0)}\frac{d}{dt_j}(B_j^{(0)}\theta_j)=B_j^{(0)}\frac{\partial\theta_j}{\partial t_j}+B_j^{(0)}\Big(\frac{dB_j^{(0)}}{dt_j}\Big)\theta_j,$$

which follows immediately from Lemma 6.5 and the Leibnitz formula.

The solutions of Eq. (6.36) and the transfer equation are essentially the same (cf. Sec. 6, Ch. IV). By Lemma 6.4 we have

$$B_k^{(0)}=E_k+\delta_k,\quad \delta_k\in\mathscr{P}_j,\quad k=1,\ \ldots,\ l.$$

Therefore, on applying the lemmas of the beginning of this section we can put (6.36) in the form

$$\Big(\frac{d}{dt_j}-\frac{1}{2}\operatorname{tr}\frac{\partial^2\lambda_j}{\partial q\,\partial p}+E_j\frac{dE_j}{dt_j}+E_j\frac{\partial E_j}{\partial p}\frac{\partial(\mathscr{H}^{(0)}-\lambda_j)}{\partial q}+\\ +E_j\mathscr{H}^{(1)}E_j+\Delta_j\Big)\theta_j=F_j,\quad \Delta_j\in\mathscr{P}_j. \tag{6.37}$$

Let Φ_j be the fundamental matrix of the following system of equations:

$$\left(\frac{d}{dt_j}-\frac{1}{2}\operatorname{tr}\frac{\partial^2\lambda_j}{\partial q\,\partial p}E_j+E_j\frac{dE_j}{dt_j}+\right.$$
$$\left.+E_j\frac{\partial E_j}{\partial p}\frac{\partial(\mathscr{H}^{(0)}-\lambda_j)}{\partial q}+E_j\mathscr{B}^{(1)}E_j\right)\Phi_j=0$$
$$\Phi_j\,|_{t=0}=E,$$

where E is a unit matrix. Let $\theta_j=\Phi_j\nu_j$. We obtain the following system of equations for ν_j:

$$\begin{cases}\dfrac{\partial\nu_j}{\partial t_j}+(\Phi_j^{-1}\Delta_j\Phi_j)\,\nu_j=\Phi_j^{-1}F_j,\\ \nu_j\,|_{t=0}=\theta_{0j},\end{cases}\tag{6.38}$$

where $\Phi_j^{-1}\Delta_j\Phi_j\in\mathscr{P}_j$ by Eq. (6.37). We also have

$$\nu_j=\sum_{k\geqslant 0}(I(\Phi_j^{-1}\Delta_j\Phi_j))^k\,[\theta_{0j}+I(\Phi_j^{-1}F_j)],$$

similarly to Sec. 6, Ch. IV. Turn to the solution of Eq. (6.33). Let $B_j^{(0)}\varphi_j=B_j^{(0)}\theta_j$. Then

$$B_j^{(0)}\varphi_j=B_j^{(0)}C_j^{(1)}B_j^{(0)}\varphi_j(\alpha,\ 0,\ h)+B_j^{(0)}C_j^{(2)}(1-B_j^{(0)})\,\varphi_j,\tag{6.39}$$

where

$$C_j^{(1)}B_j^{(0)}\varphi_j(\alpha,\ 0,\ h)=\Phi_j\,[\sum_{k\geqslant 0}(I(\Phi_j^{-1}\Delta_j\Phi_j))^k]\,B_j^{(0)}\varphi_j(\alpha,\ 0,\ h),$$
$$C_j^{(2)}(1-B_j^{(0)})\,\varphi_j=\Phi_j\,[\sum_{k\geqslant 0}(I(\Phi_j^{-1}\Delta_j\Phi_j))^k]\,I(\Phi_j^{-1}B_j^{(0)}\times$$
$$\times(R_j^{(1)}+g_j^{(2)})\,(1-B_j^{(0)})\,\varphi_j).\tag{6.40}$$

Substituting the value of $B_j^{(0)}\varphi_j$ into (6.33), we obtain

$$(1-B_j^{(0)})\,R_j\,(1-B_j^{(0)})\,\varphi_j+(1-B_j^{(0)})\,R_jB_j^{(0)}C_j^{(1)}B_j^{(0)}\varphi_j(\alpha,\ 0,\ h)+$$
$$+(1-B_j^{(0)})\,R_jB_j^{(0)}C_j^{(2)}(1-B_j^{(0)})\,\varphi_j=0.\tag{6.41}$$

Applying Lemmas 6.1-6.4, Theorem 6.1 and (6.40), we can put (6.41) into the form

$$\sum_{k\neq j}(\lambda_k-\lambda_j)\,(B_k^{(0)}\varphi_j)+\sum_{k\neq j}B_k^{(0)}R_jB_j^{(0)}C_j^{(1)}B_j^{(0)}\varphi_j(\alpha,\ 0,\ h)+$$
$$+\sum_{m\neq j}\sum_{k\neq j}B_k^{(0)}\mathscr{E}_{kj}h^{1/2}C_j^{(2)}\,(B_m^{(0)}\varphi_j)=0,\tag{6.42}$$
$$\mathscr{E}_{kj}\in\mathscr{L}_j,\quad k=1,\ \ldots,\ l,\quad k\neq j.$$

By Lemma 6.2 the following equality is true:

$$h^{1/2}C_j^{(2)}=\Phi_j\,[\sum_{k\geqslant 0}(I(\Phi_j^{-1}\Delta_j\Phi_j))^k]\,I(C_j^{(3)}(1-B_j^{(0)})\,\varphi_j),$$

where

$$C_j^{(2)} = h^{1/2}\,\Phi_j^{-1} B_j^{(0)}(R_j^{(1)} + g_j^{(2)}) \in \mathcal{N}_j.$$

Let $\mathcal{A}$ be a matrix with elements of the form

$$(\mathcal{A})_{k,\,m} = \frac{1}{(\lambda_k - \lambda_j)} B_k^{(0)} \mathcal{E}_{kj}\,(h^{1/2} C_j^{(2)}) \tag{6.43}$$

and Y and F be vectors with coordinates of the form

$$Y_i = B_i^{(0)} \varphi_j, \quad F_i = -\frac{1}{(\lambda_k - \lambda_j)} B_i^{(0)} R_j B_j^{(0)} C_j^{(1)} B_j^{(0)} \varphi_j(\alpha,\,0,\,h). \tag{6.44}$$

Then (6.42) may be put in the form

$$(E + \mathcal{A})\,Y = F.$$

By Lemma 6.2 and (6.42), (6.43) the operator $E + \mathcal{A}$ is a quasi-identity, therefore, by Lemma 4.2 there exists the operator $(E + \mathcal{A})^{-1}$ and the equality

$$Y = (E + \mathcal{A})^{-1} F$$

is true.

Choose the initial conditions θ_{0j}, $j = 1, 2, \ldots, l$ for Eq. (6.36) to satisfy Eq. (6.29). We have

$$\sum_{j=1}^{l} \varphi_j\,|_{t=0} = \sum_{j=1}^{l} \sum_{k=1}^{l} (B_k^{(0)} \varphi_j)\,|_{t=0} =$$

$$= \sum_{j=1}^{l} \Big(B_j^{(0)} \theta_{0j} + \sum_{k \neq j} B_k^{(0)} \varphi_j\,|_{t=0}\Big). \tag{6.45}$$

Define the vector $B_k^{(0)}\,\varphi_j\,|_{t=0}$, $k \neq j$ by (6.33). First we shall put (6.33) in the form

$$\sum_{k \neq j} B_k^{(0)} R_j (B_j^{(0)} \theta_{0j}) + \sum_{k \neq j} B_k^{(0)} R_j \sum_{m \neq j} B_m^{(0)} (\varphi_j\,|_{t=0}) = 0. \tag{6.46}$$

By Lemma 6.4 and Theorem 6.1 we have

$$B_k^{(0)} R_j B_m^{(0)} = B_k^{(0)}\,[(\lambda_k - \lambda_j)\,\delta_{km} + G_{km}]\,B_m^{(0)}, \tag{6.47}$$

where $G_{mk} \in \mathcal{L}$. Note that the index j of operator class is omitted, since the introduced operator classes are obviously independent of j when $t = 0$.

Let Y and F be vectors with coordinates of the form

$$Y_k = B_k^{(0)}\,(\varphi_j\,|_{t=0}), \qquad F_k = \frac{1}{\lambda_j - \lambda_k} B_k^{(0)} R_j\,(B_j^{(0)} \theta_{0j}),$$

$$k = 1, \ldots, l, \quad k \neq j,$$

and G be an operator matrix with elements of the form

$$(G)_{k,\,m} = \frac{1}{(\lambda_k - \lambda_j)} G_{km}; \quad k,\, m = 1, \ldots, l,\ k \neq j,\ m \neq j.$$

By (6.47) and Lemmas 6.3, 6.4, Eq. (6.46) can be put in the form

$$(1+G)\,Y=F.$$

By Lemma 6.3 the operator $1+G$ is a quasi-identity. Hence the following equations are true:

$$B_k^{(0)}(\varphi_j\,|_{t=0})=B_k^{(0)}\sum_{m\neq j}P_{km}^{(j)}B_m^{(0)}\theta_{0m},\quad k\neq j \tag{6.48}$$

Note that by Lemma 6.5

$$B_k^{(0)}R_jB_j^{(0)}\in\mathscr{L}.$$

Hence the operators P_{km} of (6.48) are of the class $\mathscr{L}$ as well.

Turn to (6.45) again:

$$\sum_{j=1}^{l}\Big(B_j^{(0)}\theta_{0j}+\sum_{k\neq j}B_k^{(0)}\sum_{m\neq j}P_{km}^{(j)}B_m^{(0)}\theta_{0m}\Big)=\sum_{j=1}^{l}B_j^{(0)}\varphi_0, \tag{6.49}$$

where

$$P_{km}\in\mathscr{L},\quad k,\,m=1,\,\ldots,\,l;\quad k\neq j,\quad m\neq j.$$

To define the elements $\theta_{0m}\in\mathscr{A}^m(\Lambda^n,\,r^n)$, consider the following system of equations

$$\theta_{0l}+\sum_{m\neq j}P_{lm}^{(j)}B_m^{(0)}\theta_{0m}=B_l^{(0)}\varphi_0. \tag{6.50}$$

The elements $B_m^{(0)}\theta_{0k}$ obviously satisfy Eq. (6.49) if the elements $\theta_{01},\,\ldots,\,\theta_{0l}\in\mathscr{A}^m(\Lambda^n,\,r^n)$ satisfy (6.50).

System (6.50) has a unique solution, since the operators $P_{lm}^{(j)}B_m^{(0)}$ are of the class $\mathscr{L}$. This statement is proved similarly to the solution of Eq. (6.46). Theorem 6.2 is proved.

In the proof of Theorem 6.2 there is a method of constructing the solutions of equations (6.28), (6.29). It is easy to see that the problem of solving equations (6.28), (6.29) reduces to that of constructing inverse operators for some quasi-identities. This construction is given in the proof of Lemma 4.2.

The following theorem is an obvious corollary of Theorems 6.1 and 6.2. Applying the same notation as in Theorems 6.1 and 6.2 we can formulate it in the following form.

Theorem 6.3. *Let $\varphi_1,\,\ldots,\,\varphi_l$ be such elements of $\mathscr{A}^m(M_j^{n+1})$ that the following equations are valid:*

(i) $R_j\varphi_j=0$

(ii) $\Big(\sum_{j=1}^{l}\varphi_j\Big)\Big|_{t=0}=\varphi_0,\quad \varphi_0\in\mathscr{A}^m(\Lambda^n,\,r^n),\quad j=1,\,\ldots,\,l.$

Then an h-asymptotic series $\Psi=\sum_{j=1}^{l}\mathscr{K}_{M_j^{n+1}}\varphi_j$

is a formal asymptotic solution of the problem

$$\begin{cases} -ih\dfrac{\partial\Psi}{\partial t}+\mathscr{H}\left(\overset{2}{x},\ \overset{1}{\hat{p}},\ h\right)\Psi=0 \\ \Psi\,|_{t=0}=\mathscr{K}_{\Lambda^n}\varphi_0. \end{cases}$$

Now we shall reformulate the theorems proved in this section for the operator case.

Let $\mathscr{H}\,(x, p, h)$ be the same matrix as in (6.1).

Construct an h-asymptotic operator $\Psi\left(\overset{2}{x},\ \overset{1}{\hat{p}},\ t,\ h\right)=\hat{\Psi}$ satisfying the equation

$$\begin{cases} -ih\dfrac{\partial\hat{\Psi}}{\partial t}+[\![\mathscr{H}\left(\overset{2}{x},\ \overset{1}{\hat{p}},\ h\right)]\!][\![\Psi\left(\overset{2}{x},\ \overset{1}{\hat{p}},\ t,\ h\right)]\!]=0, \\ \hat{\Psi}|_{t=0}=Ev\,(x), \end{cases} \tag{6.51}$$

where $\hat{p}=-ih\dfrac{\partial}{\partial x}$, E is a unit $(m\times m)$-matrix $v\,(x)\in C_0^\infty(\mathbf{R}^n)$.

Let $B^m\,(M_j^{n+1})$ be the set of equivalence classes of $D^{(j)}$-asymptotic series in $(m\times m)$-smooth matrices, depending on points of M_j^{n+1} and h.

Let $B^m\,(\mathbf{R}^n\times[0,\ T])$ be the set of equivalence classes of h-asymptotic series in $(m\times m)$-matrices defined in $\mathbf{R}^n\times[0,\ T]$. Within the previous notation the following theorem is valid.

Theorem 6.4. *The solution of Problem* (6.51) *has the form*

$$\psi\left(\overset{2}{x},\ \overset{1}{\hat{p}},\ t,\ h\right)=\sum_{j=1}(\mathscr{K}_{M_j^{n+1}}\varphi_j)\left(\overset{2}{x},\ \overset{1}{\hat{p}},\ t,\ h\right),\qquad \textit{where}$$

$$\varphi_j\in B^m\,(M_j^{n+1}),\quad j=1,\ \ldots,\ l.$$

The proof of Theorem 6.4 is similar to Theorem 6.3.

Sec. 7. Quasi-Inverse of Operators with Matrix Symbol

In this section the statement of the main theorem is introduced for the case when the symbol of the operator to be quasi-inversed is an $(s\times s)$-matrix, s is any.*

Definition 7.1. *Let $\mathscr{L}$ be an $\mathscr{A}$-module with a μ-structure, $A_0, \ldots$ $\ldots, A_n, B\in X$. An element $\hat{F}=g\left(\overset{2}{B}\right)\overset{1}{A_0}+f\left(\overset{1}{A_1},\ \ldots,\ \overset{n}{A_n},\ \overset{n+1}{B}\right)\in$ $\in\mathscr{A}$ is quasi-invertible if there exist such two sequences $\varkappa_k\in\mathscr{L}$, $\varkappa_k'\in\mathscr{L}$ that the products $\hat{F}\varkappa_k$ and $\varkappa_k'\hat{F}$ have the form*

$$\hat{F}\varkappa_k=1+R_k\left(\overset{1}{A_0},\ \ldots,\ \overset{n+1}{A_n},\ \overset{n+2}{B}\right),\quad \varkappa_k'\hat{F}=1+R_k'\left(\overset{n+2}{A_0},\ \ldots,\ \overset{2}{A_n},\ \overset{1}{B}\right),$$

* Operaton $\hat{F}$ with matrix symbol F is a matrix consisting of operators $\hat{F}_{ij}$ with symbols F_{ij}. Product $\hat{H}=\hat{F}\cdot\hat{G}$ is the matix with elements $\hat{H}_{ij}=\sum_k\hat{F}_{ik}\,\hat{G}_{kj}$.

where the functions R_k (x_0, x, α) *and* R'_k (x_0, x, α) *satisfy the estimates*

$$R_k = O_{\mathscr{L}}(|x|^{-k}), \quad R'_k = O_{\mathscr{L}}(|x|^{-k}), \quad x = (x_1, \ldots, x_n).$$

For the case $g(B) = 0$ this definition obviously coincides with that of Sec. 9 of Introduction. The case $g(B) \equiv 1$ is most useful for various problems. The concept of a quasi-invertible operator is changed for a Δ-quasi-invertible one when the parameter manifold M^{m+1} is not compact. Let $\Delta \in \mathbf{R}^{m+1}$ be a compact domain.

Definition 7.1′. *An element* $\hat{F} = g\left(\overset{2}{B}\right)\overset{1}{A}_0 + f\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}\right)$ *is* Δ*-quasi-invertible if there exist such sequences* $\varkappa_k$, $\varkappa'_k \in \mathscr{L}$ *that the product* $\hat{F}\varkappa_k$ *and* $\varkappa'_k\hat{F}$ *may be put in the form*

$$\hat{F}\varkappa_k = P(B) + R_k\left(\overset{1}{A}_0, \ldots, \overset{n+1}{A}_n, \overset{n+2}{B}\right),$$

$$\varkappa'_k\hat{F} = P'(B) + R'_k\left(\overset{n+2}{A}_0, \ldots, \overset{2}{A}_n, \overset{1}{B}\right),$$

where $P(\alpha)$, $P'(\alpha) \in C_0^\infty$ *equal 1 when* $\alpha \in \Delta$ *and equal* 0 *outside a small neighborhood* Δ. *The functions* R_k (x_0, x, α) *and* R'_k (x_0, x, α) *satisfy the estimates*

$$R_k = O_{\mathscr{L}}(|x|^{-k}), \quad R'_k = O_{\mathscr{L}}(|x|^{-k}), \quad x = (x_1, \ldots, x_n).$$

Let the matrix elements $f_{ij}(x, \alpha)$ belong to $\mathscr{S}^\infty(\mathbf{R}^n \times M^{m+1})$ and the matrix $f(x, \alpha)$ is asymptotically ρ-quasi-homogeneous of degree r_0 in x, i. e.,

$$f(x, \alpha) = \sum_{i=0}^{M} f^{(i)}(x, \alpha) + \sigma(x, \alpha),$$

where the matrices $f^{(i)}(x, \alpha)$ are positively quasi-homogeneous in x of degree r_i with a unique set of $\rho_1, \ldots, \rho_n$, $r_i \geqslant r_{i+1} + \delta$, $\delta > 0$. The matrix elements of $\sigma(x, \alpha)$ satisfy the conditions

$$\left|\frac{\partial^{|k|}}{\partial x^k}\sigma_{ij}(x, \alpha)\right| \leqslant c_k\left(\sum_{i=1}^{n}(x_i)^{2/\rho_i} + 1\right)^{r_0 - 1 - |k|}.$$

Similarly to Sec. 9 of Introduction, the matrix $f^{(0)}(x, \alpha)$ is called a leading term of the matrix $f(x, \alpha)$ and the matrix $f_1(x, \alpha) = \sum_{i=0}^{M} f^{(i)}(x, \alpha)$ is a essential part of the matrix $f(x, \alpha)$. The number r_0 is defined as an asymptotics to the quasi-homogeneous degree of the matrix $f(x, \alpha)$. Suppose that the function $g(\alpha)$ is smooth and uniformly bounded together with derivatives. The

matrix

$$F(x_0, x, \alpha) = g(\alpha_0)\, x_0 + f(x, \alpha)$$

is obviously asymptotically quasi-homogeneous in the variables x_0, x and of the same degree as $f(x, \alpha)$.

Without any loss of generality assume that the variable x_0 has the weight $\rho_0 = r_0$.

Let $\overset{1}{L}_0, \ldots, \overset{n+1}{L}_n, \overset{n+2}{\alpha}$ be a left-ordered representation of the operators $\overset{1}{A}_0, \ldots, \overset{n+1}{A}_n, \overset{n+2}{B}$. Suppose that the matrix

$$F(L_0(y, \eta), \ldots, L_n(y, \eta), \alpha)$$

(variables y_i, $i=1, \ldots, n$ and y_{n+k}, $k=1, \ldots, m+1$, corresponding to x_i, $i=1, \ldots, n$ and $-i\dfrac{\partial}{\partial \alpha_{k-1}}$, $k=1, \ldots, m+1$, are introduced as on p. 128) is asymptotically ρ-quasi-homogeneous in y of degree r with the weights $\rho_0, \ldots, \rho_{n+m+1}$, such that $\rho_i = 1$ for some $i \in \{n+1, \ldots, m+n+1\}$. Let $\pi(y, \eta, \alpha)$ be the leading term of the matrix $F(L_0(y, \eta), \ldots, L_n(y, \eta), \alpha)$. Let J_1 be a subset of all such numbers of the set $(0, \ldots, n)$ that $\rho_i = 1$ when $i \in J_1$. Let J_2 be a subset of all such numbers j belonging to the set $(0, \ldots, m)$ that $\rho_{j+n+1} = 1$ if $j \in J_2$. Introduce the following notation:

$$y_{n+j+1} = p_j, \quad j \in J_2; \quad y_{n+j+1} = 0, \quad j \bar{\in} J_2 \wedge j \in (0, \ldots, m),$$

$$y_i = -p_{m+i+1} + \omega_i, \quad j \in J_1, \quad y_i = \omega_i, \quad i \bar{\in} J_1 \wedge i \in (0, \ldots, n),$$

$$\alpha_i = q_i, \quad i = 0, \ldots, m; \quad \eta_j = q_{m+j+1}, \quad j = 0, \ldots, \eta.$$

On substituting the new notation into the matrix $\pi(y, \eta, \alpha)$ we obtain the matrix $\mathscr{H}(p, q, \omega)$.

Let $\mathscr{H}_1$ be an essential part of the matrix $F(L_0(y, \eta), \ldots$ $\ldots, L_n(y, \eta)\, \alpha)$ for the new variables (p, q, ω). We shall suppose that the matrix $\mathscr{H}_1(p, q, \omega)$ is diagnalizable by a smooth transformation and its eigenvalues $\mu_1(p, q, \omega), \ldots$ $\ldots, \mu_l(p, q, \omega)$ $(l \leqslant s)$ are of the class C^∞ and of constant multiplicity.

The following inequality is assumed to be true:

$$\operatorname{Im} \mu_i \leqslant 0, \quad i = 1, \ldots, l.$$

Let $\lambda_1(p, q, \omega), \ldots, \lambda_l(p, q, \omega)$ be eigenvalues of the matrix $\mathscr{H}(p, q, \omega)$ and let $\Omega_\varepsilon^{(i)}$ be a manifold in the space of the variables

p, q, ω defined by

$$p=0,\ (q_0, \ldots, q_m) \in M^{m+1}, \quad \sum_{j=0}^{n} q_{m+j+1}^2 < \varepsilon,$$

$$\sum_{i=1}^{n} \omega_i^{2/\rho_i} = 1,$$

$$\omega_0 \in \mathbf{R}^1, \quad |\lambda_i(0, q, \omega)| < \varepsilon.$$

Note. If $M^{m+1} = \mathbf{R}^{m+1}$, instead of $\Omega_\varepsilon^{(i)}$ the set $\Omega_\varepsilon^{(i)}(\Delta)$ defined by

$$p_0 = 0,\ (q_0, \ldots, q_m) \in \Delta_\varepsilon, \quad \sum_{j=0}^{n} q_{m+j+1}^2 < \varepsilon,$$

$$\sum_{i=1}^{n} \omega_i^{2/\rho_i} = 1,\ \omega_0 \in \mathbf{R}^1,\ |\lambda_i(0, q, \omega)| < \varepsilon$$

is used, where Δ_ε is ε-neighborhood of domain Δ used for the construction of a Δ-quasi-inverse.

In the sequel, the eigenvalues $\mu_1(p, q, \omega), \ldots, \mu_l(p, q, \omega)$ of the matrix $\mathscr{H}_1(p, q, \omega)$, $(p, q, \omega) \in \Omega_\varepsilon^{(k)}$ are supposed to satisfy a condition

$$\inf_{(p, q, \omega) \in \Omega_\varepsilon^{(k)}} \min_{i \neq j,\, k} |\mu_i(p, q, \omega) - \mu_j(p, q, \omega)| \geqslant \delta > 0, \tag{7.1}$$

where $\delta > 0$ is a constant.

Suppose the matrix $\mathscr{H}(p, q, \omega)$ to be diagonalizable and the eigenvalue multiplicities $\lambda_1, \ldots, \lambda_l$ to be independent of p, q, ω, which involves that the set $\Omega_\varepsilon^{(i)}$ does not intersect for sufficiently small ε, i. e.,

$$\Omega_\varepsilon^{(i)} \cap \Omega_\varepsilon^{(j)} = \varnothing, \quad i, j = 1, \ldots, l, \quad i \neq j.$$

Introduce the notation

$$H^{(i)}(p, q, \omega) = \operatorname{Re} \lambda_i(p, q, \omega),$$

$$\widetilde{H}^{(i)}(p, q, \omega) = \operatorname{Im} \lambda_i(p, q, \omega).$$

Definition 7.2. *The ith family of bicharacteristics of the operator F is a solution of the system*

$$\left.\begin{aligned} \dot{q}^{(i)} &= \frac{\partial H^{(i)}}{\partial p}(p^{(i)}, q^{(i)}, \omega), \\ \dot{p}^{(i)} &= -\frac{\partial H^{(i)}}{\partial q}(p^{(i)}, q^{(i)}, \omega), \\ &(p^{(i)}(0), q^{(i)}(0), \omega) \in \Omega_\varepsilon^{(i)}. \end{aligned}\right\} \tag{7.2}$$

Definition 7.3. *An operator $F\left(\overset{1}{A}_0, \ldots, \overset{n+1}{A}_n, \overset{n+2}{B}\right)$ satisfies the absorption condition if such constants $T > 0$, $\varepsilon > 0$, $\tau' > 0$, $0 < \tau' < T$ exist that for any $i \in \{1, \ldots, l\}$ with $\Omega_\varepsilon^{(i)} \neq \varnothing$ we have:*

(1) *on the segment* $0 \leqslant \tau \leqslant T$ *there exists a solution of the class* C^∞ *of system* (7.2)

$$q^{(i)}(q^{(i)}(0),\ \omega),\qquad p^{(i)}(q^{(i)}(0),\ \omega);$$

(2) *when* $0 \leqslant \tau \leqslant T$ *the function* $\widetilde{H}^{(i)}(p^{(i)}, q^{(i)}, \omega)$ *is non-positive and when* $\tau = \tau'$ *it is strictly negative.*

Theorem 7.1. *Let an operator* $F\left(\overset{1}{A}_0, \ldots, \overset{n+1}{A}_n, \overset{n+2}{B}\right)$ *satisfy the absorption condition, then it is quasi-invertible.*

The proof is essentially the same as of the Main Theorem of Sec. 5. The symbol of a quasi-inverse operator being a matrix, the construction of Sec. 6 to solve the initial value problem is applied.

We introduce a version of the Main Theorem useful for the study of the Cauchy problem.

Definition 7.4. *An operator* $F\left(\overset{1}{A}_0, \ldots, \overset{n+1}{A}_n, \overset{n+2}{B}\right)$ *satisfies the global absorption conditions, if there exist constants* $T > 0$, $\varepsilon > 0$, $\tau' > 0$, $0 < \tau' < T$ *such that for any* $i \in \{1, \ldots, l\}$ *we have*:

(1) *there exists a smooth solution of system* (7.2) *with initial conditions satisfying the equations*

$$\begin{gathered} p^{(i)}(0) = 0,\quad (q_0^{(i)}, \ldots, q_n^{(i)})\,|_{\tau=0} \in M^{m+1}, \\ \sum_{j=0}^{n} (q_{m+j+1}^{(i)})^2\,|_{\tau=0} < \varepsilon,\quad \sum_{i=1}^{n} \omega_i^{2/\rho_i} = 1,\quad \omega_0 \in \mathbf{R}^1; \end{gathered} \tag{7.3}$$

(2) *when* $0 \leqslant \tau \leqslant T$ *the function* $\widetilde{H}^{(i)}(p^{(i)}, q^{(i)}, \omega)$ *is non-positive and when* $\tau = \tau'$ *it is strictly negative.*

Note that when $M^{m+1} = \mathbf{R}^{m+1}$ Eqs. (7.3) in Definition 7.4 are changed for

$$p^{(i)}(0) = 0,\ (q_0^{(i)}, \ldots, q_n^{(i)})\,|_{\tau=0} \in \Delta_\varepsilon;$$

$$\sum_{j=0}^{n} (q_{m+j+1}^{(i)})^2\,|_{\tau=0} < \varepsilon,\quad \sum_{i=1}^{n} \omega_i^{2/\rho_i} = 1,\ \omega_0 \in \mathbf{R}^1.$$

The global absorption conditions being satisfied, we can construct the symbol of the quasi-inverse operator solving the problem

$$\begin{gathered} -i\overset{1}{\Lambda}{}^{r-1}\frac{\partial \Psi}{\partial \tau}\left(\alpha,\ \tau,\ -i\frac{\overset{2}{\partial}}{\partial x'},\ \overset{1}{x'}\right) + [\![F\left(\overset{1}{L}_0, \ldots, \overset{n+1}{L}_n, \overset{n+2}{\alpha}\right)]\!] \times \\ \times [\![\Psi\left(\alpha,\ \tau,\ -i\frac{\overset{2}{\partial}}{\partial x'},\ \overset{1}{x'}\right)]\!] \doteq 0, \\ \Psi\,|_{\tau=0} = E\nu\left(-i\frac{\partial}{\partial x'}\right), \end{gathered} \tag{7.4}$$

where $\Lambda = \Lambda(x)$, E is a unit $(m \times m)$-matrix, $\nu(\eta) \in C^\infty(\mathbf{R}^{n+1})$ $1 = \nu(\eta)$ when $|\eta| < \varepsilon$, and $x' = (x_0, x)$.

Following the arguments of Sec. 5 and applying the K-formula we may put the operator $F\left(\overset{1}{L}_0, \ldots, \overset{n+1}{L_n}, \overset{n+2}{\alpha}\right)$ in the form

$$F\left(\overset{1}{L}_0, \ldots, \overset{n+1}{L_n}, \overset{n+2}{\alpha}\right) \doteq \overset{1}{\Lambda}{}^{r_0} [\![\mathscr{H}_2\left(\overset{1}{x'}\overset{1'}{\Lambda}{}^{-\rho}, -i\overset{1'}{\Lambda}{}^{-\rho}\overset{1}{\frac{\partial}{\partial\alpha}}, -i\overset{2}{\frac{\partial}{\partial x'}}, \overset{2}{\alpha}, \overset{1'}{\Lambda}{}^{\varepsilon}\right) + \left(-i\overset{1'}{\Lambda}{}^{-1}\right)\mathscr{H}_3\left(\overset{1}{x'}\overset{1'}{\Lambda}{}^{-\rho}, -i\overset{1'}{\Lambda}{}^{-\rho}\overset{1}{\frac{\partial}{\partial\alpha}}, -i\overset{2}{\frac{\partial}{\partial x}}, \overset{2}{\alpha}, \Lambda^{\varepsilon}\right)]\!].$$

By this formula the Problem (7.4) is reduced to the following one:

$$-i\overset{1}{\Lambda}{}^{-1}\frac{\partial\Psi}{\partial\tau}\left(\alpha, \tau, -i\overset{2}{\frac{\partial}{\partial x'}}, \overset{1}{x'}\right) + [\![\left(\mathscr{H}_2 + \left(-i\overset{1'}{\Lambda}{}^{-1}\mathscr{H}_3\right)\right) \times$$
$$\times\left(\overset{1}{x'}\overset{1'}{\Lambda}{}^{-\rho}, -i\overset{1'}{\Lambda}{}^{-\rho}\overset{1}{\frac{\partial}{\partial\alpha}}, -i\overset{2}{\frac{\partial}{\partial x'}}, \overset{2}{\alpha}, \overset{1'}{\Lambda}{}^{\varepsilon}\right)]\!] \times$$
$$\times [\![\Psi\left(\alpha, \tau, -i\overset{2}{\frac{\partial}{\partial x'}}, \overset{1}{x'}\right)]\!] \doteq 0,$$
$$\Psi|_{\tau=0} = E\nu\left(-i\frac{\partial}{\partial x'}\right).$$

Let $\mu_i\left(\overset{1}{x'}\overset{1'}{\Lambda}{}^{-\rho}, -i\overset{1'}{\Lambda}{}^{-\rho}\overset{1}{\frac{\partial}{\partial\alpha}}, -i\overset{2}{\frac{\partial}{\partial x'}}, \overset{2}{\alpha}, \overset{1'}{\Lambda}{}^{\varepsilon}\right)$ be the ith eigenvalue of the matrix $\mathscr{H}_2$. Consider a function

$$\gamma_i(X_0, X; P_0, P; \omega, \varkappa_1, \varkappa_2) \overset{\text{def}}{=} \mu_i(\omega - \Lambda^{1-\rho}P, \Lambda^{1-\rho}P_0; X, X_0, \Lambda^{-\varepsilon}),$$

where

$$X_0 = (q_0, \ldots, q_m) \in \mathbf{R}^{m+1}, \quad X = (q_{m+1}, \ldots, q_{n+m+2}) \in \mathbf{R}^{n+1},$$
$$P_0 = (p_0, \ldots, p_m) \in \mathbf{R}^{m+1}, \quad P = (p_{m+1}, \ldots, p_{n+m+2}) \in \mathbf{R}^{n+1},$$
$$\omega_i = \Lambda^{-\rho_i}x_i, \quad i = 1, \ldots, n, \quad \omega_0 = \Lambda^{-r_0}x_0,$$
$$\varkappa_1 \overset{\text{def}}{=} (\Lambda^{\varepsilon_1}, \ldots, \Lambda^{\varepsilon_n}), \quad \varkappa_2 \overset{\text{def}}{=} (\Lambda^{1-\rho_1}, \ldots, \Lambda^{1-\rho_n}).$$

The function γ_i is a smooth function of the parameters ω, $\varkappa_1$, $\varkappa_2$ for Λ sufficiently large. The eigenvalue λ_i of the matrix $\mathscr{H}(p, q, \omega)$ is a limit of the function γ_i as $\Lambda \to \infty$. Note that $\operatorname{Im}\gamma_i \leqslant 0$.

Let $(\Lambda^{n+m+1}, r^{n+m+1})$ be a germ defined by equations

$$\Lambda^{n+m+1}=\{P_0=P=0,\ |X|<\varepsilon,\ X_0\in M^{m+1}\},\ r^{n+m+1}\equiv 0.$$

We shall construct $m+n+2$-dimensional manifolds M_i^{n+m+2} with a complex germ $w^{(i)}$, $z^{(i)}$, a dissipative function $D^{(i)}$ and a potential $E^{(i)}$ by a canonical transformation corresponding to the Hamiltonian function $\gamma_i\,(X_0, X, P_0, P, \omega, \varkappa_1, \varkappa_2)$. We shall follow the ideas of Sec. 4 to the effect.

Let $\mathscr{K}_{M_i^{m+n+2}}$ be a canonical operator on M_i^{m+n+2}. Similarly to Sec. 5 and Sec. 6 we may prove that a formal asymptotic solution of the Problem (7.4) is defined by the equation

$$\Psi\left(\overset{2}{\alpha},\ \tau,\ -i\frac{\overset{2}{\partial}}{\partial x'},\ \overset{1}{x'}\right)=\left(\sum_{j=1}^{l}\mathscr{K}_{M_j^{n+m+2}}\varphi_j\right)\left(\overset{2}{\alpha},\ \tau, -i\frac{\overset{2}{\partial}}{\partial x'},\ \overset{1}{x'}\right),$$

where $\varphi_j\in B^m\,(M_j^{n+m+2})$.

Lemma 7.1. *Let an operator* $f\left(\overset{1}{A_1},\ \ldots,\ \overset{n}{A_n},\ \overset{n+1}{B}\right)$ *satisfy the global absorption conditions. Then it has quasi-inverse and the symbol of its quasi-inverse has the form*

$$\varkappa(x,\ \alpha)=i\int_0^T\left\{[\![\overset{1}{\Lambda}{}^{(r-1)}\sum_{j=1}^{l}(\mathscr{K}_{M_j^{n+m+2}}\varphi_j)\left(\alpha,\ \tau,\right.\right.$$
$$\left.\left.-i\frac{\overset{2}{\partial}}{\partial x'},\ \overset{1}{x'}\right)]\!]\,1\,(x)\right\}d\tau,$$

where $1\,(x)$ *is a function of* x *equal to unity.*

There exist analogues of Theorem 7.1 and Lemma 7.1 for Δ-quasi-inverse operators; the absorption conditions should be restated following the previous remarks.

Example 7.1. Consider an equation (cf. Example in Sec. 9 of Introduction)

$$-i\frac{\partial u}{\partial t}+i\frac{\partial u}{\partial y}-i\left(\nu(t)\sqrt{1-\frac{\partial^2}{\partial y^2}}\right)u=f, \tag{7.5}$$

where $\nu\,(t)$ is such a smooth bounded function that

$$\nu\,(t)=0,\quad t\in[-T,\ T],$$
$$\nu\,(t)>0,\quad t\,\overline{\in}\,[-T,\ T],\qquad T>0,$$
$$f\,(y,\ t)\in L_2,\quad \operatorname{supp} f\,(y,\ t)\subset\Delta\subset\mathbf{R}^1\times[-T,\ T].$$

Let

$$A_0=-i\frac{\partial}{\partial t},\quad A_1=-i\frac{\partial}{\partial y},\quad B_0=t,\quad B_1=y;$$

we have

$$[\![A_0 - A_1 - i\nu(B_0)\sqrt{1+A_1^2}]\!]\, u = f. \tag{7.6}$$

Now construct a quasi-inverse operator for the operator

$$F\left(\overset{1}{A}_0,\ \overset{1}{A}_1,\ \overset{2}{B}\right) \overset{\text{def}}{=\!=} A_0 - A_1 - i\nu(B_0)\sqrt{1+A_1^2}.$$

Take the ordered representation of the operators

$$L_0 = x_0 - i\frac{\partial}{\partial\alpha_0}, \quad L_1 = x_1 - i\frac{\partial}{\partial\alpha_1}, \ \alpha.$$

Hence, the Hamiltonian of the operator $A_0 - A_1 - i\nu(B_0)\sqrt{1+A_1^2}$ has the form

$$F\left(\overset{1}{L}_0,\ \overset{2}{L}_1,\ \overset{2}{\alpha}\right) = \left(x_0 - i\frac{\partial}{\partial\alpha_0}\right) - \left(x_1 - i\frac{\partial}{\partial\alpha_1}\right) -$$

$$- i\nu(\alpha_0)\sqrt{1+\left(x_1 - i\frac{\partial}{\partial\alpha_1}\right)^2};$$

thus,

$$F(L_0(y,\ \eta),\ L_1(y,\ \eta),\ \alpha) =$$

$$= (y_0 + y_2) - (y_1 + y_3) - i\nu(\alpha_0)\sqrt{1+(y_1+y_3)^2}.$$

Consider the case

$$\rho_1 = \rho_2 = \rho_3 = \rho_0 = 1.$$

We have

$$\pi(y,\ \eta,\ \alpha) = (y_0 + y_2) - (y_1 + y_3) - i\nu(\alpha_0)\,|\,y_1 + y_3\,|,$$

or, with respect to the variables $(p,\ q,\ \omega)$,

$$\mathscr{H}(p,\ q,\ \omega) = (\omega_0 + p_0 - p_2) - (\omega_1 + p_1 - p_3) -$$

$$- i\nu(q_0)\,|\,\omega_1 + p_1 - p_3\,|. \tag{7.7}$$

To construct the bicharacteristics of the operator F consider the Hamiltonian system

$$\dot{q}_0 = \dot{q}_2 = 1$$

$$\dot{q}_1 = \dot{q}_3 = -1$$

$$\dot{p}_i = 0, \quad i = 0, 1, 2, 3$$

having the initial values $p\,|_{t=0} = p^{(0)}$, $q\,|_{t=0} = q^{(0)}$ satisfying Eqs. (7.3). The following functions are obviously solutions of this system

$$\left.\begin{aligned} &p_0 = p_1 = p_2 = p_3 = 0,\\ &q_0 = t + q_0^{(0)},\ q_2 = t + q_2^{(0)},\\ &q_1 = -t + q_1^{(0)},\quad q_3 = -t + q_3^{(0)}. \end{aligned}\right\} \tag{7.8}$$

By Eq. (7.5) the imaginary part of $\widetilde{H}$ (p, q, ω) of the Hamiltonian function is defined by the formula

$$\widetilde{H}(p, q, \omega) = -\nu(q_0) \mid \omega_1 + p_1 + p_3 \mid = -\nu(q_0). \tag{7.9}$$

By Eqs. (7.8), (7.9) the operator F satisfies the global absorption conditions. Hence the construction of the symbol of the quasi-inverse operator of F is reduced to the problem

$$\begin{gathered} -i\frac{\partial\Psi}{\partial\tau}\left(\alpha, \tau, -i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}\right) + \\ + [\![F\left(\overset{1}{L_0}, \overset{1}{L_1}, \overset{2}{\alpha}\right)]\!] [\![\Psi\left(\alpha, \tau, -i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}\right)]\!] \doteq 0, \qquad (7.10) \\ \Psi\left(\alpha, 0, -i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}\right) = \rho\left(-i\frac{\partial}{\partial x}\right) P(\alpha), \end{gathered}$$

where $\rho(\eta) \in C_0^\infty$, $\rho(\eta) = 0$, when $|\eta| > \varepsilon$, $\rho(\eta) = 1$, when $|\eta| < \frac{\varepsilon}{2}$, $P(\alpha) \in C_0^\infty$, $P(\alpha) = 1$, when $\alpha \in \Delta$, $P(\alpha) = 0$, when $\alpha \bar{\in} \Delta_\varepsilon$.

We may put the function $\Psi\left(\alpha, \tau, -i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}\right)$ in the form

$$\Psi\left(\alpha, \tau, -i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}\right) = e^{i\overset{1}{\Lambda} S\left(\overset{1}{x}, -i\overset{2}{\frac{\partial}{\partial x}}, \alpha, \tau\right)} \theta\left(\overset{1}{x}, -i\overset{2}{\frac{\partial}{\partial x}}, \alpha, \tau\right),$$

where $\Lambda = |x_1|$.

To satisfy the initial conditions of Problem (7.10) we may put the functions S and θ in the form

$$S(x, \eta, \alpha, 0) = 0,$$
$$\theta(x, \eta, \alpha, 0) = \rho(\eta) P(\alpha).$$

On substituting the function Ψ into Eq. (7.10) and applying the commutation formula, we obtain

$$\begin{gathered} e^{i\overset{1}{\Lambda} S\left(\overset{1}{x}, -i\overset{4}{\frac{\partial}{\partial x}}, \overset{4}{\alpha}, \tau\right)} [\![\overset{3}{\Lambda}\overset{1'}{\vphantom{\Lambda}}\frac{\partial S}{\partial\tau} - i\frac{\partial}{\partial\tau} + x_0 + \overset{1'}{\Lambda}\frac{\partial S}{\partial\alpha_0} - \\ -\overset{1'}{\Lambda}\frac{\partial S}{\partial\eta_0} - i\frac{\partial}{\partial\alpha_0} - x_1 - \overset{1'}{\Lambda}\frac{\partial S}{\partial\alpha_1} - i\frac{\partial}{\partial\alpha_1} + \overset{1'}{\Lambda}\frac{\partial S}{\partial\eta_1} - \\ - i\nu(\alpha_0)\left(1 + \left(x_1 + \overset{1'}{\Lambda}\frac{\partial S}{\partial\alpha_1} - \overset{1'}{\Lambda}\frac{\partial S}{\partial\eta_1} - i\frac{\partial}{\partial\alpha_1}\right)^2\right)^{1/2}]\!] [\![\overset{2}{\theta}]\!] = 0. \end{gathered} \tag{7.11}$$

Ordering the operators by the K-formula, we obtain

$$\overset{3}{[\![}\overset{1'}{\Lambda}\frac{\partial S}{\partial\tau}-i\frac{\partial}{\partial\tau}+x_0+\overset{1'}{\Lambda}\frac{\partial S}{\partial\alpha_0}-\overset{1'}{\Lambda}\frac{\partial S}{\partial\eta_0}-i\frac{\partial}{\partial\alpha_0}-$$
$$-\left(x_1+\overset{1'}{\Lambda}\frac{\partial S}{\partial\alpha_1}-\overset{1'}{\Lambda}\frac{\partial S}{\partial\eta_1}-i\frac{\partial}{\partial\alpha_1}\right)-i\nu(\alpha_0)\times$$
$$\times\left(1+\left(x_1+\overset{1'}{\Lambda}\frac{\partial S}{\partial\alpha_1}-\overset{1'}{\Lambda}\frac{\partial S}{\partial\eta_1}-i\frac{\partial}{\partial\alpha_1}\right)^2\right)^{1/2}]\!]\times$$
$$\times\overset{2}{[\![}\theta\left(\overset{1}{x},\ -i\overset{2}{\frac{\partial}{\partial x}},\ \alpha,\ \tau\right)]\!]=$$
$$=\Lambda\left\{\left[\frac{\partial S}{\partial\tau}+\omega_0+\frac{\partial S}{\partial\alpha_0}+\frac{\partial S}{\partial\eta_0}-\omega_1-\frac{\partial S}{\partial\alpha_1}+\frac{\partial S}{\partial\eta_1}-\right.\right.$$
$$\left.\left.-i\nu(\alpha_0)\left|\omega_1+\frac{\partial S}{\partial\alpha_1}+\frac{\partial S}{\partial\eta_1}\right|\right]+\left(\sum_{k\geqslant 0}(-i\Lambda^{-1})^k R_k\right)\theta\right\}\times$$
$$\times\left(\overset{1}{x},-i\overset{2}{\frac{\partial}{\partial x}},\ \alpha,\ \tau\right). \tag{7.12}$$

Here R_k are differential operators, which we apply to the symbol of the operator $\theta\,(x,\ -i\frac{\partial}{\partial x},\ \alpha,\ \tau)$. The operators R_k are of an order no greater than k, with the coefficients being smooth functions of ω_1, α, τ and the derivatives of the functions S; $\omega_0 = x_0\Lambda^{-1}$, $\omega_1 = x_1\Lambda^{-1}$.

Let $S_1 = \operatorname{Re} S$, $S_2 = \operatorname{Im} S$

$$\frac{\partial S}{\partial\tau}+\omega_0+\frac{\partial S}{\partial\alpha_0}-\frac{\partial S}{\partial\eta_0}-\omega_1-\frac{\partial S}{\partial\alpha_1}+\frac{\partial S}{\partial\eta_1}-$$
$$-i\nu(\alpha_0)\left|\omega_1+\frac{\partial S}{\partial\alpha_1}-\frac{\partial S}{\partial\eta_1}\right|=O_{S_2}(\Lambda^{-3/2}). \tag{7.13}$$

Eq. (7.13) is a Hamilton-Jacobi equation having the dissipative function $\mathscr{H}(p,\ q,\ \omega)$ defined by Eq. (7.7).

On applying to Eq. (7.13) the methods of Ch. IV and equating to zero the right-hand terms in the parentheses in Eq. (7.12) we obtain the dissipative transfer equation for the function θ.

This equation is solved in Theorem 6.2 of Ch. IV.

Consider

$$D=-\int_0^\tau \operatorname{Im}\mathscr{H}\,(p\,(q^{(0)},\ \omega,\ \tau'),\ q\,(q^{(0)},\ \omega,\ \tau'),\ \omega)\,d\tau'\,|_{q=q\,(q^{(0)},\omega,\tau)}.$$

We proved in Ch. IV that the solution of the equation (7.13) satisfies the dissipativity inequality:

$$\operatorname{Im} S\,(x,\ \alpha,\ \tau)\geqslant\gamma D,$$

where $\gamma = \text{const}.$

By the last inequality and the absorption condition we have $D \geqslant \delta > 0$, $\delta = \text{const}$, when $\tau = 2T$. Hence the function

$$\Psi(\alpha, \tau, \eta, x) = e^{i\Lambda S(x, \eta, \alpha, \tau)}\theta(x, \eta, \alpha, \tau)$$

satisfies the estimate

$$|\Psi(\alpha, T, \eta, x)| \leqslant \text{const}\ e^{-\delta|x_1|}.$$

Thus, by the reduction rule (cf. Introduction, p. 109), the operator having the symbol

$$\varkappa(x, \alpha) = i\int_0^{2T} \left\{ [\![\Psi\left(\alpha, \tau, -i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}\right)]\!] 1(x)\right\} d\tau \tag{7.14}$$

is a quasi-inverse for the operator F.

Note that by Eq. (7.13) the function S has the form

$$S = -\omega_0\tau + S^1(x_1, \alpha, \tau), \tag{7.15}$$

where S^1 satisfies an equation

$$\frac{\partial S^1}{\partial \tau} + \frac{\partial S^1}{\partial \alpha_0} - \omega_1 - \frac{\partial S^1}{\partial \alpha_1} - i\nu(\alpha)\left|\omega_1 + \frac{\partial S^1}{\partial \alpha_1}\right| = O_{S_2}(\Lambda^{-3/2}).$$

We have indicated that the coefficients of the operators R_k in Eq. (7.12) do not depend on x_0. Hence the function θ does not depend on x_0. Besides, the coefficients of R_k do not depend on η. Hence the function $\theta(x_1, \eta, \alpha, \tau)$ has the form (cf. (7.8))

$$\theta(x_1, \eta, \alpha, \tau) = \theta_1(x_1, \alpha, \tau)\,\rho(\eta_0 + \tau, \eta_1 + \tau), \tag{7.16}$$

where $\theta_1(x_1, \alpha, \tau)$ satisfies the same transport equation as the function $\theta(x_1, \eta, \alpha, \tau)$ and the initial condition

$$\theta_1(x_1, \alpha, 0) = P(\alpha).$$

The following equation is a corollary of Eqs. (7.15), (7.16):

$$[\![\Psi\left(\alpha, \tau, -i\overset{2}{\frac{\partial}{\partial x}}, \overset{1}{x}\right)]\!] 1(x) = e^{-ix_0\tau}[\![e^{i\Lambda S_1\left(\overset{1}{x_1}, \alpha, \tau\right)}\theta_1\left(\overset{1}{x_1}, \alpha, \tau\right) \times$$

$$\times \rho\left(0, -i\overset{2}{\frac{\partial}{\partial x_1}} + \tau\right)]\!] 1(x) \overset{\text{def}}{=} e^{-ix_0\tau}\psi'(x_1, \alpha, \tau).$$

Thus Eq. (7.14) may be put in the form

$$\varkappa(x, \alpha) = i\int_0^{2T} e^{-ix_0\tau}\psi'(x_1, \alpha, \tau)\, d\tau. \tag{7.17}$$

Let the function $f(y, t)$, the right-hand side of Eq. (7.5), satisfy the condition $f(y, t) = 0$ when $t < 0$.

Then prove that the following equation is true

$$\left\{ [\![\varkappa \left(-i \overset{1}{\frac{\partial}{\partial t}}, \ -i \overset{1}{\frac{\partial}{\partial y}}, \ \overset{2}{t}, \ \overset{2}{y} \right)]\!] f(y, t) \right\} \Big|_{t=0} = 0, \tag{7.18}$$

where $\varkappa \left(-i \overset{1}{\frac{\partial}{\partial t}}, \ -i \overset{1}{\frac{\partial}{\partial y}}, \ \overset{2}{t}, \ \overset{2}{y} \right)$ is an operator with a symbol defined by Eq. (7.14). On applying Eq. (7.17) we obtain

$$[\![\varkappa \left(-i \overset{1}{\frac{\partial}{\partial t}}, \ -i \overset{1}{\frac{\partial}{\partial y}}, \ \overset{2}{t}, \ \overset{2}{y} \right)]\!] f(y, t) =$$

$$= i \int_0^{2T} \left\{ [\![e^{-\tau \overset{1}{\frac{\partial}{\partial t}}} \psi' \left(-i \overset{1}{\frac{\partial}{\partial y}}, \ \overset{2}{t}, \ \overset{2}{y} \right)]\!] f(y, t) \right\} d\tau =$$

$$= i \int_0^{2T} \left\{ [\![\psi' \left(-i \overset{1}{\frac{\partial}{\partial y}}, \ \overset{2}{t}, \ \overset{2}{y} \right)]\!] f(y, t-\tau) \right\} d\tau.$$

At the point $t=0$ we have

$$\left\{ [\![\varkappa \left(-i \overset{1}{\frac{\partial}{\partial t}}, \ -i \overset{1}{\frac{\partial}{\partial y}}, \ \overset{2}{t}, \ \overset{2}{y} \right)]\!] f(y, t) \right\}_{t=0} =$$

$$= i \int_0^{2T} \left\{ [\![\psi' \left(-i \overset{1}{\frac{\partial}{\partial y}}, \ \overset{2}{t}, \ \overset{2}{y} \right)]\!] f(y, -\tau) \right\} d\tau = 0.$$

Thus the function $u(t, y)$, $0 \leqslant t \leqslant T$

$$u(t, y) = [\![\varkappa \left(-i \overset{1}{\frac{\partial}{\partial t}}, \ -i \overset{1}{\frac{\partial}{\partial y}}, \ \overset{2}{t}, \ \overset{2}{y} \right)]\!] f(y, t)$$

is a formal asymptotic solution of the problem

$$\begin{cases} -i \dfrac{\partial u}{\partial t} + i \dfrac{\partial u}{\partial y} = f(y, t) \\ u|_{t=0} = 0, \end{cases}$$

where $f(y, t) = 0$, when $t < 0$.

Note that to prove Eq. (7.18) we have used only the additive dependence of the operator $F\left(\overset{1}{A}_0, \overset{1}{A}_1, \overset{2}{B}\right)$ on A_0

$$F\left(\overset{1}{A}_0, \ \overset{1}{A}_1, \ \overset{2}{B}\right) = A_0 + F'\left(\overset{1}{A}_1, \ \overset{2}{B}\right).$$

Hence the Main Theorem permits to construct a solution of the Cauchy problem in a more general situation.

Example 7.2. Consider an equation

$$-i\frac{\partial u}{\partial t}+G\left(\overset{2}{y},\ -i\overset{1}{\frac{\partial}{\partial y}}\right)u-i\nu(t)\left(1-\sum\frac{\partial^2}{\partial y_i}\right)^{1/2}u=f(y,\ t). \tag{7.19}$$

Here $G(y, \xi)$ is an $m \times m$-matrix positively homogeneous of the first order in ξ, $f(y, t) \in L_2$, $\operatorname{supp} f(y, t) \subset \Delta \subset \mathbf{R}^n \times [0, T]$, $f(y, t) = 0$, when $t < 0$,

$\nu(t)$ is the same as in the previous example,

Δ is a compact domain.

Let the matrix elements of $G(y, \xi)$ belong to the space $C^\infty(\mathbf{R}^n \times \times (\mathbf{R}^n \setminus 0))$ and be uniformly bounded with the derivatives when $|\xi| = 1$. Suppose that the matrix $G(y, \xi)$ is diagonalizable for all values of its arguments by a smooth transformation and its eigenvalues $\gamma_1, \ldots, \gamma_l$ $(l \leqslant m)$ are of the space $C^\infty(\mathbf{R}^n \times (\mathbf{R}^n \setminus 0))$ and satisfy the conditions $\operatorname{Im} \gamma_i \leqslant 0$, $i = 1, \ldots, l$, $\inf |\gamma_i(y, \xi) - \gamma_j(y, \xi)| \geqslant \geqslant \text{const}$ when $|\xi| = 1$.

Let the Hamilton system

$$\frac{dq^{(i)}}{d\tau}=\frac{\partial \operatorname{Re}\gamma_i}{\partial p}(q^{(i)},\ p^{(i)}), \qquad (q_0^{(i)}(0),\ \ldots,\ q_n^{(i)}(0)) \in \Delta_\varepsilon$$

$$\frac{dp^{(i)}}{d\tau}=-\frac{\partial \operatorname{Re}\gamma_i}{\partial q}(q^{(i)},\ p^{(i)}), \qquad |p^{(i)}(0)|=1$$

have the solution for $0 \leqslant \tau < \infty$. Then we shall construct a formal asymptotic solution of the problem

$$\begin{cases} -i\dfrac{\partial u}{\partial t}+G\left(\overset{2}{y},\ -i\overset{1}{\dfrac{\partial}{\partial y}}\right)u=f(y,\ t) \\ u(y,\ 0)=0 \end{cases}$$

with respect to the powers of $\left(1+\left|\frac{\partial}{\partial y}\right|^2\right)^{-1}$ when $0 \leqslant t \leqslant T$. We shall use a Δ-quasi-inverse of the operator

$$-i\frac{\partial}{\partial t}+G\left(\overset{2}{y},\ -i\overset{1}{\frac{\partial}{\partial y}}\right)-i\nu(t)\left(1-\sum_{i=1}^{n}\frac{\partial^2}{\partial y_i^2}\right)^{1/2}.$$

We mentioned in Introduction that solution of partial differential equations reduces by quasi-inversion to that of integral equations (in the present case of the Volterra type) with smooth kernels.

Take the operators

$$A_0 = -i\frac{\partial}{\partial t},\ A_1 = -i\frac{\partial}{\partial y_1},\ \ldots,\ A_n = -i\frac{\partial}{\partial y_n},\ B = (t,\ y),$$

$$F\left(\overset{1}{A}_0,\ \overset{1}{A}_1,\ \ldots,\ \overset{1}{A}_n,\ \overset{2}{B}\right) = A_0 + G\left(\overset{2}{B},\ \overset{1}{A}_1,\ \ldots,\ \overset{1}{A}_n\right) - \\ - i\nu(B_0)\cdot\left(1+\sum_{i=1}^{n} A_i^2\right)^{1/2}.$$

The Hamiltonian of the operator F has the form:

$$F\left(\overset{1}{L}_0,\ \ldots,\ \overset{1}{L}_n,\ \overset{2}{\alpha}\right) = \left(x_0 - i\frac{\partial}{\partial\alpha_0}\right) + \\ + G\left(\overset{2}{\alpha},\ \overset{1}{L}_1,\ \ldots,\ \overset{1}{L}_n\right) - i\nu(\alpha_0)\left(1+\overset{1}{L}{}_1^2+\ldots+\overset{1}{L}{}_n^2\right)^{1/2},$$

$$L_i = x_i - i\frac{\partial}{\partial\alpha_i},\quad i = 1,\ \ldots,\ n.$$

Hence,

$$F(L_0(y,\ \eta),\ \ldots,\ L_n(y,\ \eta),\ \alpha) = (y_0 + y_{n+1}) + \\ + G(\alpha,\ y_1 + y_{n+2},\ \ldots,\ y_n + y_{2n+1}) - \\ - i\nu(\alpha_0)\left(1+\sum_{i=1}^{n}(y_i + y_{n+i+1})^2\right)^{1/2}.$$

Take $\rho_1 = \ldots = \rho_{2n+1} = 1$ and change the variables y, η, α for the new variables p, q, ω. We have

$$\mathcal{H}(p,\ q,\ \omega) = (\omega_0 + p_0 - p_{n+1}) + \\ + G(q_1,\ \ldots,\ q_n,\ \omega_1 + p_1 - p_{n+2},\ \ldots,\ \omega_n + p_n - p_{2n+1}) - \\ - i\nu(q_0)\left(\sum_{i=1}^{n}(\omega_i + p_i - p_{n+i+1})^2\right)^{1/2}.$$

The eigenvalues $\lambda_1, \ldots, \lambda_l$ of the matrix $\mathcal{H}(p, q, \omega)$ are obviously defined by the equations

$$\lambda_i(p,\ q,\ \omega) = \omega_0 + p_0 - p_{n+1} + \\ + \gamma_i(q_1,\ \ldots,\ q_n;\ \omega_1 + p_1 - p_{n+2},\ \ldots,\ \omega_n + p_n - p_{2n+1}) - \\ - i\nu(q_0)\left(\sum_{i=1}^{n}(\omega_i + p_i - p_{n+i+1})^2\right)^{1/2}.$$

We shall now verify that the operator $F\left(\overset{1}{A}_0, \overset{1}{A}_1, \ldots, \overset{1}{A}_n, \overset{2}{B}\right)$ satisfies the global absorption conditions. Consider the ith family

of the bicharacteristics of the operator F

$$\begin{aligned}
&\dot q_0^{(i)} = 1\\
&\dot q_j^{(i)} = \frac{\partial \operatorname{Re} \lambda_i}{\partial p_j}(p^{(i)}, q^{(i)}, \omega), \quad j = 1, \ldots, 2n+2\\
&\dot p_0^{(i)} = 0\\
&\dot p_j^{(i)} = -\frac{\partial \operatorname{Re} \lambda_i}{\partial q_i}(p^{(i)}, q^{(i)}, \omega), \quad j = 1, \ldots, n\\
&\dot p_k^{(i)} = 0, \quad n < k \leqslant 2n+1,\\
&p^{(i)}(0) = 0, \quad (q_0^{(i)}, \ldots, q_n^{(i)})|_{\tau=0} \in \Delta_\varepsilon, \quad \sum_{i=1}^{n} \omega_i^2 = 1, \quad \omega_0 \in \mathbf{R}^1\\
&\sum_{j=0}^{n} (q_{m+j+1}^{(i)})^2 |_{\tau=0} < \varepsilon.
\end{aligned} \tag{7.20}$$

The following equations are obviously true:

$$p_k^{(i)} = 0 \quad \text{as} \quad n < k \leqslant 2n+1, \tag{7.21}$$
$$q_0^{(i)} = \tau + q_0^{(i)}(0).$$

The remaining part of Eqs. (7.20) can be solved with the help of the following Hamilton system

$$\begin{aligned}
&\dot q_j^{(i)} = \frac{\partial \operatorname{Re} \gamma_i}{\partial p_j}(q_1^{(i)}, \ldots, q_n^{(i)}, \omega_1 + p_1^{(i)}, \ldots, \omega_n + p_n^{(i)}),\\
&\dot p_j^{(i)} = -\frac{\partial \operatorname{Re} \gamma_i}{\partial q_j}(q_1^{(i)}, \ldots, q_n^{(i)}, \omega_1 + p_1^{(i)}, \ldots, \omega_n + p_n^{(i)}),\\
&j = 1, \ldots, n, \quad (p_1^{(i)}, \ldots, p_n^{(i)})|_{\tau=0} = 0,\\
&(q_1^{(i)}, \ldots, q_n^{(i)})|_{\tau=0} \in \Delta_\varepsilon.
\end{aligned} \tag{7.22}$$

The functions $q_j^{(i)}$, when $n \leqslant j \leqslant 2n+1$, are determined by the formulas

$$q_j^{(i)} = q_j^{(i)}(0) + \int_0^\tau \frac{\partial \operatorname{Re} \lambda_i}{\partial p_j}(p_1^{(i)}, \ldots, p_n^{(i)}, q_1^{(i)}, \ldots, q_n^{(i)}, \omega)\, d\tau'.$$

By our assumptions Eqs. (7.22) and, therefore, Eqs. (7.20), have a solution. Hence the operator $F\left(\overset{1}{A}_0, \ldots, \overset{1}{A}_n, \overset{2}{B}\right)$ satisfies the global absorption conditions by Eq. (7.21) and the choice of the function $\nu(q_0)$. Hence by Lemma 7.1 the symbol of the Δ-quasi-inverse operator $\varkappa\left(\overset{1}{A}_0, \ldots, \overset{1}{A}_n, \overset{2}{B}\right)$ has the form

$$\varkappa(x_0, x, \alpha) = i \int_0^{2T} \sum_{j=1}^{l} [\![(\mathscr{K}_{M_{j(\omega)}^{2n+3}} \varphi_j) \left(\alpha, \tau, -i\frac{\overset{2}{\partial}}{\partial x'}, \overset{1}{x'}\right)]\!]\, 1\, d\tau.$$

Here $M_j^{2n+3}(\omega)$ is a family of $2n+3$-dimensional Lagrangean manifolds depending on the parameters

$$\omega_1 = \frac{x_1}{\Lambda}, \quad \ldots, \quad \omega_n = \frac{x_n}{\Lambda}, \qquad \Lambda = |x|.$$

The family $M_j^{2n+3}(\omega)$ is associated with a family of Lagrangean manifolds with a complex germ:

$$\{M_{j,\tau}^{2n+2}(\omega),\ r_{j,\tau}^{2n+2}(\omega)\} = D_{\lambda_j}^{\tau}\{M_0^{2n+2},\ r_0^{2n+2}\}, \qquad 0 \leqslant \tau \leqslant 2T,$$

where $r_0^{2n+2} \equiv 0$, $M_0^{2n+2} = \{p_0 = \ldots = p_n = 0,\ (q_0, \ldots, q_n) \in \Delta_\varepsilon\}$.

Finally, $\mathscr{K}_{M^{2n+3}}\varphi_j$ is the value on the matrix φ_j of the canonical operator defined on the family M_j^{2n+3}. The elements of φ_j belong to the space of equivalence classes of D-asymptotic series on $M_j^{2n+3}(\omega)$.

The formula

$$\varkappa(x_0, x, \alpha) = i \int_0^{2T} e^{-ix_0\tau}\varkappa'(x, \alpha, \tau)\, d\tau$$

can be proved as in Example 7.1 by using the expression of the eigenvalue

$$\lambda_j(p, q, \omega) = \omega_0 + p_0 + p_{n+1} + \\ - \lambda_j'(p_1, \ldots, p_{2n+1}, q_1, \ldots, q_n, \omega_1, \ldots, \omega_n).$$

Then, again applying the method of Example 7.1, we obtain

$$\left\{ [\![\varkappa\left(-i\frac{\overset{1}{\partial}}{\partial t},\ -i\frac{\overset{1}{\partial}}{\partial y},\ \overset{2}{t},\ \overset{2}{y}\right)]\!]\, f(y, t)\right\}\Bigg|_{t=0} = 0$$

under the assumption $f(y, t) = 0$ when $t < 0$.

Hence the function

$$u(y, t) = [\![\varkappa\left(-i\frac{\overset{1}{\partial}}{\partial t},\ -i\frac{\overset{1}{\partial}}{\partial y},\ \overset{2}{t},\ \overset{2}{y}\right)]\!]\, f(y, t)$$

is a formal asymptotic solution of the problem

$$\begin{cases} -i\dfrac{\partial u}{\partial t} + G\left(\overset{2}{y},\ -i\dfrac{\overset{1}{\partial}}{\partial y}\right) u = f(y, t) \\ u(y, 0) = 0 \end{cases}$$

when $0 \leqslant t \leqslant T$.

Example 7.3. The Main Theorem affords the construction of a kind of asymptotic solution of equations with partial derivatives with growing coefficients. Consider an example of a formal asymptotic (with respect to powers of the operator $\left(\overset{2}{y^2} + \dfrac{\overset{1}{\partial^2}}{\partial y^2}\right)^{-1/2}$) solu-

tion of the problem

$$\begin{cases} -\dfrac{\partial^2 u}{\partial t^2}+\dfrac{\partial^2 u}{\partial y^2}-a(y)\,y^2u=f(t,\,y), \\ u\,|_{t=0}=u_t'\,|_{t=0}=0. \end{cases} \tag{7.23}$$

Suppose that

$$f(t,\,y)\in L_2,\quad \operatorname{supp} f(t,\,y)\subset\Delta\subset[0,\,T]\times\mathbf{R},$$

where Δ is a compact domain, $a(y)$ is a smooth real function satisfying the conditions $0<c_1\leqslant a(y)\leqslant c_2$ for some constants c_1 and c_2.

Consider the operators

$$\overset{1}{A}_0=-i\frac{\partial}{\partial t},\quad \overset{1}{A}_1=-i\frac{\partial}{\partial y},\ \overset{2}{A}_2=y,$$

$$\overset{2}{B}_0=t,\quad \overset{2}{B}_1=y.$$

The left-order representatives of these operators have the form

$$\overset{1}{L}_0=\left(x_0-i\frac{\partial}{\partial\alpha_0}\right),\ \overset{1}{L}_1=x_1-i\frac{\partial}{\partial\alpha_1}-i\frac{\partial}{\partial x_2},$$

$$\overset{2}{L}_2=x_2;\ \overset{2}{\alpha}.$$

Eq. (7.23) may be put in the form

$$[\![A_0^2-A_1^2-a(B_1)\,A_2^2]\!]\,u=f.$$

We shall reduce Eq. (7.24) to a system of equations. Consider the equations

$$\begin{gathered} [\![A_0]\!]\,u(B)=v, \\ [\![\left(\overset{1}{A}{}_1^2+a\left(\overset{2}{B}_1\right)\overset{2}{A}{}_2^2\right)^{1/2}]\!]\,u=w. \end{gathered} \tag{7.25}$$

There is an equation

$$[\![A_0]\!]\,w-[\![\left(\overset{1}{A}{}_1^2+a\left(\overset{2}{B}_1\right)\overset{2}{A}{}_2^2\right)^{1/2}]\!]\,v=0. \tag{7.26}$$

The product of the operators

$$[\![\left(\overset{1}{A}{}_1^2+a\left(\overset{2}{B}_1\right)\overset{2}{A}{}_2^2\right)^{1/2}]\!]\,[\![\left(\overset{1}{A}{}_1^2+a\left(\overset{2}{B}_1\right)\overset{2}{A}{}_2^2\right)^{1/2}]\!]$$

can be transformed by the commutation formula into the form

$$\begin{aligned} &[\![\left(\overset{1}{A}{}_1^2+a\left(\overset{2}{B}_1\right)\overset{2}{A}{}_2^2\right)^{1/2}]\!]\,[\![\left(\overset{1}{A}{}_1^2+a\left(\overset{2}{B}_1\right)\overset{2}{A}{}_2^2\right)^{1/2}]\!]= \\ &\qquad =A_1^2+a(B_1)\,A_2^2+R\left(\overset{1}{A}_1,\,\overset{2}{A}_2,\,\overset{2}{B}\right), \end{aligned} \tag{7.27}$$

where the symbol $R(x_1, x_2, \alpha)$ is an asymptotic sum

$$R(x_1, x_2, \alpha) = \sum_{k \geqslant 0} R_k(x_1, x_2, \alpha),$$

where the functions $R_k(x_1, x_2, \alpha)$ are homogeneous of a positive degree $(-k+1)$ with respect to x_1, x_2.

The following equation is a corollary of Eq. (7.27)

$$[\![\left(\overset{1}{A}{}_1^2 + a\left(\overset{2}{B}_1\right)\overset{2}{A}{}_2^2\right)^{1/2}]\!] w = [\![A_1^2 + a(B_1)A_2^2]\!] u + \\ + [\![R\left(\overset{1}{A}_1, \overset{2}{A}_2, \overset{2}{B}\right)]\!] u.$$

Substitute the function $u = [\![\left(\overset{1}{A}{}_1^2 + a\left(\overset{2}{B}_1\right)\overset{2}{A}{}_2^2\right)^{1/2}]\!]^{-1} w$ into the last equation. On applying the commutation formula

$$[\![\left(\overset{1}{A}{}_1^2 + a\left(\overset{2}{B}_1\right)\overset{2}{A}{}_2^2\right)^{1/2}]\!] w = [\![A_1^2 + a(B_1)A_2^2]\!] u + \\ + [\![R^1\left(\overset{1}{A}_1, \overset{2}{A}_2, \overset{2}{B}\right)]\!] w, \tag{7.28}$$

where the symbol $R^1(x_1, x_2, \alpha)$ is an asymptotic sum

$$R^1(x_1, x_2, \alpha) = \sum_{k \geqslant 0} R'_k(x_1, x_2, \alpha).$$

The functions $R'_k(x_1, x_2, \alpha)$ are homogeneous of degree $-k$ with respect to x.

Thus, Eq. (7.24) is reduced to the system

$$[\![A_0]\!] w - [\![\left(\overset{1}{A}{}_1^2 + a\left(\overset{2}{B}_1\right)\overset{2}{A}{}_2^2\right)^{1/2}]\!] v = 0, \tag{7.29}$$

$$[\![A_0]\!] v - [\![\left(\overset{1}{A}{}_1^2 + a\left(\overset{2}{B}_1\right)\overset{2}{A}{}_2^2\right)^{1/2}]\!] w + \\ + [\![R^1\left(\overset{1}{A}_1, \overset{2}{A}_2, \overset{2}{B}\right)]\!] w = f.$$

Similarly to the previous examples the construction of a formal asymptotic solution of Eq. (7.29) is reduced in the case to the construction of a Δ-quasi-inverse operator of the operator

$$F\left(\overset{1}{A}_0, \overset{1}{A}_1, \overset{2}{A}_2, \overset{2}{B}\right) = \\ = \begin{pmatrix} A_0 & -\left(\overset{1}{A}{}_1^2 + a\left(\overset{2}{B}_1\right)\overset{2}{A}_2\right)^{1/2} \\ -\left(\overset{1}{A}{}_1^2 + a\left(\overset{2}{B}_1\right)\overset{2}{A}{}_2^2\right)^{1/2} + R^1\left(\overset{1}{A}_1, \overset{2}{A}_2, \overset{2}{B}\right) & A_0 \end{pmatrix} - \\ - i\nu\left(\overset{2}{B}_0\right)\left(\overset{1}{A}{}_1^2 + \overset{2}{A}{}_2^2\right)^{1/2},$$

where the function $\nu(\alpha_0)$ is the same as above. The Hamiltonian of the operator F has the form

$$F\left(\overset{1}{L}_0, \overset{1}{L}_1, \overset{2}{\alpha}\right) =$$

$$= \begin{pmatrix} \left(x_0 - i\dfrac{\partial}{\partial\alpha_0}\right) & -\left(\left(x_1 - i\dfrac{\overset{1}{\partial}}{\partial\alpha_1} - \dfrac{\overset{1}{\partial}}{\partial x_2}\right)^2 + a\left(\overset{2}{\alpha}_1\right)\overset{2}{x}{}_2^2\right)^{1/2} \\ -\left(\left(x_1 - i\dfrac{\overset{1}{\partial}}{\partial\alpha_1} - i\dfrac{\overset{1}{\partial}}{\partial x_2}\right)^2 + a\left(\overset{2}{\alpha}_1\right)\overset{2}{x}{}_2^2\right)^{1/2} + & \\ + R^1\left(\overset{1}{L}_1, \overset{2}{L}_2, \overset{2}{\alpha}\right) & \left(x_0 - i\dfrac{\partial}{\partial\alpha_0}\right) \end{pmatrix} -$$

$$- i\nu(\alpha_0)\left(\left(x_1 - i\frac{\partial}{\partial\alpha_1} - i\frac{\overset{1}{\partial}}{\partial x_2}\right)^2 + \overset{2}{x}{}_2^2\right)^{1/2}.$$

Taking the variables (p, q, ω), we obtain

$$\mathscr{H}(p, q, \omega) =$$

$$= \begin{pmatrix} \omega_0 + p_0 - p_2 & -((\omega_1 + p_1 - p_3)^2 + \\ & + a(q_1)(\omega_2 - p_4)^2)^{1/2} \\ -((\omega_1 + p_1 - p_3)^2 + & \\ + a(q_1)(\omega_2 - p_4)^2)^{1/2} & \omega_0 + p_0 - p_2 \end{pmatrix} -$$

$$- i\nu(q_0)((\omega_1 + p_1 - p_3)^2 + (\omega_2 - p_4)^2)^{1/2}.$$

The eigenvalues λ_1, λ_2 of the matrix $\mathscr{H}(p, q, \omega)$ have the form

$$\lambda_{1,2}(p, q, \omega) = \omega_0 + p_0 - p_2 \pm ((\omega_1 + p_1 - p_3)^2 +$$
$$+ a(q_1)(\omega_2 - p_4)^2)^{1/2} - i\nu(q_0)((\omega_1 + p_1 - p_3)^2 + (\omega_2 - p_4)^2)^{1/2}.$$

The bicharacteristics of the operator F obviously satisfy the conditions

$$p_0^{(i)} = p_2^{(i)} = p_2^{(i)} = p_3^{(i)} = 0, \quad i = 1, 2.$$

Hence the equation is true

$$\lambda_i(p^{(i)}, q^{(i)}, \omega) = \omega_0 \pm ((\omega_1 + p_1^{(i)})^2 + a(q_1^{(i)})\omega_2^2)^{1/2} -$$
$$- i\nu(q_0^{(i)})((\omega_1 + p_1)^2 + \omega_2^2)^{1/2}, \quad i = 1, 2.$$

The following equation is easy to verify

$$q_0^{(i)} = \tau + q_0^{(i)}(0).$$

Hence the operator F satisfies the global absorption conditions and therefore is Δ-quasi-invertible. The initial value conditions in this case are verified similarly to the previous one.

We shall consider an analogue of Theorem 9.1 of Introduction in the case of operators with matrix symbols.

Take a one-parameter families of m-dimensional matrix symbols $f(x_1, \ldots, x_n, \alpha, \xi)$, $g(\alpha, \xi, x_0)$ where ξ is a parameter $0 < \xi < \infty$. Suppose that

$$\lim_{\xi \to \infty} f(x_1, \ldots, x_n, \alpha, \xi) = \delta_0(x_1, \ldots, x_n, \alpha) \in \mathcal{S}^\infty.$$

Consider a family of operators

$$\hat{F} = g\left(\overset{2}{B}, \xi, \overset{1}{A_0}\right) + f\left(\overset{1}{A_1}, \ldots, \overset{n}{A_n}, \overset{n+1}{B}, \xi\right),$$

where g is an $m \times m$-matrix with elements of $\mathcal{S}^\infty$. Let $\overset{1}{L_0}, \ldots$ $\ldots, \overset{n+1}{L_n}, \overset{n+2}{\alpha}$ be a left ordered representation of the operators $A_0, \ldots, A_n, B$. Suppose that the function

$$F(L_0(y, \eta), \ldots, L_n(y, \eta), \alpha, \xi),$$

$$y_i \Longleftrightarrow x_i, \; i = 0, \ldots, n;$$

$$y_{n+k} \Longleftrightarrow -i \frac{\partial}{\partial \alpha_{k-1}}, \qquad k = 1, 2, \ldots, m+1$$

is asymptotically ρ-quasi-homogeneous in y, ξ of degree r with the weights

$$\rho_0, \ldots, \rho_n, \rho_{n+1}, \ldots, \rho_{n+m+1}, \rho_{n+m+2}$$

such that $\rho_i = 1$ at least for one i, $i \in \{n+1, \ldots, m+n+1\}$.

Suppose that the spectrum of the set of the operators

$$\xi^{-\rho_0} \overset{1}{A_0}, \ldots, \xi^{-\rho_n} \overset{n}{A_n}$$

is contained in a ball, $|x| < d$, d is a constant.

Take the new variables. Let J_1 be such a subset of $(0, \ldots, n)$ that $\rho_i = 1$ only for $i \in J_1$. Let J_2 be such a subset of $(0, \ldots, m)$ that $\rho_{n+j+1} = 1$ only if $j \in J_2$.

Take

$$y_{n+j+1} = p_j, \; j \in J_2, \quad y_{n+j} = 0, \; j \bar{\in} J_2 \wedge j \in (0, \ldots, m),$$

$$y_i = -p_{m+i+1} + \omega_i, \quad i \in J_1, \; y_i = \omega_i, \; i \bar{\in} J_1, \; i \in (0, \ldots, n),$$

$$\alpha_i = q_i, \; i = 0, \ldots, m, \quad \eta_i = q_{m+j+1}, \; j = 0, \ldots, n, \; \xi = \nu.$$

Let $\pi(y, \eta, \alpha, \xi)$ be the leading part of the matrix $F(L_0, (y, \eta), \ldots, L_n(y, \eta), \alpha, \xi)$ and let $\mathcal{H}(p, q, \omega, \nu)$ be the matrix $\pi(y, \eta, \alpha, \xi)$ transformed by the introduced change of variables.

Let $\mathcal{H}_1$ be the essential part of the matrix $F(L_0(y, \eta), \ldots$ $\ldots, L_n(y, \eta), \alpha, \xi)$ with respect to the variables p, q, ω, ν.

Just as at the beginning of this section, introduce the absorption conditions. The only difference constitutes a subsidiary inequality $|\nu| < d$ due to the presence of an additional parameter ν. The conditions defining the manifolds $\Omega_\varepsilon^{(i)}$, $\Omega_\varepsilon^{(i)}(\Delta)$ and Eqs. (7.3) remain valid.

Theorem 7.2. Let an operator $\hat{F} = g\left(\overset{2}{B}, \xi, \overset{1}{A}_0\right) + f\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}, \xi\right)$ satisfy the absorption conditions. Then there exists such a sequence of symbols $\varkappa_N(x_0, x, \alpha, \xi) \in C^\infty_{\mathscr{L}}$ depending on a parameter ξ that

$$[\![g\left(\overset{2}{B}, \xi, \overset{1}{A}_0\right) + f\left(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}, \xi\right)]\!]\,[\![\varkappa_N\left(\overset{1}{A}_0, \overset{2}{A}_1, \ldots, \overset{n+1}{A}_n, \overset{n+2}{B}, \xi\right)]\!] =$$
$$= P_0\left(\xi^{-\rho_0}\overset{1}{A}_0, \ldots, \xi^{-\rho_n}\overset{n}{A}_n\right) + R_N\left(\overset{1}{A}_0, \ldots, \overset{n+1}{A}_n, \overset{n+2}{B}, \xi\right),$$

where $R_N(x_0, x_1, \ldots, x_n, \alpha, \xi) = O_{\mathscr{L}}(|x|^{-N})$ for any $N > 0$ uniformly with respect to ξ, α and x_0; $P_0(x_1, \ldots, x_n)$ is a function of the class C_0^∞ which is equal to 1 in the domain $|x| < d$.

A version of Lemma 7.1 and the corresponding statements about Δ-quasi-inverse sequence are also true for operators with symbols depending on a parameter.

As an application of the theorem, consider an example of a formal asymptotic solution of the Cauchy problem for finite difference schemes. Take a net Ω_h in the space $\mathbf{R}^1 \times \mathbf{R}^n$

$$(t, y) \in \Omega_h \Longleftrightarrow (t = kh_0,\ y_1 = j_1 h,\ \ldots,\ y_n = j_n h),$$

where $k, j_1, \ldots, j_n$ are integers.

Consider the following finite difference Cauchy problem

$$\frac{U^{k+1}_{j_1, \ldots, j_n} - U^k_{j_1, \ldots, j_n}}{h_0} = \sum_{\nu=1}^{n} C^{(\nu)}_{j_1, \ldots, j_n} \times$$
$$\times \frac{U^{k+1}_{j_1, \ldots, j_{\nu+1}, \ldots, j_n} - U^{k+1}_{j_1, \ldots, j_{\nu-1}, \ldots, j_n}}{2h} + f^{k+1}_{j_1, \ldots, j_n},$$
$$U^0_{j_1, \ldots, j_n} = 0 \tag{7.30}$$

where $C^{(\nu)}_{j_1, \ldots, j_n}$ are the values on the net of the $(m \times m)$ matrices $C_\nu(y)$ with smooth bounded elements and $U^n_{j_1, \ldots, j_n}$, $f^n_{j_1, \ldots, j_n}$ are net functions defined on the net. Suppose that the function $f^k_{j_1, \ldots, j_n}$ equals zero at all knots of the net outside a compact domain $\Delta \subset [0, T] \times \mathbf{R}^n$.

On applying the Kotelnikov theorem, construct for a net function $\varphi^k_{j_1, \ldots, j_n}$ satisfying the condition

$$h_0 \cdot h^n \sum_{k, j_1, \ldots, j_n} |\varphi^k_{j_1, \ldots, j_n}|^2 < \infty, \tag{7.31}$$

an integral function $\varphi_h(t, y)$ satisfying the conditions

(1) $\varphi_h(k\tau, j_1 h, \ldots, j_n h) = \varphi^k_{j_1, \ldots, j_n}$;

(2) $\displaystyle\int_{\mathbf{R}^{n+1}} |\varphi_h(t, y)|^2\, dy\, dt < \infty$

Take

$$\varphi_h(t, y) = \left(\frac{1}{2\pi}\right)^{n+1} \int\limits_{\substack{|\xi_i h| \leqslant \pi,\ i=1, \ldots, n \\ |\xi_0 h| \leqslant \pi}} e^{i\langle y, \xi\rangle + i\langle t, \xi_0\rangle} \times$$

$$\times h_0 \cdot h^n \sum_{k, j_1, \ldots, j_n} \varphi^k_{j_1, \ldots, j_n} \times$$

$$\times e^{-i\xi_0 k h_0 - i \sum\limits_{k=1}^{n} j_k h \xi_k} d\xi_0\, d\xi_k. \qquad (7.32)$$

It is easy to verify that the given function $\varphi_h(t, y)$ satisfies all the necessary conditions.

Let $f_h(t, y)$ be a function of continuous argument defined by Eq. (7.32) from the net function $f^k_{j_1, \ldots, j_n}$.

Suppose we have constructed a continuous function which is a solution of the problem

$$\frac{u(t+h_0, y, h, h_0) - u(t, y, h, h_0)}{h_0} = \sum_{\nu=1}^{n} C_\nu(y) \times$$

$$\times \frac{u(t+h_0, y_1, \ldots, y_\nu + h, \ldots, y_n, h, h_0) - u(t+h_0, y_1, \ldots, y_n, h, h_0)}{h} +$$

$$+ f_h(t+h_0, y) \qquad (7.33)$$

$$u(0, y, h, h_0) = 0.$$

It is clear that the restriction of the function $u(t, y, h, h_0)$ on the net Ω_h is a solution of Problem (7.30). Thus, the construction of a formal asymptotic solution of Problem (7.30) is reduced to the construction of a formal asymptotic solution of Problem (7.33).

We infer from the estimate

$$\left\| h \frac{\partial}{\partial y_i} f_h(t, y) \right\|_{L_2} \leqslant \pi \| f_h(t, y) \|_{L_2}$$

that the joint spectrum of the operators $-ih\frac{\partial}{\partial y_1}, \ldots, -ih\frac{\partial}{\partial y_n}$ is contained in a ball $|x| \leqslant \pi\sqrt{n}$.

Take

$$\overset{1}{A}_0 = -i\frac{\partial}{\partial t}, \ \overset{1}{A}_1 = -i\frac{\partial}{\partial y_1}, \ \ldots, \ \overset{1}{A}_n - = i\frac{\partial}{\partial y_n}, \ \xi = \frac{1}{h},$$

$$\overset{2}{B}_0 = t, \ \overset{2}{B}_1 = y_1, \ \ldots, \ \overset{2}{B}_n = y_n.$$

It is easy to verify that the left ordered representation of the indicated operators is the set of operators

$$\overset{1}{L}_0 = x_0 - i\frac{\partial}{\partial \alpha_0}, \ \ldots, \quad \overset{n+1}{L}_n = x_n - i\frac{\partial}{\partial \alpha_n}; \ \overset{n+2}{\alpha}.$$

Applying this notation we may put Eq. (7.33) in the from

$$[\![\xi(1-e^{-i\xi^{-1}A_0})+\xi\sum_{\nu=1}^{n}C_\nu\left(\overset{2}{B}\right)\sin\xi^{-1}\overset{1}{A}_\nu]\!]u=f_h. \tag{7.34}$$

Similarly to the previous examples, a formal asymptotic solution of Eq. (7.34) is constructed by a Δ-quasi-inverse operator. Indeed, take an operator

$$F\left(\overset{1}{A}_0,\overset{1}{A}_1,\ldots,\overset{1}{A}_n,\overset{2}{B},\xi\right)\overset{\text{def}}{=}\xi[1-e^{-i\xi^{-1}A_0}]+$$
$$+\xi\sum_{\nu=1}^{n}C_\nu\left(\overset{2}{B}\right)\sin\xi^{-1}\overset{1}{A}_\nu-i\nu(B_0)\cdot\left(1+\sum_{\nu=1}^{n}A_\nu^2\right)^{1/2},$$

where the function $\nu(\alpha_0)$ is the same as in the previous examples. Taking the variables p, q, ω, ν, we obtain

$$\mathscr{H}(p,q,\omega,\nu)=\frac{1}{\nu}[1-e^{-i\nu(\omega_0+p_0-p_{n+1})}]+$$
$$+\frac{1}{\nu}\sum_{j=1}^{n}C_j(q_1,\ldots,q_n)\sin\nu(\omega_j+p_j-p_{n+j+1})-$$
$$-i\nu(\alpha_0)\left(\sum_{j=1}^{n}(\omega_j+p_j-p_{n+j+1})^2\right)^{1/2}.$$

Let $\lambda_1,\ldots,\lambda_l\,(l\leqslant m)$ be the eigenvalues of the matrix $\mathscr{H}(p,q,\omega,\nu)$. The following equations are obviously true

$$\lambda_i(p,q,\omega,\nu)=\frac{1}{\nu}[1-e^{-i\nu(\omega_0+p_0-p_{n+1})}]+$$
$$+\gamma_i(q_1,\ldots,q_n,\omega_1+p_1-p_{n+1},\ldots,\omega_n+p_n-p_{n+1},\nu)-$$
$$-i\nu(q_0)\left(\sum_{j=1}^{n}(\omega_j+p_j-p_{n+j+1})^2\right)^{1/2}, \tag{7.35}$$

where γ_i are eigenvalues of the matrix

$$\frac{1}{\nu}\sum_{j=1}^{n}C_j(q_1,\ldots,q_n)\sin\nu(\omega_j+p_j-p_{n+j+1}).$$

Take

$$H^{(i)}=\operatorname{Re}\gamma_i,\qquad \widetilde{H}^{(i)}=\operatorname{Im}\gamma_i.$$

Suppose the following inequalities to be verified

$$\widetilde{H}^{(i)}\leqslant 0,\qquad i=1,\ldots,l. \tag{7.36}$$

We shall define the sufficient conditions so that the absorption conditions were satisfied in the present example.

By definition, the ith family of bicharacteristics of the operator is a solution of the Hamilton system

$$\left\{\begin{aligned} &\dot{q}_0^{(i)} = \cos\nu\,(\omega_0 + p_0 - p_{n+1}),\\ &\dot{q}_j^{(i)} = \frac{\partial \operatorname{Re}\lambda_i}{\partial p_j}(p^{(i)}, q^{(i)}, \omega, \nu), \quad j = 1, \ldots, 2n+1,\\ &\dot{p}_0^{(i)} = \dot{p}_{n+j}^{(i)} = 0,\\ &\dot{p}_j^{(i)} = -\frac{\partial H^{(i)}}{\partial q_j}(q_1^{(i)}, \ldots, q_n^{(i)}, \omega_1 + p_1^{(i)} - p_{n+1}^{(i)}, \ldots\\ &\qquad \ldots, \omega_n + p_n^{(i)} - p_{2n+1}^{(i)}, \nu), \quad 1 \leqslant j \leqslant n. \end{aligned}\right. \tag{7.37}$$

By Eq. (7.37) the equation

$$\begin{aligned} &q_0^{(i)} = \tau\cos\nu\,\omega_0 + q_0^{(i)}(0),\\ &p_j^{(i)} \equiv 0, \qquad n < j \leqslant 2n+1 \end{aligned} \tag{7.38}$$

is evidently true. Hence for any τ there exists a solution of Eq. (7.37) if for any τ there exists a solution of the following Hamilton system

$$\begin{aligned} &\dot{q}_j^{(i)} = \frac{\partial H^{(i)}}{\partial p_j}(q_1^{(i)}, \ldots, q_n^{(i)}, \omega_1 + p_1^{(i)}, \ldots, \omega_n + p_n^{(i)}, \nu)\\ &\dot{p}_j^{(i)} = -\frac{\partial H^{(i)}}{\partial q_j}(q_1^{(i)}, \ldots, q_n^{(i)}, \omega_1 + p_1^{(i)}, \ldots, \omega_n + p_n^{(i)}, \nu)\\ &p_1^{(i)}(0) = \ldots = p_n^{(i)}(0), \ (q_1^{(i)}(0), \ldots, q_n^{(i)}(0)) \in \Delta_\varepsilon,\\ &\sum_{i=1}^{n} \omega_i^2 = 1, \ |\nu| < \pi\sqrt{n}. \end{aligned} \tag{7.39}$$

Note that $\widetilde{H}^{(i)} \leqslant \dfrac{1 - \cos\nu\,(\omega_0 + p_0 - p_{n+1})}{\nu}$. So if $|\cos\nu\,\omega_0| \leqslant \delta$, where δ is a sufficiently small positive number, then $\widetilde{H}^{(i)} \leqslant \dfrac{1-\delta}{\nu} < 0$. Thus, by Eqs. (7.35), (7.36), (7.38) the operator F satisfies the global absorption conditions if Eq. (7.37) has a smooth solution for any τ and $i = 1, \ldots, l$. Thus we have proved the following proposition.

Proposition. An operator $F\left(\overset{1}{A}_0, \ldots, \overset{1}{A}_n, \overset{2}{B}, \xi\right)$ is a Δ-quasi-invertible if inequalities (7.35) are verified and the Hamilton system (7.39) has a smooth solution.

Suppose the conditions of the proposition are satisfied. Then the operator $F\left(\overset{1}{A}_1, \ldots, \overset{1}{A}_n, \overset{2}{B}, \xi\right)$ is Δ-quasi-invertible. Let $\varkappa(x_0, x_1, \alpha)$ be a symbol of its Δ-quasi-inverse. Similarly to the

previous examples we can show that the following equation is true

$$\varkappa(x_0, x, \alpha, \xi) = \int_0^{2T} \exp\{\tau\xi(e^{-i\xi^{-1}x_0} - 1)\}\, \varkappa'(x, \alpha, \tau, \xi)\, d\tau.$$

Prove the equality

$$[\![\varkappa(\overset{1}{A}_0, \overset{1}{A}_1, \ldots, \overset{1}{A}_n, \overset{2}{B}, \xi)]\!] f_h |_{t=0} = 0. \tag{7.40}$$

We have

$$[\![\varkappa(\overset{1}{A}_0, \overset{1}{A}_1, \ldots, \overset{1}{A}_n, \overset{2}{B}, \xi)]\!] f_h =$$

$$= [\![\int_0^{2T} \exp\{\tau\xi(e^{-i\xi^{-1}\overset{1}{A}_0} - 1)\}\, \varkappa'(\overset{1}{A}_1, \ldots, \overset{1}{A}_n, \overset{2}{B}, \tau, \xi)\, d\tau]\!] f =$$

$$= \int_0^{2T} \left\{ \overset{2}{[\![} \varkappa'(\overset{1}{A}_1, \ldots, \overset{1}{A}_n, \overset{2}{B}, \tau, \xi)]\!] \overset{1}{[\![} \exp\{\tau\xi(e^{-i\xi^{-1}\overset{1}{A}_0} -$$

$$- 1)\}]\!] f_h \right\} d\tau.$$

Thus the problem is reduced to the equation

$$J = \left\{ \exp\left\{ -\tau \frac{1 - e^{h_0 \frac{\partial}{\partial t}}}{h_0} \right\} f_h(t, y) \right\} \Big|_{t=0},$$

where $\tau \geqslant 0$, $f_h(t, y) = 0$ when $t = kh_0$, $k \leqslant 0$.

We have

$$J = \int_{-\pi/h}^{\pi/h} e^{it\zeta_0 - \tau \frac{1 - e^{-i\zeta_0 h_0}}{h_0}} \tilde{f}_h(\zeta_0, y)\, d\zeta_0, \tag{7.41}$$

where $\tilde{f}_h(\zeta_0, y)$ is a Fourier transform of the function $f_h(t, y)$ with respect to t

$$\tilde{f}_h(\zeta_0, y) = h_0 \sum_k e^{-ik\zeta_0 h_0} f_k(y, h). \tag{7.42}$$

By Eq. (7.32)

$$f_k(y, h) = \left(\frac{1}{2\pi}\right)^n \int_{|\xi_i h| \leqslant \pi,\ i=1, \ldots, k} e^{i\langle \xi, y\rangle} h^n \times$$

$$\times \sum_{j_1, \ldots, j_n} f^k_{j_1, \ldots, j_n} e^{-i \sum_{k=1}^{n} j_k h \xi_k}\, d\xi.$$

Denote $e^{-it\zeta_0}$ by z. We may put Eq. (7.41) in the form

$$J=\oint\limits_{|z|=1} e^{\frac{\tau}{h_0}z}\left(h\sum_{k\geq 1} f_k(y,h)\,z^k\right)\frac{dz}{z}=0.$$

Thus Eq. (7.40) is proved. Hence the restriction of the function

$$u(t, y, h, h_0)=[\![\varkappa\left(\overset{1}{A}_0, \ldots, \overset{1}{A}_n, \overset{2}{B}, \xi\right)]\!]\, f_h(t, y)$$

on the net Ω_h is a formal asymptotic (in the differentiable sense) solution of Eq. (7.30) in the stripe $[0, T] \times \mathbf{R}^n$.

Using this asymptotic expansion with respect to the smoothness, the asymptotic power expansion in h of the solution of Problem (7.33) (hence, of Problem (7.30) as well) can be obtained as for the crystal equation in Sec. 8 of Introduction.

APPENDIX

Spectral Expansion of T-products

Here we shall outline a rigorous approach to the Feynman operator calculus wlich will generalize the concept of T-products, widely used in modern quantum physics.

Define a T-product of functions of two self-adjoint operators A and B in the Hilbert space H.

Let $\Phi_k(x, y)$ be complex-valued functions such that for any arbitrarily small ε, $[\Phi_k(x, y)]^{\varepsilon} \in \mathscr{B}_0(\mathbf{R}^2)$. After Feynman we shall use non-integer numbers over operators:

$$\Phi\left(\overset{s_1}{A}, \overset{s_2}{B}\right) = \Phi\left(\overset{1}{A}, \overset{2}{B}\right)$$

if $s_1 < s_2$. Consider a sequence of bounded operators

$$V_N = \prod_{k=1}^{N} \Phi_k^{T/N}\left(\overset{k}{A}, \overset{k+1/2}{B}\right)$$

as $N \to \infty$ and $T > 0$. Set $\Delta t = \frac{T}{N}$ and let

$$\Phi(t, x, y) = \Phi_k(x, y) \text{ for } t = t_k \overset{\text{def}}{=} k\cdot\Delta t.$$

Then

$$V_N = \prod_{k=1}^{N} \Phi^{\Delta t}\left(t_k, \overset{t_k}{A}, \overset{t_k+\Delta t/2}{B}\right) = e^{\sum \Delta t \ln \Phi\left(t_k, \overset{t_k}{A}, \overset{t_k+\Delta t/2}{B}\right)} \tag{1}$$

if $\Phi(t, x, y) \neq 0$.

Assume $\ln \Phi(t, x, y) = \Phi(t, x) + \varphi(t, x, y)$.

For the symbol in the right-hand side of (1) we obtain $\exp\left\{\sum \Delta t \ln \Phi(t_k, x(t_k), y(t_k))\right\}$. Hence it is evident that the limiting symbol must be a functional on trajectories $x(t)$, $y(t)$.

When determining the spectrum of two operators, by virtue of Chapter 2 it suffices to consider the products of spectral families $E_{\Delta, x^0}(B) E_{\Delta, y^0}(A)$, ($E_{\Delta, z^0}(z)$ being the characteristic function of the interval $z_0 - \frac{\Delta}{2} \leqslant z < z_0 + \frac{\Delta}{2}$ centered on the point z^0) for any $(x^0, y^0) \in \mathbf{R}^2$ and sufficiently small Δ.

We shall apply the same approach for the T-product. Divide the interval $(-\infty, \infty)$ into subintervals of length δ centered on x_i^0 and setting $1 = \sum_i E_{\delta, y_i^0}(B)$ and in the same way $1 = \sum_j E_{\delta, x_j^0}(A)$ we obtain

$$V_N = \sum_{\substack{j_1, \ldots, j_N \\ i_1, \ldots, i_N}} \prod_{k=1}^{N} E_{\delta, x_{j_k}^0} \left(\overset{t_k}{A} \right) \Phi^{\Delta t} \left(t_k, \overset{t_k}{A}, \overset{t_k + \Delta t/2}{B} \right) \times$$
$$\times E_{\delta, y_{i_k}^0} \left(\overset{t_k + \Delta t/2}{B} \right). \tag{2}$$

Here the sum is taken with respect to every possible sequence $\{j_1, \ldots, j_N, i_1, \ldots, i_N\}$. By setting

$$x(t_k) \overset{\text{def}}{=\!=} x_{j_k}^0, \; y(t_k) \overset{\text{def}}{=\!=} y_{i_k}^0$$

we can say that the sum is taken with respect to the totality of discrete paths. In the limit it is natural to consider the set of all the trajectories in $\mathbf{R}^2$. Under broad assumptions regarding the operators A and B the spectrum of a T-product happens to be on the trajectories whose projection onto x is a set of stepwise trajectories. But the y-component is discontinuous everywhere and unbounded on any finite interval. When $A = i \frac{\partial}{\partial q}$, $B = q$, this fact reflects the well-known uncertainty principle in quantum mechanics.

The value of (2) on the family of n-step x-trajectories with arbitrary values at the points of discontinuity is equal to the n-th term of the perturbation series in φ*. Therefore the spectral expansion (2) in stepwise trajectories is identical with the expansion (1) in the perturbation theory series in φ. We shall proceed to the proof of the above assertions.

Let A and B be two self-adjoint vector operators $A = A^{(1)}, \ldots, A^{(r)}$, $B = B^{(1)}, \ldots, B^{(r)}$ in a Hilbert space H, A being a generator of degree l in the scale generated by the degree of B and vice versa.

* This fact is closely connected with the Feynman diagrams concept (see V.V. Belov and V.P. Maslov, Appendix in V.V. Belov, E.M. Vorobyov, V.E. Shatalov, *The Theory of Graphs*. Moscow, Vysshaya shkola, 1976.)

Let H'_s (H''_s) be the scale of Hilbert spaces with the norm

$$\|f\|'_s \stackrel{\text{def}}{=} \sum_j \|(B^{(j)}+i)^s f\|, \quad \left(\|f\|''_s = \sum_j \|(A^{(j)}+i)^s f\|\right), \qquad s \in \mathbf{Z}.$$

By assumption A is a generator from H'_s into H'_s of degree l for any integer $s \leqslant r+1$, $s > 0$. In the same way B is a generator from H''_M into H''_M of degree l. Let $E_\Omega(x)$ be a characteristic function of the set Ω belonging to the spectrum $\sigma(A)$ of the vector operator A and Mes Ω is the Lebesgue measure.

Definition. The spectra of $\overset{1}{A}$, $\overset{2}{B}$ will be called incompatible if the following two conditions are fulfilled:

(1) there exists an integer M and a constant $c > 0$ such that when Mes $\Omega \to 0$, $E_\Omega(A)$ as an operator from H'_M into H approaches zero like c (Mes $\Omega)^{1/2}$:

$$\|E_\Omega(A)\|_{H'_M \to H} \leqslant c\,|\mathrm{Mes}\,\Omega|^{1/2}; \tag{3}$$

(2) for all $|p| \geqslant c_1$ (Mes $\Omega)^{1/2+\varepsilon}$ ($\varepsilon > 0$ being an arbitrarily small number and c_1 being a constant) there exists a constant c_2 such that

$$\|E_\Omega(A)\, e^{iBp} E_\Omega(A)\| \leqslant c_2\ (\mathrm{Mes}\ \Omega)^{1/2+\varepsilon}, \quad p = p_1, \ldots p_r \tag{4}$$

for Mes $\Omega \to 0$.

Examples. (1) Let $A = x$, $B = i\dfrac{\partial}{\partial x}$, $H = L_2(\mathbf{R})$. Show that the spectra of these operators are incompatible. We have

$$\int \left| E_\Omega(x) \left(i\frac{\partial}{\partial x}+i\right)^{-2} f(x)\right|^2 dx \leqslant$$

$$\leqslant \Delta \max \left|\left(i\frac{\partial}{\partial x}+i\right)^{-2} f(x)\right|^2 \leqslant \Delta\, \|f(x)\|^2, \quad \Delta = \mathrm{Mes}\ \Omega,$$

i.e. $\left\|E_\Omega(x)\left(-i\dfrac{\partial}{\partial x}+i\right)^{-2}\right\| \leqslant \sqrt{\Delta}$. Besides

$$E_\Omega(x)\, e^{-t\frac{\partial}{\partial x}} E_\Omega(x) = E_\Omega(x)\, E_\Omega(x-t) = 0$$

for $t > c_1\Delta$.

(2) Let $A = x^2$, $B = i\dfrac{\partial^2}{\partial x^2}$. Then for $t > \Delta^{1/2+\varepsilon}$

$$\left\| E_\Omega(x^2)\, e^{itB} E_\Omega(x^2) f(x)\right\|^2 \leqslant$$

$$\leqslant \frac{c}{t}\int_\Omega dx \left|\int_\Omega e^{\frac{i(x-\xi)^2}{2t}} f(\xi)\, d\xi\right|^2 \leqslant \frac{c}{t}\,\Delta^2\, \|f(x)\|^2 \leqslant c\Delta^{3/2-\varepsilon}\, \|f\|^2,$$

i.e. condition (4) is fulfilled for $\varepsilon < 1/6$.

Condition (3) is fulfilled by virtue of the previous evaluation.

For simplicity, below we shall set $r=1$. Now assume that $F(t, x)$ is a function such that there exists a limit $f(t, x)$ of the expression

$$\prod_{k=1}^{N} F(t_k, x)^{\Delta t}, \qquad N=\frac{t}{\Delta t}\to\infty$$

belonging to the space $\mathscr{B}_l(\mathbf{R})$ for all $t>0$ and $\varphi(t, x, y)\in\mathscr{B}_l(\mathbf{R}^2)$ is continuous in t with the function $\int e^{-ipy}\varphi(t, x, y)\,dy$ being continuous on the line $p=0$*. To simplify the proof we shall suppose in addition that $F(t, x)\neq 0$ and let $\Phi(t, x)=\ln F(t, x)$. Thus we assume that

$$e^{\int_0^t \Phi(\tau, x)d\tau}\in\mathscr{B}_l.$$

Consider the T-product.

$$\lim_{N\to\infty} V_N=\lim_{N\to\infty}\prod_{k=1}^{N}\exp\left\{\left[\Phi\left(t_k, \overset{t_k}{A}\right)+\varphi\left(t_k, \overset{t_k}{A}, \overset{t_k+\Delta t/2}{B}\right)\right]\Delta t\right\} \tag{5}$$

with A and B satisfying the spectra incompatibility conditions. It is known that under the above conditions $\lim\limits_{N\to\infty} V_N\overset{\text{def}}{=\!=}u(t)$ exists and satisfies the equation

$$\frac{\partial u(t)}{\partial t}=(\hat{\Phi}+\hat{\varphi})\,u(t), \tag{6}$$

$$\hat{\varphi}=\varphi\left(t, \overset{1}{A}, \overset{2}{B}\right), \quad \hat{\Phi}=\Phi(t, A).$$

The proof of this assertion is left to the reader.

Let $E_{\Delta, x^0}(x)$ be a characteristic function of the interval

$$\left[x^0+\frac{\Delta}{2},\ x^0-\frac{\Delta}{2}\right).$$

First we shall point out that under the above conditions

$$\| E_{\Delta, x^0}\hat{\varphi}E_{\Delta, x^0}\|\leqslant c_3\Delta^{1/2+\varepsilon}.$$

Here $E_{\Delta, x^0}=E_{\Delta, x^0}(A)$. Hence

$$\| E_{\Delta, x^0}\hat{\varphi}E_{\Delta, x^0}\|_{H_M'\to H}\leqslant\| E_{\Delta, x^0}\hat{\varphi}E_{\Delta, x^0}\|\,\| E_{\Delta, x^0}\|_{H_M'\to H}\leqslant c_4\Delta^{1+\varepsilon}, \tag{7}$$

---

* Instead of the above conditions we can use the following:

$|\varphi|\leqslant(|x|+|y|)^{-M}$, $\tilde{\varphi}\in L_1$, and $f(x, t)\in C$.

c_3 and c_4 being independent of x. In fact,

$$\Big\| \iint\limits_{|p_1|\leqslant c_1\Delta^{1/2+\varepsilon}} E_{\Delta, x^0}(A)\, e^{iBp_1} E_{\Delta, x^0}(A) \times$$

$$\times e^{iAp_2}\widetilde{\varphi}(t, p_1, p_2)\, dp_1\, dp_2 \Big\| \leqslant$$

$$\leqslant \int\limits_{|p_1|\leqslant c_1\Delta^{1/2+\varepsilon}} \Big\| \int e^{-ip_1 y} \varphi(t, A, y)\, dy \Big\|\, dp_1 \leqslant$$

$$\leqslant c_1 \Delta^{1/2+\varepsilon} \max_{x,\, |p_1|\leqslant c_1\Delta^{1/2+\varepsilon}} \Big| \int e^{-ip_1 y}\varphi(t, x, y)\, dy \Big| \leqslant$$

$$\leqslant \text{const}\ \Delta^{1/2+\varepsilon},$$

$$\Big(h_1^*, \iint\limits_{|p_1|\geqslant c_1\Delta^{1/2+\varepsilon}} E_{\Delta, x^0}(A)\, e^{iBp_1}\, E_{\Delta, x^0}(A) \times$$

$$\times e^{iAp_2}\widetilde{\varphi}(t, p_1, p_2)\, dp_1\, dp_2 h\Big) \leqslant$$

$$\leqslant \max_{|p_1|\geqslant c_1\Delta^{1/2+\varepsilon},\ p_2\in(-\infty, \infty)} \Big| (h_1^*, E_{\Delta, x^0}(A) \times$$

$$\times e^{iBp_1} E_{\Delta, x^0}(A)\, e^{iAp_2} h\Big) \Big|\, \|\widetilde{\varphi}\|_{C+(|p_1|\geqslant c_1\Delta^{1/2+\varepsilon})} \leqslant$$

$$\leqslant c\, \|h_1^*\|\ \|h\|\, \Delta^{1/2+\varepsilon};\quad h_1^*,\ h\in H.$$

This completes the proof of the assertion. Consider

$$V_{N, x^0}(\Delta) = \prod_{k=1}^{N} E_{\Delta, x^0}\Big(\overset{t_k}{A}\Big) \exp\Big\{\Big[\Phi\Big(t_k, \overset{k}{A}\Big) +$$

$$+ \varphi\Big(t_k, \overset{t_k}{A}, \overset{t_k+\Delta t/2}{B}\Big)\Big]\Delta t\Big\}.$$

Prove that $\lim\limits_{N\to\infty} V_{N, x^0}(\Delta) = E_{\Delta, x^0}(A)\, \psi_{\Delta, x^0}(t)$, with $\psi_{\Delta, x^0}(t)$ satisfying the equation

$$\frac{\partial \psi_{\Delta, x^0}}{\partial t} = E_{\Delta, x^0}[\hat{\Phi} + \hat{\varphi}]\, E_{\Delta, x^0}\psi_{\Delta, x^0}, \tag{9}$$

$$\hat{\Phi} = \Phi(t, A),\quad \hat{\varphi} = \varphi\Big(t, \overset{1}{A}, \overset{2}{B}\Big).$$

Since $(E_{\Delta, x^0})^n = E_{\Delta, x^0}$

$$E_{\Delta, x^0}\left(\overset{1}{A}\right) \exp\left\{\left[\Phi(t, A) + \varphi\left(t, \overset{1}{A}, \overset{2}{B}\right)\right]\Delta t\right\} E_{\Delta, x^0} =$$
$$= E_{\Delta, x^0}\left(\overset{1}{A}\right) \exp\left\{E_{\Delta, x^0}\left(\overset{3}{A}\right)\left[\Phi\left(t, \overset{1}{A}\right) + \varphi\left(t, \overset{1}{A}, \overset{2}{B}\right)\right] \times\right.$$
$$\left.\times E_{\Delta, x^0}\left(\overset{1}{A}\right)\Delta t\right\} = E_{\Delta, x^0}\left(\overset{1}{A}\right)\left\{1 + \left[E_{\Delta, x^0}\left(\overset{1}{A}\right) \times\right.\right.$$
$$\left.\left.\times \Phi\left(t, \overset{1}{A}\right) + \varphi\left(t, \overset{1}{A}, \overset{2}{B}\right)\right] E_{\Delta, x^0}\left(\overset{3}{A}\right)\Delta t\right\} +$$
$$+ (\Delta t)^2 \sum_{n=0}^{\infty} \frac{(\Delta t)^n}{(n+2)!} \times$$
$$\times \left\{E_{\Delta, x^0}\left(\overset{3}{A}\right)\left[\Phi\left(t, \overset{1}{A}\right) + \varphi\left(t, \overset{1}{A}, \overset{2}{B}\right)\right] E_{\Delta, x^0}\left(\overset{1}{A}\right)\right\}^{n+2}. \tag{10}$$

But

$$\left\|\left\{E_{\Delta, x^0}\left(\overset{1}{A}\right)\left[\Phi\left(t, \overset{1}{A}\right) + \varphi\left(t, \overset{1}{A}, \overset{2}{B}\right)\right] E_{\Delta, x^0}\left(\overset{3}{A}\right)\right\}^n\right\| \leqslant$$
$$\leqslant \sum C_k^n \,\| E_{\Delta, x^0}\hat{\Phi}^k \| \cdot \left\| \varphi^k\left(t, \overset{1}{A}, \overset{2}{B}\right)\right\| \leqslant$$
$$\leqslant \sum C_k^n \left(\max_{x \in \mathbf{R}} |\Phi|\right)^k \|\varphi\|_{\mathscr{B}_l}^{n-k} = \left(\max_x |\Phi| + \|\varphi\|_{\mathscr{B}_l}\right)^n.$$

Therefore the term relative to the sign $\sum$ in (10) is bounded in the norm, hence

$$V_N(\Delta) = \prod_{k=1}^{N} E_{\Delta, x^0}\left(\overset{t_k}{A}\right)\left\{1 + \Delta t E_{\Delta, x^0}\left(\overset{t_k}{A}\right)\left[\Phi\left(t_k, \overset{t_k}{A}\right) +\right.\right.$$
$$\left.\left.+ \varphi\left(t_k, \overset{t_k}{A}, \overset{t_k+\Delta t/2}{B}\right)\right] E_{\Delta, x^0}\left(\overset{t_{k+1}}{A}\right)\right\} + O(\Delta t) =$$
$$= E_{\Delta, x^0}\left(\overset{t_N}{A}\right) \prod_{k=1}^{N}\left\{1 + \Delta t E_{\Delta, x^0}\left(\overset{t_k}{A}\right)\left[\Phi\left(t_k, \overset{t_k}{A}\right) +\right.\right.$$
$$\left.\left.+ \varphi\left(t_k, \overset{t_k}{A}, \overset{t_k+\Delta t/2}{B}\right)\right] E_{\Delta, x^0}\left(\overset{t_{k+1}}{A}\right)\right\} + O(\Delta t) =$$
$$= E_{\Delta, x^0}(A) \prod_{k=1}^{N} \exp\left\{\overset{t_k}{[\![} E_{\Delta, x^0}\left(\overset{1}{A}\right)\left[\Phi\left(t_k, \overset{1}{A}\right) +\right.\right.$$
$$\left.\left.+ \varphi\left(t_k, \overset{1}{A}, \overset{2}{B}\right)\right] E_{\Delta, x^0}\left(\overset{3}{A}\right)]\!]\,\Delta t\right\} + O(\Delta t)$$

due to the boundedness of the norm of the operator in the autonomous brackets. Hence (9) easily follows.

Now sum expression (2) with respect to all $E_{\delta, y_i^0}(B)$. We obtain

$$V_N = \sum_{j_1, \ldots, j_N} \prod_{k=1}^{N} E_{\delta, x_{j_k}^0}\left(\overset{t_k}{A}\right) \exp\left\{\left[\Phi\left(t_k, \overset{t_k}{A}\right) + \right.\right.$$

$$\left.\left. + \varphi\left(t_k, \overset{t_k}{A}, \overset{t_k+\Delta t/2}{B}\right)\right]\Delta t\right\}. \qquad (11)$$

Now consider one term $V_N(\delta)$ of the sum corresponding to the interval $\{x_1^0\} \times (0, T)$. Namely

$$V_{N, x_1^0}(\delta) = \prod_{k=1}^{N} E_{\delta, x_1^0}\left(\overset{t_k}{A}\right) \exp\left\{\left[\Phi\left(t_k, \overset{t_k}{A}\right) + \right.\right.$$

$$\left.\left. + \varphi\left(t_k, \overset{t_k}{A}, \overset{t_k+\Delta t/2}{B}\right)\right]\Delta t\right\}.$$

Let $\psi_{\delta, x_1^0}|_{t=0} = \psi_0 \in H_s''$ where $(i+A)^s \psi_0 \in H_M', M$, $s = r+1 = 2$.

Set $\psi = e^{\int_0^t \Phi(\tau, A)\, d\tau} E_{\delta, x_1^0}(A)\, \psi_0$.

We obtain

$$\frac{\partial\left[\psi_{\delta, x_1^0} - \psi\right]}{\partial t} + E_{\delta, x_1^0}\hat{\Phi}\left(\psi_{\delta, x_1^0} - \psi\right) = -E_{\delta, x_1^0}\hat{\varphi}E_{\delta, x_1^0}\psi_{\delta, x_1^0}.$$

Hence

$$\psi_{\delta, x_1^0} = \psi - \int_0^t \left[\exp\left\{\int_{t'}^{t} \Phi(\tau, A)\, d\tau\right\} E_{\delta, x_1^0}\hat{\varphi}E_{\delta, x_1^0}\right] \times$$

$$\times\, \psi_{\delta, x_1^0}(t')\, dt' \overset{\text{def}}{=} \psi - \int_0^t K_\delta(t, t')\, \psi_{\delta, x_1^0}(t')\, dt' =$$

$$= \psi - \int_0^t K_\delta(t, t')\, \psi(t')\, dt' +$$

$$+ \int_0^t K_\delta(t, t')\, dt' \int_0^{t'} K_\delta(t', t'')\, \psi_{\delta, x_1^0}(t'')\, dt''. \qquad (12)$$

Evaluate the operator $K_\delta(t, t')$ from space H''_s into H''_s. We have

$$\|(i+A)^s K_\delta(t, t')(i+A)^{-s}\| \leqslant \left\| \exp\left\{\int_{t'}^{t} \Phi(\tau, A)\, d\tau\right\} E_{\delta, x_1^0} \right\| \times$$

$$\times \|(i+A)^s \hat{\varphi}(i+A)^{-s}\| \leqslant \left\| \exp\left\{\int_{t'}^{t} \Phi(\tau, x)\, d\tau\right\|_{\mathscr{B}_l} \times$$

$$\times \|\varphi(t, x, y)\|_{\mathscr{B}_l}.$$

Hence it follows that $\psi_{\delta, x_1^0} \in H''_s$ and its norm is independent of x_1^0 and δ. Therefore

$$\|\psi_{\delta, x_1^0} - \psi\| \leqslant c_1 \|E_{\delta, x_1^0}\hat{\varphi}E_{\delta, x_1^0}\| \times$$

$$\times \|E_{\delta, x_1^0}\|_{H'_M \to H} \|E_{\delta, x_1^0}(A)(i+A)^{-s}\|' \times$$

$$\times \left\| \exp\left\{\int_{t'}^{t} \Phi(\tau, A)\, d\tau\right\}\right\|_{H'_M \to H'_M} \|(i+A)^s \psi_0\|'_M +$$

$$+ c_2 \|E_{\delta, x_1^0}\hat{\varphi}E_{\delta, x_1^0}\|^2 \|E_{\delta, x_1^0}(A)(i+A)^{-s}\| \|\psi_{\delta, x_1^0}\|''_s.$$

Hence by virtue of (7) we have

$$\|\psi_{\delta, x_1^0} - \psi\| \leqslant c_6 \delta^{1+\varepsilon}(1+|x_1^0|)^{-s}.$$

We shall evaluate one term of the sum (11) which corresponds to a straight trajectory in the x-plane, i.e. to the interval $\{x_1^0\} \times \times (0, T)$. Now take the sum of the terms corresponding to the intervals

$$\{x_j^0\} \times (0, T), \quad x_j^0 = (x_1^0 + j\delta), \quad j = 1, \; -1, \; \ldots.$$

We have

$$\left\| \sum_j \psi_{\delta, x_1^0 + j\delta} - \exp\left\{\int_{t'}^{t} \Phi(\tau, A)\, d\tau\right\} \sum_j E_{\delta, x_1^0 + j\delta}\psi_0 \right\| =$$

$$= \left\| \sum_j \psi_{\delta, x_1^0 + j\delta} - \exp\left\{\int_{t'}^{t} \Phi(\tau, A)\, d\tau\right\} \psi_0 \right\| \leqslant$$

$$\leqslant c_6 \delta^{\varepsilon} \sum_j \delta_j (1 + |x_1^0 + \delta j|)^{-s} \leqslant c_7 \delta^{\varepsilon}.$$

As $\delta \to 0$ we obtain that the sum of all the terms corresponding to straight trajectories in (11) is equal to the first term of the perturbation series in operator $\hat{\varphi}$ for equation (5).

Consider now the step $\{x_i^0\}\times(0, t')\cup\{x_n^0\}\times(t', T)$ $i\neq n$ with arbitrary values at the points of discontinuity. We put a tube around it $x_i^0-\frac{\delta}{2}\leqslant x\leqslant x_i^0+\frac{\delta}{2}$ when $0\leqslant t\leqslant t'-\delta'$ and $-\infty\leqslant x\leqslant\infty$ when $t'-\delta'<t<t+\delta'$ and $x_n^0-\frac{\delta}{2}\leqslant x\leqslant x_n^0+\frac{\delta}{2}$ when $t'+\delta'\leqslant\leqslant t\leqslant T$. We must now take the product in (11), which corresponds to $\{x_i^0\}\times(0, t'-\delta')$, sum with respect to all $E_{\delta,\, x_k^0}$ for $t\in(t-\delta', t+\delta')$, take again the product corresponding to the interval $\{x_n^0\}\times\times(t'+\delta', T)$ and proceed to the limit as $N\to\infty$. Let $V_{in}(\delta, \delta')$ be the resulting limit. This means that we must solve equation (9) for $x^0=x_i^0$, $\Delta=\delta$ up to the point $t'-\delta'$, set the resulting solution at the moment $t=t'-\delta'$ as the initial condition for equation (6). The resulting solution must be put again as the initial condition for (9) at the moment $t'+\delta'$, for $x^0=x_n^0$, $\Delta=\delta$. We have an integral equation of type (12) for $t\in(t'-\delta', t'+\delta')$

$$\psi_i(t'+\delta')=\psi^1(t'+\delta')-\int_{t'-\delta'}^{t'+\delta'} K_\infty(t'+\delta', t'')\,\psi^1(t'')\,dt''+$$

$$+\int_{t'-\delta'}^{t'+\delta'} dt\int_{t'-\delta'}^{t} K_\infty(t'+\delta', t)\,K_\infty(t, t'')\,\psi_i(t'')\,dt'';$$

$$\psi^1(t)=\exp\left\{\int_{t'-\delta'}^{t}\Phi(\tau, A)\,d\tau\right\}\psi_{\delta,\, x_i^0}(t'-\delta');$$

$$\psi_i(t'-\delta')=\psi_{\delta,\, x_i^0}(t'-\delta').$$

At the moment $t'+\delta'$ we act on ψ_i with the operator $E_{\delta,\, x_n^0}(A)$, but $\psi^1=E_{\delta,\, x_i^0}(A)\,\psi^1$. Hence

$$E_{\delta,\, x_n^0}\psi_i(t'+\delta')=\int_{t'-\delta'}^{t'+\delta'}\left[\exp\left\{\int_{t}^{t'+\delta'}\Phi(\tau, A)\,d\tau\right\}\times\right.$$

$$\left.\times E_{\delta,\, x_n^0}\hat{\varphi}E_{\delta,\, x_i^0}\psi^1(t)\right]dt+$$

$$+E_{\delta,\, x_n^0}\int_{t'-\delta'}^{t'+\delta'} dt\int_{t'-\delta'}^{t} K_\infty(t'+\delta', t)\,K_\infty(t, t'')\,\psi_i(t'')\,dt''.$$

Therefore

$$V_{in}(\delta,\ \delta') = \int\limits_{t'-\delta'}^{t'+\delta'} dt \Big[\exp \Big\{ \int\limits_{t}^{T} \Phi(\tau,\ A)\, d\tau \Big\} \times$$

$$\times \{1 + O_1(\delta^{1+\varepsilon} |\, x_n^0 \,|^{-s})\} E_{\delta,\ x_n^0} \hat{\varphi} \times$$

$$\times E_{\delta,\ x_i^0} \exp \Big\{ \int\limits_{0}^{t} \Phi(\tau,\ A)\, d\tau \Big\} \{1 +$$

$$+ O_2(\delta^{1+\varepsilon} |\, x_i^0 \,|^{-s})\} \psi_0 \Big] + \exp \Big\{ \int\limits_{t'+\delta'}^{T} \Phi(\tau,\ A)\, d\tau \Big\} \times$$

$$\times E_{\delta,\ x_n^0} \{1 + O_1(\delta^{1+\varepsilon} |\, x_n^0 \,|^{-s})\} \int\limits_{t'-\delta'}^{t'+\delta'} dt \times$$

$$\times \int\limits_{t'-\delta'}^{t} K_\infty(t' + \delta',\ t)\, K_\infty(t,\ t'')\, \psi_i(t'')\, dt''. \tag{14}$$

Here O_1 is independent of i, O_2 is independent of n and $K(t,\ t')$ is independent of i and n. In the second term only the function $\psi_i(t')$ is dependent of i. By virtue of (13) we have

$$\psi_i = e^{\int\limits_0^t \Phi(\tau,\ A) d\tau} E_{\delta,\ x_i^0} \{1 + O_2(\delta^{1+\varepsilon} |\, x_i^0 \,|^{-s})\} \psi_0 +$$

$$+ \int\limits_{t'-\delta'}^{t'+\delta'} K_\infty(t' + \delta',\ t)\, \psi_i(t)\, dt.$$

Summing both sides of the equation with respect to i we obtain the equation for $G = \sum \psi_i$:

$$G = e^{\int\limits_0^t \Phi(\tau,\ A) d\tau} (1 + O_3(\delta^\varepsilon))\, \psi_0 + \int\limits_{t'-\delta'}^{t'+\delta'} K_\infty(t' + \delta',\ t)\, G(t)\, dt$$

where O_3 is independent of n. Sum with respect to i and n both sides of (14) for $i \neq n$. For this we must sum with respect to all n and i and subtract the sum in $n = i$ from the expression obtained. By virtue of the above the latter sum is equal to $O(\delta^\varepsilon)\,\delta'$.

Hence

$$\sum_{i\neq n} V_{in}(\delta,\ \delta') = O(\delta^{\varepsilon})\,\delta' + \sum_{i,\ n} V_{in}(\delta,\ \delta') =$$

$$= \int\limits_{t'-\delta'}^{t'+\delta'} \Big[\exp\Big\{\int\limits_{t}^{T} \Phi(\tau,\ A)\,d\tau\Big\}\{1+O(\delta^{\varepsilon})\}\,\hat{\varphi}\times$$

$$\times \exp\Big\{\int\limits_{0}^{t} \Phi(\tau,\ A)\,d\tau\Big\}\{1+O(\delta^{\varepsilon})\}\,\psi_0\Big]\,dt +$$

$$+\exp\Big\{\int\limits_{t+\delta'}^{T} \Phi(\tau,\ A)\,d\tau\Big\}\{1+O(\delta^{\varepsilon})\}\int\limits_{t'-\delta'}^{t'+\delta'} dt \int\limits_{t'-\delta'}^{t} K_{\infty}\times$$

$$\times (t'+\delta',\ t)\,K_{\infty}(t,\ t'')\,G(t'')\,dt''.$$

Divide the interval $(0,\ T)$ into subintervals of the length $2\delta'$. Setting $t' = t_k' = 2k\delta'$, sum with respect to k. It is easily seen that approaching the limit as $\delta' \to 0$ and $\delta \to 0$ gives an expression of the form

$$\int\limits_{0}^{T} \Big[\exp\Big\{\int\limits_{t}^{T} \Phi(\tau,\ A)\,d\tau\Big\}\,\hat{\varphi}\exp\Big\{\int\limits_{0}^{t} \Phi(\tau,\ A)\,d\tau\Big\}\,\psi_0\Big]\,dt$$

which is to be proved.

Similarly for the sum corresponding to the steps with n discontinuities we obtain the n-th term of the perturbation theory series in $\hat{\varphi}$.

Since the sum of the n terms of the perturbation series differs from the solution of equation (5) by $O\left(\frac{1}{n!}\right)$, we obtain that the sum (11) with respect to stepwise trajectories with no more than n discontinuities differs from the sum (11) with respect to all trajectories by $O\left(\frac{1}{n!}\right)$. This signifies that the projection of the T-product spectrum onto the space of the trajectory $x(t)$ consists of stepwise trajectories. The spectral expansion (defined in the Introduction) of the T-product in stepwise trajectories is identical with the expansion in the perturbation series in $\hat{\varphi}$.

Further note that since $E_{\Delta,\,x^0}(A)\,E_{\Delta,\,y^0}(B)\,E_{\Delta,\,x^0}(A)$ is less than unity for sufficiently small Δ by virtue of (4), the terms of the sum (2) containing the straight interval $\{x^0\}\times(t_1,\ t_2)$ decrease as $e^{-\alpha r}$ (α being a constant), where r is the number of the terms of

the form E_{Δ, y_i^0}, for r of different values of time and finite Δ. This means that the spectrum consists of trajectories which are everywhere discontinuous and unbounded on any time interval not equal to zero.

It is easily seen that for sufficiently small δ the value $E_{\delta, x_i^0} \exp\{\Phi(t_k, A)\Delta t\}$ is close to the value $\exp\{\Phi(t_k, x_i^0)\Delta t\} E_{\delta, x_i^0}$, hence the sum (11) can be interpreted as an integral of the functional $\exp\left\{\int\limits_0^T \Phi(t, x(t))\,dt\right\}$ relative to complex spectral measure * concentrated on stepwise trajectories; the variation of the measure on trajectories is bounded. It is established as above if we put $\Phi \equiv 0$.

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* The exact results involving this measure when $A = i\frac{\partial}{\partial x}$, $B = x$ under less limiting conditions obtained jointly by the author and A. G. Chebotarev are given in the author's book *Complex Markov Chains and the Feynman Continuum Integral*. M., Nauka, 1976.

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